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SIMULTANEOUS ESTIMATION OF LOCATION PARAMETERS OF THE DISTRIBUTION WITH FINITE SUPPORT

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Summary

Let X_i, $i=1,\cdots,p$ be the ith component of the $p\times 1$ vector $X=(X_1, X_2,\cdots, X_p)'$. Suppose that $X_1, X_2,\cdots, X_p$ are independent and that X_i has a probability density which is positive on a finite interval, is symmetric about θ_i and has the same variance. In estimation of the location vector $\theta=(\theta_1, \theta_2,\cdots, \theta_p)'$ under the squared error loss function explicit estimators which dominate X are obtained by using integration by parts to evaluate the risk function. Further, explicit dominating estimators are given when the distributions of X_i's are mixture of two uniform distributions. For the loss function $L(\hat{\theta}, \theta)=\|\hat{\theta}-\theta\|^4$ such an estimator is also given when the distributions of X_i's are uniform distributions.

1. Introduction

Stein [7] and Brown [3] proved that the best invariant estimator of the location vector of three or more dimensions are inadmissible, and there has been considerable interest in how to improve it. James and Stein [5] presented an explicit estimator $\{1-(p-2)/\|X\|^2\}X$, which is better than X under squared error loss if X has a normal distribution with covariance matrix I, the identity matrix. They also showed that the estimator

$$\delta_1(X)=\left\{1-\frac{b}{a+\|X\|^2}\right\}X\,, \tag{1.1}$$

is better than X, without the normality assumption, for sufficiently small b and sufficiently large a. They did not, however, determine explicitly the values of these constants.

When $X=(X_1, X_2,\cdots, X_p)'$ is an observed value from a spherically

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symmetric p-dimensional distribution, explicit estimators of a location vector which dominate X are given. See Brandwein and Strawderman [2] and the papers in their references. Shinozaki [6] obtained similar results in the case where $X_1, X_2, \cdots, X_p$ are independent, identically and symmetrically distributed p random variables, by applying integration by parts to three typical distributions; uniform, double exponential and t. Since Stein [8] used integration by parts for estimating the location parameter of the normal distribution, it has been shown to apply to simultaneous estimation problems in general continuous exponential family by many authors. See Hudson [4].

In this paper, let X_i, $i=1,\cdots,p$ be the ith components of the $p\times 1$ vector $X=(X_1, X_2, \cdots, X_p)'$. Suppose that $X_1, X_2, \cdots, X_p$ are independent and that X_i has a probability density which is positive on a finite interval, is symmetric about θ_i, the center of the interval, and has the same variance, and we estimate the location vector $\theta=(\theta_1, \theta_2, \cdots, \theta_p)'$ by the estimator δ_1 of (1.1) under the squared error loss function.

Berger [1] showed some results for losses which are polynomial in the coordinates of $(\hat{\theta}-\theta)$ for the normal case, and Brandwein and Strawderman [2] for the spherically symmetric distribution when the loss is a nondecreasing concave function of quadratic loss. Here we also study a special form of Berger's loss function for the uniform distribution.

In Section 2, some sufficient conditions on the constants a and b for the estimator δ_1 to dominate X are given. Further the constants in another estimator

$$\delta_2(X)=\left\{1-\frac{b(I-B)}{a+X'(I-B)X}\right\}X, \tag{1.2}$$

where B is a $p\times p$ projection matrix and I is the identity matrix, are determined. In Section 3, the results in Section 2 are applied to the truncated normal, the parabola and the cusp shaped distributions which are defined by (3.1), (3.3) and (3.5), respectively. The cusp shaped distribution is the distribution of the best invariant estimator of the location parameter of the uniform distribution.

In Section 4, the values of a and b for the estimator δ_1 to dominate X are given when the distributions of X_i's are mixture of two uniform distributions with a common center.

In Section 5, we give sufficient conditions on the constants a and b for the estimator δ_1 to dominate X under the loss $L(\delta_1, \theta)=\|\delta_1-\theta\|^4$ when the distributions of X_i's are uniform distributions.

2. Estimation of location parameters of the distributions with finite support

Let X_i, $i=1,\cdots,p$, be the ith component of the $p\times 1$ vector $X=(X_1, X_2,\cdots, X_p)'$ and a random variable from a probability density of the form

$$f_i(x_i-\theta_i)=\begin{cases} f_i(|x_i-\theta_i|)>0\,, & \text{if } |x_i-\theta_i|\leqq c_i\,,\\ 0\,, & \text{otherwise}\,. \end{cases} \tag{2.1}$$

Suppose that $X_1, X_2,\cdots, X_p$ are independent and have the same variance V. Set $Z_i=X_i-\theta_i$, and assume $\mathrm{E}\,Z_i=0$. In estimating the location vector $\theta=(\theta_1, \theta_2,\cdots, \theta_p)'$ by $\delta_1(X)=(\delta_{11}(X), \delta_{12}(X),\cdots, \delta_{1p}(X))'$, some sufficient conditions on the constants a and b of (1.1) are given such as the risk $R(\delta_1, \theta)=\mathrm{E}_X\,\|\delta_1-\theta\|^2$ is uniformly smaller than $R(X, \theta)$, where

$$\|\delta_1(X)-\theta\|^2=\sum_{i=1}^{p}(\delta_{1i}(X)-\theta_i)^2\,.$$

E_X denotes the expectation with respect to X.

THEOREM 2.1. *Let X_i have a probability density of the form* (2.1). *Assume the following conditions be satisfied for $i=1,\cdots,p$:*

$$\mathrm{E}\,Z_i^4<\infty\,, \tag{2.2}$$

and there exists a constant $d_i>0$ such that

$$\left|\int_{-c_i}^{z}\int_{-c_i}^{y}\left(\int_{-c_i}^{u} tf_i(t)dt/V+f_i(u)\right)dudy\right|\leqq d_if_i(z)\,, \tag{2.3}$$

where $V=\int_{-c_i}^{c_i} z^2f_i(z)dz$. Then the risk of δ_1 is uniformly smaller than that of X if $a\geqq 6\sum_{i=1}^{p} d_i/p$ and $0<b\leqq 2(p-2)V$.

PROOF.

$$R(X, \theta)-R(\delta_1, \theta)=2b\,\mathrm{E}\left\{\sum_{i=1}^{p}\frac{X_i(X_i-\theta_i)}{a+\|X\|^2}-\frac{b\|X\|^2}{2(a+\|X\|^2)^2}\right\}\,. \tag{2.4}$$

The conditional expectation of a term of the summation in (2.4) equals

$$I\equiv\mathrm{E}_{X_i}\left\{\frac{X_i(X_i-\theta_i)}{a+\|X\|^2}\right\}=\mathrm{E}_{Z_i}\left\{\frac{Z_i(Z_i+\theta_i)}{a+\|Z+\theta\|^2}\right\}\,.$$

Integration by parts gives

$$I=-\int_{-c_i}^{c_i} g_1(z_i)\int_{-c_i}^{z_i} yf_i(y)dydz_i=V\,\mathrm{E}_{Z_i}\{g_1(Z_i)\}+I_1\,, \tag{2.5}$$

where

$$g_1(z_i)=\frac{1}{a+\|z+\theta\|^2}-\frac{2(z_i+\theta_i)^2}{(a+\|z+\theta\|^2)^2} \tag{2.6}$$

and

$$I_1=-V\int_{-c_i}^{c_i} g_1(z_i)\Big(\int_{-c_i}^{z_i} yf_i(y)dy/V+f_i(z_i)\Big)dz_i\ .$$

Integration by parts is applied to I_1 twice, then

$$I_1=6V\int_{-c_i}^{c_i} g_2(z_i)\Big\{\int_{-c_i}^{z_i}\int_{-c_i}^{y}\Big(\int_{-c_i}^{u} tf_i(t)dt/V+f_i(u)\Big)dudy\Big\}dz_i\ , \tag{2.7}$$

where

$$g_2(z_i)=\frac{1}{(a+\|z+\theta\|^2)^2}-\frac{8(z_i+\theta_i)^2}{(a+\|z+\theta\|^2)^3}+\frac{8(z_i+\theta_i)^4}{(a+\|z+\theta\|^2)^4}\ .$$

Note that $\int_{-c_i}^{y}\Big(\int_{-c_i}^{u} tf_i(t)dt/V+f_i(u)\Big)du$ is an odd function and that $\int_{-c_i}^{z}\int_{-c_i}^{y}\Big(\int_{-c_i}^{u} tf_i(t)dt/V+f_i(u)\Big)dudy$ is an even function.

The inequality $|g_2(z_i)|\leqq 1/(a+\|z+\theta\|^2)^2$ and the condition (2.3) show that

$$I_1\geqq -6d_iV\,\mathrm{E}_{X_i}\left\{\frac{1}{(a+\|X\|^2)^2}\right\}\ , \tag{2.8}$$

and from the last expression of (2.5) and (2.8),

$$I\geqq V\,\mathrm{E}_{X_i}\left\{\frac{1}{a+\|X\|^2}-\frac{2X_i^2}{(a+\|X\|^2)^2}-\frac{6d_i}{(a+\|X\|^2)^2}\right\}\ .$$

Hence from (2.4),

$$R(X,\theta)-R(\delta_1,\theta)\geqq 2bV\,\mathrm{E}\left\{\frac{ap+(p-2)\|X\|^2-6\sum_{i=1}^{p} d_i}{(a+\|X\|^2)^2}-\frac{b\|X\|^2}{2V(a+\|X\|^2)^2}\right\}\ .$$

which is nonnegative if $a\geqq 6\sum_{i=1}^{p} d_i/p$ and $0<b\leqq 2(p-2)V$.

Remark 2.1. Note that $-\int_{-\infty}^{y} tf_i(t)dt/V=f_i(y)$ in the normal case. It is shown, by using integration by parts also, that

$$-\int_{-c_i}^{c_i}\int_{-c_i}^{z}\int_{-c_i}^{y}\Big(\int_{-c_i}^{u} tf_i(t)dt/V+f_i(u)\Big)dudydz=\{\mathrm{E}\,Z_i^4-3(\mathrm{E}\,Z_i^2)^2\}/6\,\mathrm{E}\,Z_i^2\ .$$

Under a stronger condition an alternative sufficient condition is obtained.

THEOREM 2.2. *Let f_i satisfy the conditions of Theorem 2.1 and*

$$\int_{-c_i}^{z}\int_{-c_i}^{y}\left(\int_{-c_i}^{u} tf_i(t)dt/V+f_i(u)\right)dudy\geqq 0\,,\qquad |z|\leqq c_i\,. \tag{2.9}$$

Then the risk of δ_1 is uniformly smaller than that of X if $a\geqq 24(2-\sqrt{2})\max_{1\leqq i\leqq p} d_i/p$ and $0<b\leqq 2(p-2)V$.

PROOF. The inequality

$$g_2(z_i)\geqq -4(2-\sqrt{2})(z_i+\theta_i)^2/(a+\|z+\theta\|^2)^3\,,$$

(2.7) of Theorem 2.1 and the condition (2.9) give

$$I_1\geqq -24(2-\sqrt{2})V\,\mathrm{E}_{X_i}\{d_iX_i^2/(a+\|X\|^2)^3\}\,. \tag{2.10}$$

Therefore from the last expression of (2.5) of Theorem 2.1 and (2.10), $R(X,\theta)-R(\delta_1,\theta)\geqq 0$ if the conditions on a and b are satisfied.

In Theorems 2.1 and 2.2 it is shown that the estimator δ_1, which pulls X_i's towards the origin, dominates X. Here the estimator δ_2 of (1.2), which pulls the estimators towards a sub-space spanned by B, is considered, and some sufficient conditions on the constants a and b of (1.2) are given such as $R(X,\theta)-R(\delta_2,\theta)$ is nonnegative.

THEOREM 2.3. *Let f_i satisfy the conditions of Theorem 2.1. Then the risk of δ_2 is uniformly smaller than that of X if*

$$a\geqq\frac{6\left\{8\sum_{i=1}^{p} b_{ii}d_i+\sum_{i=1}^{p}(1-b_{ii})^2d_i\right\}}{\sum_{i=1}^{p}(1-b_{ii})}$$

and if

$$0<b\leqq 2\left(p-\sum_{i=1}^{p} b_{ii}-2\right)V\,,$$

where $B=(b_{ij})$.

PROOF. Theorem 2.3 can be proved in a similar way as Theorem 2.1 and the proof is omitted.

THEOREM 2.4. *Let f_i satisfy the conditions of Theorem 2.2. Then the risk of δ_2 is uniformly smaller than that of X if*

$$a\geqq 6\sum_{i=1}^{p}(1-b_{ii})^2d_i\Big/\sum_{i=1}^{p}(1-b_{ii})$$

and if

$$0<b\leqq 2\Big(p-\sum_{i=1}^{p} b_{ii}-2\Big)V .$$

COROLLARY 2.5. *Let f_i satisfy the conditions of Theorem 2.2. Assume that the diagonal elements of B are equal. Then the risk of δ_2 is uniformly smaller than that of X if $a\geqq 24(2-\sqrt{2})\max d_i/p$ and $0<b\leqq 2\{p(1-b_{ii})-2\}V$.*

Remark 2.2. If $B=(b_{ij})$, $b_{ij}=1/p$ for all i and j, the estimator δ_2 pulls the estimators towards their average $\bar{X}=(X_1+\cdots+X_p)/p$.

3. Examples

In this section Theorem 2.1 and Theorem 2.2 are applied to the truncated normal, the parabola and the cusp shaped distributions to obtain the estimator δ_1 which dominates X.

Example 3.1. Suppose that Z_i has the common density of the form

$$f_1(z)=\frac{1}{\sqrt{2\pi}}e^{-z^2/2}I_{[-c,c]}(z)/(\Phi(c)-\Phi(-c)) , \tag{3.1}$$

where Φ is a c.d.f. of standard normal distribution and $I_{[-c,c]}(z)$ is an indicator function of the interval $[-c, c]$. (i.e. $I_{[-c,c]}(z)=1$ if $|z|\leqq c$, and $I_{[-c,c]}(z)=0$ if $|z|>c$).

Then, integration by parts shows that

$$p_1(z_i)\equiv-\int_{-c}^{z_i}\int_{-c}^{y}\Big(\int_{-c}^{u} tf_1(t)dt/V+f_1(u)\Big)dudy=\frac{e^{-c^2/2}}{(2\Phi(c)-1)V}q_1(z_i)f_1(z_i) ,$$

where

$$q_1(z_i)=\{2cz_i\Phi(z_i)+2c^2\Phi(-c)-(z_i^2-c^2)(2\Phi(c)-1)/2-c(z_i+c)\}e^{z_i^2/2}$$
$$+2c(1-e^{(z_i^2-c^2)/2})/\sqrt{2\pi} .$$

Note that $q_1(z_i)\leqq 0$ for $0\leqq z_i\leqq c$. The inequality $\Phi(z_i)\geqq(2\Phi(c)-1)z_i/2c+1/2$ gives

$$-q_1(z_i)\leqq(-k_1z_i^2/2+c^2k_1/2+k_2)e^{z_i^2/2}-2c/\sqrt{2\pi} \qquad \text{for } 0\leqq z_i\leqq c ,$$

where $k_1=2\Phi(c)-1$ and $k_2=2ce^{-c^2/2}/\sqrt{2\pi}$, and

$$-p_1(z_i)\leqq(k_1e^{-1+k_2/k_1}-k_2)f_1(z_i)/k_1V . \tag{3.2}$$

By Theorem 2.2 and (3.2), the risk of δ_1 is uniformly smaller than that of X if

$$a\geqq a_0 , \qquad \text{where } a_0=24(2-\sqrt{2})(k_1e^{-1+k_2/k_1}-k_2)/k_1pV$$

and if

$$0<b\leqq 2(p-2)V.$$

Table 3.1 gives the value of a_0 for $c=1$; $c=2$ and $c=\infty$.

Table 3.1. a_0: The lower bound of a

c	1	2	∞
a_0	$6.40V/p$ ($V=0.7559$)	$5.52V/p$ ($V=0.8214$)	$24(2-\sqrt{2})/ep$ ($V=1$)

Example 3.2. Suppose that Z_i has the common density of the form

$$f_2(z)=\frac{3}{4c^2}(c^2-z^2)I_{[-c,c]}(z)\ . \tag{3.3}$$

Without loss of generality we set $c=1$. Then

$$-\int_{-1}^{z_i}\int_{-1}^{y}\left(\int_{-1}^{u} tf_2(t)dt/V+f_2(u)\right)dudy=-\frac{1}{24}(z_i^2-1)^2f_2(z_i)\geqq -f_2(z_i)/24\ . \tag{3.4}$$

According to Theorem 2.2 and (3.4), $R(X,\theta)-R(\delta_1,\theta)$ is nonnegative if $a\geqq 5(2-\sqrt{2})V/p$ and $0<b\leqq 2(p-2)V$.

Example 3.3. Suppose that Z_i has the common density of the form

$$f_3(z)=\frac{k+1}{2c^{k+1}}(c-|z|)^kI_{[-c,c]}(z)\ , \tag{3.5}$$

where $k\geqq 0$. Assume $c=1$. Then

$$\begin{aligned} p_3(z_i)&\equiv -\int_{-1}^{z_i}\int_{-1}^{y}\left(\int_{-1}^{u} tf_3(t)dt/V+f_3(u)\right)dudy\\ &=\frac{(1-z_i)^2}{2(k+1)(k+2)(k+4)}q_3(z_i)f_3(z_i)\ , \qquad \text{for } 0\leqq z_i\leqq 1\ , \end{aligned}$$

where $q_3(z_i)=-(k+1)(k+2)z_i^2+(k+2)(k-2)z_i+k-2$. To evaluate $p_3(z_i)$, we consider the following two cases: $0\leqq k\leqq 2$ and $k>2$.

Case 1: $0\leqq k\leqq 2$. Noting that $q_3(z_i)\leqq 0$, we can show that

$$-p_3(z_i)\leqq l_1(l_2+l_3l_4^{1/2})Vf_3(z_i)\ , \tag{3.6}$$

where

$l_1=(k+3)/2048(k+1)^4(k+2)^2(k+4)$,

$l_2=-27k^6-252k^5-620k^4+640k^3+4992k^2+7168k+1024$,

$l_3=9k^4+62k^3+72k^2-96k+128$, and

$$l_4=(k+2)^2(5k-4)^2-16k(k-2)(k+1)(k+2)\ .$$

Therefore from Theorem 2.2 and (3.6), $R(X,\theta)-R(\delta_1,\theta)\geqq 0$ if $a\geqq 24(2-\sqrt{2})l_1(l_2+l_3l_4^{1/2})V/p$ and $0<b\leqq 2(p-2)V$.

Case 2: $k>2$. Noting that

$$q_3(z_i)\begin{cases}\geqq 0\ , & \text{if } 0\leqq z_i\leqq \dfrac{(k+2)(k-2)+\{(k+2)^2(k-2)^2}{2(k+1)}\\ & \qquad\qquad \dfrac{+4(k-2)(k+1)(k+2)\}^{1/2}}{\times(k+2)}\ ,\\ <0\ , & \text{otherwise}\ ,\end{cases}$$

we get

$$|p_3(z_i)|\leqq l_1(|l_2|+l_3l_4^{1/2})Vf_3(z_i)\ . \tag{3.7}$$

By Theorem 2.1 and (3.7), $R(X,\theta)-R(\delta_1,\theta)\geqq 0$ if $a\geqq 6l_1(|l_2|+l_3l_4^{1/2})V$ and $0<b\leqq 2(p-2)V$.

Remark 3.1. Note that the best invariant estimator of the location parameter of the uniform distribution has the probability density (3.5). According to Example 3.3, the estimator δ_1 has a smaller risk than the best invariant one.

4. Estimation of location parameters for the mixture of uniform distributions

Let X_i, $i=1,\cdots,p$ be the ith component of the $p\times 1$ vector $X=(X_1,X_2,\cdots,X_p)'$ and have a mixture of distributions with the same variance. Let their densities satisfy the conditions (2.2) and (2.3) of Theorem 2.1. In this case, sufficient conditions on the constants a and b of the estimator δ_1 in (1.1) to dominate X are obtained by applying Theorem 2.1. Here we study the case where Z_i has the common density which is a mixture of two uniform distributions, i.e. the density is

$$f(z)=\frac{\alpha}{2c_1}I_{[-c_1,c_1]}(z)+\frac{1-\alpha}{2c_2}I_{[-c_2,c_2]}(z)\ , \tag{4.1}$$

where $0<\alpha<1$ and $c_1\leqq c_2$.

THEOREM 4.1. *Let Z_i have the common density of the form* (4.1). *Then the risk of δ_1 is uniformly smaller than that of X if*

$$a\geqq\max\{6c_0+3(2-\sqrt{2})c_2^2/p,\ 3c_0+(9c_0^2+60c_0(2-\sqrt{2})c_2^2)^{1/2}\}$$

and if

$$0<b\leqq 2(p-2)V,$$

where

$$c_0=\max\left\{\frac{\alpha(1-\alpha)c_1c_2(c_2-c_1)(V_{c_2}-V_{c_1})}{2((1-\alpha)c_1+\alpha c_2)V},\ \frac{\alpha(c_2-c_1)^2(V_{c_2}-V_{c_1})}{2V}\right\},$$

$$V_{c_i}=\int_{-c_i}^{c_i}\frac{z^2}{2c_i}dz\,,\qquad and \qquad V=\int_{-c_i}^{c_i}z^2f(z)dz=\alpha V_{c_1}+(1-\alpha)V_{c_2}\,.$$

Note. It is easily seen that

$$c_0=\begin{cases}\dfrac{\alpha(1-\alpha)c_1c_2(c_2-c_1)(V_{c_2}-V_{c_1})}{2((1-\alpha)c_1+\alpha c_2)V}, & \text{if } \alpha\leqq\dfrac{c_1^2}{c_2(c_2-c_1)+c_1^2},\\[2ex] \dfrac{\alpha(c_2-c_1)^2(V_{c_2}-V_{c_1})}{2V}, & \text{if } \alpha>\dfrac{c_1^2}{c_2(c_2-c_1)+c_1^2}.\end{cases}$$

PROOF. From (2.4) of Theorem 2.1, the conditional expectation of a term of the summation in $\{R(X,\theta)-R(\delta_1,\theta)\}/2b$ can be written as

$$\begin{aligned}L&\equiv \mathrm{E}_{X_i}\left\{\frac{X_i(X_i-\theta_i)}{a+\|X\|^2}\right\}\\ &=\alpha\int_{-c_1}^{c_1}\frac{z_i(z_i+\theta_i)}{a+\|z+\theta\|^2}\frac{1}{2c_1}dz_i+(1-\alpha)\int_{-c_2}^{c_2}\frac{z_i(z_i+\theta_i)}{a+\|z+\theta\|^2}\frac{1}{2c_2}dz_i\,.\end{aligned}$$

Integration by parts shows that

$$(4.2)\qquad \int_{-c_j}^{z_i}\int_{-c_j}^{y}\left(\int_{-c_j}^{u}tdt/2c_jV_{c_j}+1/2c_j\right)dudy=\frac{1}{16c_j^3}(z_i^2-c_j^2)^2\qquad j=1,2\,.$$

The last expression of (2.5) of Theorem 2.1 and (4.2) give

$$(4.3)\qquad L\geqq\alpha V_{c_1}\int_{-c_1}^{c_1}\frac{t_1(z_i)}{2c_1}dz_i+(1-\alpha)V_{c_2}\int_{-c_2}^{c_2}\frac{t_1(z_i)}{2c_2}dz_i=V\,\mathrm{E}_{Z_i}\{t_1(Z_i)\}+L_1\,,$$

where

$$t_1(z_i)=\frac{1}{a+\|z+\theta\|^2}-\frac{2(z_i+\theta_i)^2}{(a+\|z+\theta\|^2)^2}-\frac{3(2-\sqrt{2})c_2^2(z_i+\theta_i)^2}{(a+\|z+\theta\|^2)^3}\,,$$

$$L_1=V\int_{-c_2}^{c_2}t_1(z_i)(\tilde f(z_i)-f(z_i))dz_i\,,$$

and

$$\tilde f(z)=\frac{\alpha V_{c_1}}{2c_1V}I_{[-c_1,c_1]}(z)+\frac{(1-\alpha)V_{c_2}}{2c_2V}I_{[-c_2,c_2]}(z)\,.$$

Note that $\tilde f(z)-f(z)>0$ if $c_1<|z|\leqq c_2$, and $\tilde f(z)-f(z)<0$ if $|z|\leqq c_1$. Integration by parts is applied to L_1 twice, then

$$L_1=-6V\int_{-c_2}^{c_2} t_2(z_i)\tilde{g}(z_i)dz_i \ ,$$

where

$$t_2(z_i)=\frac{1}{(a+\|z+\theta\|^2)^2}+\frac{(2-\sqrt{2})c_2^2-8(z_i+\theta_i)^2}{(a+\|z+\theta\|^2)^3}$$
$$+\frac{8(z_i+\theta_i)^4-15(2-\sqrt{2})c_2^2(z_i+\theta_i)^2}{(a+\|z+\theta\|^2)^4}$$
$$+\frac{24(2-\sqrt{2})c_2^4(z_i+\theta_i)^2}{(a+\|z+\theta\|^2)^5} \ ,$$

and

$$\begin{aligned}\tilde{g}(z_i)&=(1-\alpha)(V_{c_2}/V-1)z_i^2/4c_2-(1-\alpha)(V_{c_2}/V-1)|z_i|/2\\&\quad+(1-\alpha)(V_{c_2}/V-1)c_2/4\ , \qquad \text{if } c_1<|z_i|\leqq c_2\ ,\\&=\alpha(V_{c_1}/V-1)z_i^2/4c_1+(1-\alpha)(V_{c_2}/V-1)z_i^2/4c_2\\&\quad+(1-\alpha)(V_{c_2}/V-1)(c_2-c_1)/4\ , \qquad \text{if } |z_i|\leqq c_1\ .\end{aligned}$$

Note that $\tilde{g}(z_i)$ is positive, even and unimodal on $(-c_2, c_2)$ and $f(z_i)$ is a step function of the same property. Thus, $\tilde{g}(z_i)/f(z_i)$ has the maximum value when $z_i=0$ or $z_i=c_1$, and

$$\tilde{g}(z_i)\leqq c_0 f(z_i)\ . \tag{4.4}$$

The inequalities (4.4) and

$$t_2(z_i)\leqq 1/(a+\|z+\theta\|^2)^2-10(2-\sqrt{2})c_2^2/(a+\|z+\theta\|^2)^3\ ,$$

give

$$L_1\geqq -6c_0V\,\mathrm{E}_{X_i}\left\{\frac{1}{(a+\|X\|^2)^2}+\frac{10(2-\sqrt{2})c_2^2}{(a+\|X\|^2)^3}\right\}\ . \tag{4.5}$$

Therefore from (4.3) and (4.5),

$$L\geqq V\,\mathrm{E}_{X_i}\left\{\frac{1}{a+\|X\|^2}-\frac{2X_i^2+6c_0}{(a+\|X\|^2)^2}-\frac{3(2-\sqrt{2})c_2^2X_i^2+60(2-\sqrt{2})c_0c_2^2}{(a+\|X\|^2)^3}\right\}\ .$$

Hence

$$\begin{aligned}&R(X,\theta)-R(\delta_1,\theta)\\&\quad\geqq 2bV\,\mathrm{E}\left\{\frac{ap+(p-2)\|X\|^2-6c_0p-3(2-\sqrt{2})c_2^2}{(a+\|X\|^2)^2}\right.\\&\qquad\left.+\frac{3(2-\sqrt{2})ac_2^2-60(2-\sqrt{2})c_0c_2^2p}{(a+\|X\|^2)^3}-\frac{b\|X\|^2}{2V(a+\|X\|^2)^2}\right\}\ ,\end{aligned}$$

which is nonnegative if

$$a \geqq \max\{6c_0+3(2-\sqrt{2})c_2^2/p,\ 3c_0+(9c_0^2+60c_0(2-\sqrt{2})c_2^2)^{1/2}\}\ .$$

and if

$$0<b\leqq 2(p-2)V\ .$$

Table 4.1 shows the lower bound of a/V for $p=3$; $c_2/c_1=1.5, 2, 2.5$ and $\alpha=0.2, 0.4, 0.6, 0.8$. The lower bound of a/V becomes large when c_2/c_1 or α is large.

Table 4.1. The lower bound of a/V

c_2/c_1 \ α	0.2	0.4	0.6	0.8
1.5	2.63	3.82	4.84	7.51
2	3.91	6.61	12.15	23.93
2.5	4.55	9.42	19.01	44.58

Remark 4.1. Theorem 4.1 can be extended to the mixture of n-uniform distributions.

5. Estimation of location parameters for the uniform distribution under the loss $\|\delta_1-\theta\|^4$

In this section the constants a and b of (1.1) are given such as the risk of the estimator δ_1 is uniformly smaller than that of X with respect to the loss

$$\|\delta_1-\theta\|^4=\left\{\sum_{i=1}^{p}(\delta_{1i}-\theta_i)^2\right\}^2, \tag{5.1}$$

for the uniform distribution.

Following Berger's notation [1], let

$$\gamma(X)=(\gamma_1(X),\cdots,\gamma_p(X))'=\delta_1(X)-X$$

and

$$c_{n,i}=\begin{cases}1, & \text{if } i=0\,,\\ 0, & \text{if } i<0 \text{ or } i>n \text{ or } i \text{ is odd}\,,\\ \sum_{j=1}^{n-1} jc_{j-1,i-2}\,, & \text{if } 2\leqq i\leqq n\,.\end{cases}$$

If $g: R^1\to R^1$ is n times differentiable, let

$$g^{(j)}(z)=\frac{d^j}{dz^j}g(z)\,,\qquad 0\leqq j\leqq n,\ (g^{(0)}(z)=g(z))\,.$$

If $h: R^p \to R^1$ is a function with sufficient order derivatives, let

$$h^{i(k),j(l)}(z) = \frac{\partial^{(k+l)}}{\partial z_i^k \partial z_j^l} h(z) .$$

The following lemma is useful in carrying out integration by parts for the loss $\|\delta_1 - \theta\|^4$.

LEMMA 5.1. *Let Z have the density of the form*

$$f(z) = \begin{cases} f(|z|), & \text{if } |z| \leq c, \\ 0, & \text{otherwise}. \end{cases}$$

Suppose that $g(z)$ is a real-valued n times continuously differentiable function and that

$$\int_{-c}^{c} |g^{(n-j)}(z)||z|^j f_{n-j}(z) dz < \infty \qquad \text{for } 0 \leq j \leq n ,$$

where

$$f_i(z) = \int_{-c}^{z} -y f_{i-1}(y) dy \qquad \text{for } 1 \leq i \leq n \text{ and } f_0(z) = f(z) .$$

Then

$$\mathrm{E}\{g(Z)Z^n\} = \int_{-c}^{c} \sum_{i=0}^{n} c_{n,i} g^{(n-i)}(z) f_{[n-i/2]}(z) dz ,$$

where $[x]$ is the greatest integer less than or equal to x.

PROOF. This lemma can be proved by induction on n along the same line as in the proof of Lemma 1 in Berger [1].

THEOREM 5.2. *Let Z_i $(i=1, \cdots, p)$ be p independent random variables from a uniform distribution on $(-c, c)$. Then the risk of the estimator δ_1 is uniformly smaller than that of X with respect to the loss given by* (5.1) *if*

$$a \geq \frac{p+107/30}{p+4/5} b + \frac{331}{20} V$$

and if

$$0 < b \leq 2 \frac{p+4/5}{p+2} (p-2) V .$$

PROOF.

$$\|\delta_1 - \theta\|^4 = \left\{\sum_{i=1}^{p} (\gamma_i(x) + x_i - \theta_i)^2\right\}^2 = \left\{\sum_{i=1}^{p} (\gamma_i(z+\theta) + z_i)^2\right\}^2 .$$

Therefore

$$\begin{aligned}(5.2)\qquad \Delta(\theta)&=\mathrm{E}\,\|\delta_1-\theta\|^4-\mathrm{E}\,\|X-\theta\|^4\\ &=\mathrm{E}\,\{4\sum_{ij}\sum\gamma_i\gamma_j Z_i Z_j+\sum_{ij}\sum\gamma_i^2\gamma_j^2+4\sum_{ij}\sum\gamma_i Z_i Z_j^2\\ &\quad+2\sum_{ij}\sum\gamma_i^2 Z_j^2+4\sum_{ij}\sum\gamma_i\gamma_j^2 Z_i\}\;.\end{aligned}$$

Using Lemma 5.1 to evaluate the conditional expectation of a term of each summation in (5.2), we get

$$\mathrm{E}_{Z_i}\{\gamma_i^2 Z_i^2\}=2\int_{-c}^{c}\{(\gamma_i^{i(1)})^2+\gamma_i\gamma_i^{i(2)}\}f_2(z_i)dz_i+\int_{-c}^{c}\gamma_i^2 f_1(z_i)dz_i\;,$$

$$\begin{aligned}\mathrm{E}_{Z_i,Z_j}\{\gamma_i\gamma_j Z_i Z_j\}=\int_{-c}^{c}\int_{-c}^{c}\{&\gamma_i^{i(1),j(1)}\gamma_j+\gamma_i^{i(1)}\gamma_j^{j(1)}+\gamma_i^{j(1)}\gamma_j^{i(1)}\\ &+\gamma_i\gamma_j^{i(1),j(1)}\}f_1(z_i)f_1(z_j)dz_i dz_j\;,\end{aligned}$$

$$(5.3)\qquad \mathrm{E}_{Z_i}\{\gamma_i Z_i^3\}=\int_{-c}^{c}\gamma_i^{i(3)}f_3(z_i)dz_i+3\int_{-c}^{c}\gamma_i^{i(1)}f_2(z_i)dz_i\;,$$

$$\begin{aligned}\mathrm{E}_{Z_i,Z_j}\{\gamma_i Z_i Z_j^2\}=&\int_{-c}^{c}\int_{-c}^{c}\gamma_i^{i(1),j(2)}f_1(z_i)f_2(z_j)dz_i dz_j\\ &+\int_{-c}^{c}\int_{-c}^{c}\gamma_i^{i(1)}f_1(z_i)f_1(z_j)dz_i dz_j\;,\end{aligned}$$

$$\mathrm{E}_{Z_j}\{\gamma_i^2 Z_j^2\}=2\int_{-c}^{c}\{(\gamma_i^{j(1)})^2+\gamma_i\gamma_i^{j(2)}\}f_2(z_j)dz_j+\int_{-c}^{c}\gamma_i^2 f_1(z_j)dz_j\;,$$

and

$$\mathrm{E}_{Z_i}\{\gamma_i\gamma_j^2 Z_i\}=\int_{-c}^{c}\{\gamma_i^{i(1)}\gamma_j^2+2\gamma_i\gamma_j\gamma_j^{i(1)}\}f_i(z_i)dz_i\;,$$

where

$$\gamma_i(z+\theta)=-\frac{b(z_i+\theta_i)}{a+\|z+\theta\|^2}\,,\qquad f_1(z_i)=\frac{1}{4c}(-z_i^2+c^2)\;,$$

$$f_2(z_i)=\frac{1}{16c}(z_i^2-c^2)^2\,,\quad\text{and}\quad f_3(z_i)=\frac{1}{96c}(-z_i^2+c^2)^3\;.$$

Similarly in the proof of Theorem 2.1, a straightforward calculation of each term in (5.3) gives

$$\begin{aligned}(5.4)\qquad \Delta(\theta)\leq\mathrm{E}\,\Bigg\{\frac{b}{a+\|X\|^2}\Bigg\{&m_0+\frac{m_1\|X\|^2+m_2}{a+\|X\|^2}+\frac{m_3\|X\|^2+m_4}{(a+\|X\|^2)^2}\\ &+\frac{m_5\|X\|^4+m_6\|X\|^2+m_7}{(a+\|X\|^2)^3}+\frac{m_8\|X\|^2}{(a+\|X\|^2)^4}+\frac{m_9\|X\|^4}{(a+\|X\|^2)^5}\Bigg\}\Bigg\}\;,\end{aligned}$$

where

$m_0 = -4p(p+4/5)c^4/9$,

$m_1 = 2(p+2)c^2b/3 + 8(p+4/5)c^4/9$,

$m_2 = 4p(p+107/30)c^4b/9 + 61p(p+89/61)c^6/135$,

$m_3 = -4(p+2)c^2b^2/3 - 176(p+19/11)c^4b/45 - 128(p+2)c^6/135$,

$m_4 = pc^4b^2 + 4p(p-1/10)c^6b/3 + 3p(p-1)c^8/5$,

$m_5 = b^3 + 8c^2b^2 + 64c^4b/5 + 128c^6/45$,

$m_6 = 21(p-4/7)c^4b^2/5 + 1492(p-1477/1492)c^6b/75 + 186(p-1)c^8/25$,

$m_7 = 3p(p-1)c^8b/2$,

$m_8 = 352(p-1)c^8b$, and

$m_9 = 480c^8b$.

Define

$$v(y, a, b) = m_0 + \frac{m_1 y + m_2}{a+y} + \frac{m_3 y + m_4}{(a+y)^2} + \frac{m_5 y^2 + m_6 y + m_7}{(a+y)^3} + \frac{m_8 y}{(a+y)^4} + \frac{m_9 y^2}{(a+y)^5} .$$

It is clear from (5.4) that $\Delta(\theta) \leqq 0$ if a and b can be chosen so that $v(y, a, b) \leqq 0$ for all $0 \leqq y \leqq \infty$. To evaluate $v(y, a, b)$, we decompose $v(y, a, b)$ as

$$v(y, a, b) \equiv v_1(y, a, b) + v_2(y, a, b) .$$

where

$$v_1(y, a, b) = m_0 + \frac{m_1 y + m_2}{a+y} + \frac{m_4}{(a+y)^2} + \frac{m_6' y + m_7}{(a+y)^3} + \frac{m_8 y}{(a+y)^4} ,$$

$$v_2(y, a, b) = \frac{m_3 y}{(a+y)^2} + \frac{m_5 y^2 + m_6'' y}{(a+y)^3} + \frac{m_9 y^2}{(a+y)^5} ,$$

$$m_6' = 3(p-2/5)c^4b^2 + 1492(p-1477/1492)c^6b/45 + 186(p-1)c^8/25 ,$$

and

$$m_6'' = 6(p-1)c^4b^2/5 .$$

It is straightforward to verify that $v_1(y, a, b) \leqq 0$ and $v_2(y, a, b) \leqq 0$ if the conditions on a and b are satisfied. The proof is completed.

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