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ASYMPTOTIC CONSISTENCY OF FIXED-WIDTH SEQUENTIAL CONFIDENCE INTERVALS FOR A MULTIPLE REGRESSION FUNCTION

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Summary

Let $m_n(x)$ be the recursive kernel estimator of the multiple regression function $m(x)=\mathrm{E}[Y|X=x]$. For given α $(0<\alpha<1)$ and $d>0$ we define a certain class of stopping times $N=N(\alpha, d, x)$ and take $I_{N,d}(x)=[m_N(x)-d,\ m_N(x)+d]$ as a $2d$-width confidence interval for $m(x)$ at a given point x. In this paper it is shown that the probability $\mathrm{P}\{m(x)\in I_{N,d}(x)\}$ converges to α as d tends to zero.

1. Introduction

Let $Z=(X, Y)$, $Z_1=(X_1, Y_1)$, $\cdots$, $Z_n=(X_n, Y_n)$ be independent and identically distributed $R^p\times R$-valued random vectors on a probability space $(\Omega, \mathcal{B}, P)$ with an unknown joint probability density function (p.d.f.) $f^*(x, y)$ with respect to the Lebesgue measure. There have been many papers on the estimation of the nonparametric regression function $m(x)=\mathrm{E}[Y|X=x]$ (of Y on X) by

$$m_n(x)=\sum_{i=1}^{n} W_{ni}(x)Y_i\,, \tag{1.1}$$

where $W_{ni}(x)=W_{ni}(x, X_1, \cdots, X_n)$ for each i $(1\leqq i\leqq n)$ is a suitable real-valued Borel measurable function of $x, X_1, \cdots, X_n$.

Nadaraya [8] and Watson [14] proposed the estimator (1.1) with $p=1$ and

$$W_{ni}(x)=K((x-X_i)/h_n)\Big/\sum_{j=1}^{n} K((x-X_j)/h_n)\,,$$

where $K(x)$ is a suitable kernel function and $\{h_n\}$ is a sequence of window-widths tending to zero. Later on, many authors have studied

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its asymptotic properties (see Prakasa Rao [10] for example).

When data increase we may be faced with computational burdens in processing them. To decrease these burdens Ahmad and Lin [1] proposed a recursive version of (1.1) with

$$W_{ni}(x)=h_i^{-p}K((x-X_i)/h_i)\Big/\sum_{j=1}^{n} h_j^{-p}K((x-X_j)/h_j)\,,$$

or equivalently,

$$m_0(x)=f_0(x)\equiv 0$$

$$f_n(x)=(h_n/h_{n-1})^p f_{n-1}(x)+K((x-X_n)/h_n) \tag{1.2}$$

$$m_n(x)=m_{n-1}(x)+f_n^{-1}(x)\{Y_n-m_{n-1}(x)\}K((x-X_n)/h_n)$$

and they proved some pointwise results for these estimators. Devroye and Wagner [4] considered a still simpler recursive estimator than (1.2). The author [6] proposed a class of recursive estimators $m_n(x)$ defined in Section 2 and used in this paper, which contains (1.2) as a special case. Stone [13] has investigated the estimator (1.1) and discussed about sufficient conditions on $\{W_{ni}(x)\}$ for $m_n(x)$ to be consistent.

On the other hand, when one uses a recursive estimator in practical situations one may be required to terminate the computations to obtain the estimator with given accuracy. In this case the sample size is a random variable. Suppose that N_t for each $t\in(0,\infty)$ is a stopping time. Recently, Samanta [12] has shown the asymptotic normality of $m_{N_t}(x)$ by using the estimator (1.2).

In this paper we propose a class of stopping times $N=N(\alpha, d, x)$ based on the idea of Chow and Robbins [3], construct a sequence of $2d$-width sequential confidence intervals $I_{N,d}(x)=[m_N(x)-d,\ m_N(x)+d]$ for $m(x)$ and show that the probability $\mathrm{P}\{m(x)\in I_{N,d}(x)\}$ converges to α as d tends to zero.

In Section 2 we shall make some preparations and give several lemmas. In Section 3 we shall prove the asymptotic consistency of the sequential confidence intervals $I_{N,d}(x)$.

2. Preliminaries and several lemmas

In this section we shall make some preparations for Section 3.

Let $K(x)$ be a given bounded p.d.f. on R^p with respect to the Lebesgue measure satisfying $\|u\|_p^p K(u)\to 0$ as $\|u\|_p\to\infty$, where $\|\cdot\|_p$ denotes the Euclidean norm on R^p. We shall impose either of the following conditions on $K(x)$:

(K1) $\int_{R^p} u_i K(u_1,\cdots,u_p)du_1\cdots du_p=0$ for all $i=1,\cdots,p$, and

(K2) $\int_{R^p} \|u\|_p^2 K(u)du < \infty$.

Let $\{h_n\}$ be a nonincreasing sequence of positive numbers converging to zero, on which some of the following conditions are imposed:

(H1) $nh_n^p \uparrow \infty$ as $n \to \infty$;

(H2) For some a $(0<a\leqq 1)$

$$n^{1-2a}h_n^p \to 0 \quad \text{as } n \to \infty ,$$

$$n^{1-2a}h_n^p \sum_{j=1}^n j^{2(a-1)}h_j^{-p} \to \beta \quad \text{as } n \to \infty \text{ for some positive constant } \beta,$$

$$n^{3/2-3a}h_n^{3p/2} \sum_{j=1}^n j^{3(a-1)}h_j^{-2p} \to 0 \quad \text{as } n \to \infty , \text{ and}$$

$$n^{1/2-a}h_n^{p/2} \sum_{j=1}^n j^{a-1}h_j^2 \to 0 \quad \text{as } n \to \infty ;$$

(H3) For any ε (>0) there exists a positive constant δ such that $|n/m-1|<\delta$ implies $|h_n/h_m-1|<\varepsilon$;

(H4) $\sum_{n=1}^{\infty} (n^2h_n^p)^{-1} < \infty$;

(H5) $n^{1+\eta}h_n^{p+4} \to 0$ as $n \to \infty$ for some positive constant η .

Throughout this paper we use the following class of recursive estimators $m_n(x)$ of the regression function $m(x)$, which is proposed by the author [6]:

$$m_0(x)\equiv 0 , \quad f_0(x)\equiv c \quad \text{for an arbitrary positive constant } c$$

$$f_n(x)=f_{n-1}(x)+a_n\{K_n(x, X_n)-f_{n-1}(x)\}$$

$$m_n(x)=m_{n-1}(x)+a_nG(f_n(x))\{Y_n-m_{n-1}(x)\}K_n(x, X_n)$$

for each $n\geqq 1$, where

$$a_n=a/n \quad \text{with } 0<a\leqq 1 , \tag{2.1}$$

$$G(y)=\begin{cases} y^{-1} & \text{if } y>0 \\ 0 & \text{otherwise} , \end{cases}$$

and

$$K_n(x, s)=h_n^{-p}K((x-s)/h_n) \quad \text{for } x, s \in R^p .$$

In this paper, if in some term its denominator is less than or equal to zero we define the value of the term to be zero. Let

$$f(x)=\int_R f^*(x, y)dy\ , \quad q(x)=\int_R yf^*(x, y)dy\ , \quad g(x)=\int_R y^2 f^*(x, y)dy\ ,$$

$$\psi(x)=\int_R y^4 f^*(x, y)dy \quad \text{and} \quad v(x)=(g(x)/f(x))-(q(x)/f(x))^2 \quad (\geqq 0)\ .$$

Clearly, $v(x)>0$ is equivalent to $f(x)>0$ and $v(x)>0$. Throughout this paper we assume that $f(x)$, $q(x)$, $g(x)$ and $\psi(x)$ are all finite on R^p and write $m(x)=q(x)/f(x)$. Also, let C, C_1, C_2, $\cdots$ denote appropriate positive constants.

Define sequences $\{q_n(x)\}$, $\{g_n(x)\}$ and $\{v_n(x)\}$ as follows:

$$q_0(x)=g_0(x)\equiv 0\ ,$$

$$q_n(x)=q_{n-1}(x)+a_n\{Q_n(x, Z_n)-q_{n-1}(x)\}$$

$$g_n(x)=g_{n-1}(x)+a_n\{G_n(x, Z_n)-g_{n-1}(x)\}$$

$$v_n(x)=(g_n(x)/f_n(x))-(q_n(x)/f_n(x))^2$$

for each $n\geqq 1$, where for $x\in R^p$, $z=(u, y)\in R^p\times R$ and $n\geqq 1$

$$Q_n(x, z)=yK_n(x, u) \quad \text{and} \quad G_n(x, z)=y^2K_n(x, u)\ .$$

For a_n in (2.1) set

$$\gamma_0=\gamma_1=1\ , \quad \gamma_n=\prod_{j=2}^{n}(1-a_j) \quad \text{for } n\geqq 2 \quad \text{and}$$

$$\beta_{mn}=\begin{cases} \prod_{j=m+1}^{n}(1-a_j) & \text{if } n>m\geqq 0 \\ 1 & \text{if } n=m\geqq 0\ . \end{cases}$$

Obviously, $\gamma_n\downarrow 0$ as $n\to\infty$, $\gamma_n>0$ for all $n\geqq 0$ and

(2.2) $$\beta_{mn}=\gamma_m^{-1}\gamma_n \quad \text{if } n\geqq m\geqq 1\ .$$

It is known that

(2.3) $$\beta_{mn}\sim m^a n^{-a} \quad \text{as } n\geqq m\to\infty\ , \quad \text{and}$$

(2.4) $$C_1 n^{-a}\leqq\gamma_n\leqq C_2 n^{-a} \quad \text{for all } n\geqq 1\ ,$$

where "$\phi_n\sim\psi_n$ as $n\to\infty$" means $\lim_{n\to\infty}\phi_n/\psi_n=1$.

Remark 2.1. We can write $f_n(x)$, $q_n(x)$ and $g_n(x)$ as follows:

$$f_n(x)=\sum_{j=1}^{n} a_j\beta_{jn}K_j(x, X_j)+\beta_{0n}c\ ,$$

$$q_n(x)=\sum_{j=1}^{n} a_j\beta_{jn}Q_j(x, Z_j) \quad \text{and} \quad g_n(x)=\sum_{j=1}^{n} a_j\beta_{jn}G_j(x, Z_j)\ ,$$

where $\sum_{j=m}^{n}(\cdot)=0$ if $n<m$.

For any real-valued function θ on R^p let $C(\theta)$ be the set of continuity points of θ and $\|\theta\|_\infty=\sup\{|\theta(x)|: x\in R^p\}$. For any fixed $x\in R^p$ and $n\geqq 1$ let

$$U_n^{(0)}=K_n(x, X_n)-\mathrm{E}\,[K_n(x, X_n)]\,, \quad U_n^{(1)}=Q_n(x, Z_n)-\mathrm{E}\,[Q_n(x, Z_n)]\,,$$

$$W_n=a_n\gamma_n^{-1}(U_n^{(0)}, U_n^{(1)})'\,, \quad B_n=(nh_n^p)^{1/2}\gamma_n\sum_{j=1}^{n}W_j \quad \text{and}$$

$$B_n^*=(nh_n^p)^{1/2}(f_n(x)-f(x),\ q_n(x)-q(x))'\,,$$

where the prime denotes transpose.

We shall summarize some results obtained in [5] and [6].

LEMMA 2.1 ([6]). *Let $\{d_n\}$ be a sequence of positive numbers converging to zero. Let $k(x)$ be a bounded, integrable, real-valued Borel measurable function on R^p satisfying $\|u\|_p^p|k(u)|\to 0$ as $\|u\|_p\to\infty$. Let $\theta(x)$ be an integrable, real-valued Borel measurable function on R^p. Then, for each point $x\in C(\theta)$ we have*

$$\int_{R^p} d_n^{-p}k((x-u)/d_n)\theta(u)du \to \theta(x)\int_{R^p} k(u)du \qquad \text{as } n\to\infty \quad \text{and}$$

$$\sup_{n\geq 1}\int_{R^p} d_n^{-p}|k((x-u)/d_n)||\theta(u)|du\leqq C\,,$$

where C may depend on x.

LEMMA 2.2 ([6]). *Assume* $\mathrm{E}\,[Y^2]<\infty$. *Let* (H4) *be satisfied, and let x be a point with $f(x)>0$, belonging to the set $C(f)\cap C(q)\cap C(g)$. Then we have*

$$\lim_{n\to\infty} f_n(x)=f(x) \qquad \text{with probability one } (w.p.1.)\,,$$

$$\lim_{n\to\infty} q_n(x)=q(x) \quad w.p.1.\,, \quad \text{and} \quad \lim_{n\to\infty} m_n(x)=m(x) \quad w.p.1.\,.$$

LEMMA 2.3 ([5]). *Let $\{y_n\}$ be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$. Suppose that there exists a null set A such that for each $\omega\in A^c$*

$$y_n(\omega)\geqq 0 \quad \text{for all } n\geqq 1\,, \quad \lim_{n\to\infty} y_n(\omega)=1 \quad \text{and}$$

$$y_n(\omega)>0 \quad \text{for all } n\geqq m \quad \text{if } y_m(\omega)>0 \quad \text{for some } m=m(\omega)\,.$$

Let $\{b(n)\}$ be a sequence of positive numbers satisfying

$$\lim_{n\to\infty} b(n)=\infty \quad and \quad \lim_{n\to\infty} b(n)/b(n-1)=1\,.$$

For each $t\in(0,\infty)$ *define* N_t *as the smallest integer* $n\geqq 1$ *such that* $b(n)/t \geqq y_n>0$. *Then, we have*

$$\mathrm{P}\{N_t<+\infty\}=1 \quad for\ each\ t\in(0,\infty)\,,\quad \mathrm{P}\{N_t\uparrow\infty\ as\ t\to\infty\}=1 \quad and$$

$$\mathrm{P}\{b(N_t)/t\to 1\ as\ t\to\infty\}=1\,.$$

The following lemma gives the asymptotic normality of B_n.

LEMMA 2.4. *Assume* $\mathrm{E}[Y^4]<\infty$. *Let* (H2) *be satisfied. Consider a point* $x\in C(f)\cap C(q)\cap C(g)$ *with* $v(x)>0$. *If either* $x\in C(\psi)$ *or* $\|\psi\|_\infty<\infty$ *holds, then we have*

$$B_n \underset{L}{\to} N(0,\Gamma) \qquad as\ n\to\infty\ (in\ law)\,,$$

where the covariance matrix $\Gamma=\Gamma(x)$ *is given by*

$$\Gamma=a^2\beta\int_{R^p}K^2(u)du\begin{pmatrix} f(x) & q(x)\\ q(x) & g(x)\end{pmatrix}.$$

PROOF. By Hölder's inequality we get that for each $j\geqq 1$

$$\begin{aligned}
\mathrm{E}[|U_j^{(1)}|^3] &\leqq 8\,\mathrm{E}[|Q_j(x,Z_j)|^3]\\
&=8\int_{R^p\times R}\{|y|^3K_j^2(x,u)(f^*(u,y))^{3/4}\}\{K_j(x,u)(f^*(u,y))^{1/4}\}dudy\\
&\leqq 8\left[\iint_{R^p\times R} y^4\{K_j(x,u)\}^{8/3}f^*(u,y)dudy\right]^{3/4}\\
&\quad\times\left[\iint_{R^p\times R} K_j^4(x,u)f^*(u,y)dudy\right]^{1/4}=8h_j^{-2p}\times I_1\times I_2\,,
\end{aligned}$$

where

$$I_1=\left[\iint_{R^p} h_j^{-p}\{K((x-u)/h_j)\}^{8/3}\psi(u)du\right]^{3/4} \quad \text{and}$$

$$I_2=\left[\iint_{R^p} h_j^{-p}K^4((x-u)/h_j)f(u)du\right]^{1/4}.$$

As $x\in C(f)$, it follows from Lemma 2.1 that $I_2\leqq C_1$. If $x\in C(\psi)$ then by Lemma 2.1 we get $I_1\leqq C_2$, and if $\|\psi\|_\infty<\infty$ then we have

$$I_1\leqq\{\|\psi\|_\infty\|K\|_\infty^{5/3}\}^{3/4}<\infty\,.$$

In any case, $I_1\times I_2$ is bounded by a constant and therefore, we obtain

$$\mathrm{E}[|U_j^{(1)}|^3]\leqq C_3h_j^{-2p} \qquad \text{for all } j\geqq 1\,. \tag{2.5}$$

Replacing (4.17) in [6] by (2.5) we can use the same arguments as in the proof of Lemma 4.1 of [6], to complete the proof of Lemma 2.4.

LEMMA 2.5. *Let* $\{h_n\}$ *be a nonincreasing sequence of positive numbers converging to zero and satisfy* (H1) *and* (H2). *Let* $\{V_n\}$ *be a sequence of independent random variables on a probability space* $(\Omega, \mathcal{F}, P)$ *satisfying*

$$\mathrm{E}[V_n]=0 \quad and \quad h_n^p \mathrm{E}[V_n^2]\leqq C \qquad for\ all\ n\geqq 1 .$$

For any $d\in(0,\infty)$ *let* $N(d)$ *be a positive integer-valued random variable on* $(\Omega, \mathcal{F}, P)$ *and* $n(d)$ *a positive integer with* $\lim_{d\to 0} n(d)=\infty$. *Set*

$$T_n=\sum_{j=1}^{n} a_j\beta_{jn}V_j \qquad for\ each\ n\geqq 1 ,$$

where a_n *is given in* (2.1). *If*

$$N(d)/n(d) \underset{P}{\rightarrow} 1 \qquad as\ d\to 0 ,$$

then we have

$$(N(d)h^p_{N(d)})^{1/2}(T_{N(d)}-T_{n(d)}) \underset{P}{\rightarrow} 0 \qquad as\ d\to 0 .$$

PROOF. For simplicity let $N=N(d)$ and $n=n(d)$. Let any positive numbers ε and ξ be fixed. For any ρ (>0) set

$$M_1=[(1-\rho)n] \quad \text{and} \quad M_2=[(1+\rho)n] ,$$

where $[b]$ denotes the largest integer not greater than b. Since as $\rho\to 0$

$$2\rho/(1-\rho)\to 0 \quad \text{and} \quad \{(1-\rho)/(1+\rho)\}^a\to 1 ,$$

there exists a positive constant $\rho=\rho(\varepsilon,\xi)<1/2$ such that

$$(2.6) \qquad 2\rho/(1-\rho)<\varepsilon^2\xi/2 \quad \text{and} \quad (1-\{(1-\rho)/(1+\rho)\}^a)^2<\varepsilon^2\xi/2 .$$

As $\lim_{d\to 0} n=\infty$ we get $M_i\to\infty$ as $d\to 0$ for $i=1,2$ and $M_1/M_2\sim(1-\rho)/(1+\rho)$ as $d\to 0$. Hence by (2.3) and (2.6) we have

$$(2.7) \qquad (1-\beta_{M_1M_2})^2<\varepsilon^2\xi \qquad \text{for } d \text{ sufficiently small} .$$

Also, since $(M_2-M_1)/M_1\sim 2\rho/(1-\rho)$ as $d\to 0$, it follows from (2.6) that

$$(2.8) \qquad (M_2-M_1)/M_1<\varepsilon^2\xi \qquad \text{for } d \text{ sufficiently small} .$$

It is clear that

$$(2.9) \qquad M_2/M_1\leqq C_1 \qquad \text{for all } d>0 .$$

Now, we shall prove the lemma. By assumption we get

$$\text{(2.10)} \qquad \mathrm{P}\{|N/n-1|\geqq\rho\}<\xi \qquad \text{for } d \text{ sufficiently small}.$$

Put

$$I=\mathrm{P}\{(Nh_N^p)^{1/2}|T_N-T_n|\geqq\varepsilon,\ |N-n|<\rho n\}.$$

Then

$$\text{(2.11)} \quad \begin{aligned} I&\leqq\mathrm{P}\{(ih_i^p)^{1/2}|T_i-T_n|\geqq\varepsilon \text{ for some } i\in(M_1, M_2]\}\\ &\leqq\mathrm{P}\{(ih_i^p)^{1/2}(|T_i-T_{M_1}|+|T_n-T_{M_1}|)\geqq\varepsilon \text{ for some } i\in(M_1, M_2]\}\\ &\leqq\mathrm{P}\{\max_{M_1<i\leqq M_2}(ih_i^p)^{1/2}\times 2\max_{M_1<i\leqq M_2}|T_i-T_{M_1}|\geqq\varepsilon\}\\ &=\mathrm{P}\{\max_{M_1<i\leqq M_2}(ih_i^p)^{1/2}\times\max_{M_1<i\leqq M_2}|T_i-T_{M_1}|\geqq\varepsilon/2\}.\end{aligned}$$

By (2.2) and the definition of T_n we have that for $i\in(M_1, M_2]$

$$\begin{aligned}|T_i-T_{M_1}|&\leqq\left|\sum_{j=1}^{M_1}a_j(\beta_{ji}-\beta_{jM_1})V_j\right|+\left|\sum_{j=M_1+1}^{i}a_j\beta_{ji}V_i\right|\\ &\leqq(\gamma_{M_1}-\gamma_i)\left|\sum_{j=1}^{M_1}a_j\gamma_j^{-1}V_j\right|+\gamma_i\left|\sum_{j=M_1+1}^{i}a_j\gamma_j^{-1}V_j\right|,\end{aligned}$$

which, together with (2.11) and the monotonicity of h_n, nh_n^p and γ_n, implies that

$$\text{(2.12)} \quad \begin{aligned} I\leqq\mathrm{P}\Big\{&(M_2h_{M_1}^p)^{1/2}(\gamma_{M_1}-\gamma_{M_2})\left|\sum_{j=1}^{M_1}a_j\gamma_j^{-1}V_j\right|\\ &+(M_2h_{M_2}^p)^{1/2}\max_{M_1<i\leqq M_2}\gamma_i\left|\sum_{j=M_1+1}^{i}a_j\gamma_j^{-1}V_j\right|\geqq\varepsilon/2\Big\}\leqq J_1+J_2,\end{aligned}$$

where

$$J_1=\mathrm{P}\left\{(M_2h_{M_1}^p)^{1/2}(\gamma_{M_1}-\gamma_{M_2})\left|\sum_{j=1}^{M_1}a_j\gamma_j^{-1}V_j\right|\geqq\varepsilon/4\right\} \quad \text{and}$$

$$J_2=\mathrm{P}\left\{(M_2h_{M_2}^p)^{1/2}\max_{M_1<i\leqq M_2}\gamma_i\left|\sum_{j=M_1+1}^{i}a_j\gamma_j^{-1}V_j\right|\geqq\varepsilon/4\right\}.$$

From (H2), (2.4) and assumption we get

$$ih_i^p\sum_{j=1}^{i}a_j^2\beta_{ji}^2\,\mathrm{E}[V_j^2]\leqq C_2 i^{1-2\alpha}h_i^p\sum_{j=1}^{i}j^{2(\alpha-1)}h_j^{-p}\leqq C_3 \qquad \text{for all } i\geqq 1.$$

Hence by Chebychev's inequality, (2.2), (2.7) and (2.9) we obtain

$$\text{(2.13)} \qquad J_1\leqq C_4\varepsilon^{-2}(1-\beta_{M_1M_2})^2(M_2/M_1)M_1h_{M_1}^p\sum_{j=1}^{M_1}a_j^2\beta_{jM_1}^2\,\mathrm{E}[V_j^2]\leqq C_5\xi$$

for d sufficiently small. From assumption, the Hájek-Rényi inequality (see Petrov [9], page 51), the monotonicity of h_n, (2.8) and (2.9) we have that for d sufficiently small

(2.14)
$$J_2 \leqq C_6 \varepsilon^{-2} M_2 h_{M_2}^p \sum_{j=M_1+1}^{M_2} a_j^2 \, \mathrm{E}\,[V_j^2] \leqq C_7 \varepsilon^{-2} M_2 \sum_{j=M_1+1}^{M_2} j^{-2}$$
$$\leqq C_7 \varepsilon^{-2} M_2 M_1^{-2}(M_2 - M_1) \leqq C_8 \xi \, .$$

Combining (2.10), (2.12), (2.13) and (2.14) we obtain

$$\mathrm{P}\,\{(N h_N^p)^{1/2} |T_N - T_n| \geqq \varepsilon\} \leqq I + \mathrm{P}\,\{|N-n| \geqq \rho n\} \leqq C_9 \xi$$

for d sufficiently small, which concludes the proof of Lemma 2.5.

3. Main results

In this section we shall propose sequential confidence intervals $I_{n,d}(x)$ for $m(x)$ with prescribed width $2d$ and coverage probability α. Then the asymptotic consistency of these confidence intervals will be shown, that is, $\mathrm{P}\,\{m(x) \in I_{N(d),d}(x)\} \to \alpha$ as $d \to 0$, where $\{N(d)\}$ is the class of stopping times defined below.

Let any α $(0<\alpha<1)$ be given. Define $D=D(\alpha)>0$ by $\Phi(D)-\Phi(-D)=\alpha$, where $\Phi(\cdot)$ is the standard normal distribution function. Let d be any positive number, and let any $x \in R^p$ be fixed. We define the stopping time $N(d)=N(\alpha, d, x)$ as the smallest integer $n \geqq 1$ such that $(D^2 B)^{-1} d^2 n h_n^p \geqq v_n(x)/f_n(x) > 0$, where

(3.1) $\quad B = a^2 \beta \int_{R^p} K^2(u) du \ (>0) \qquad$ with β being given in (H2).

Define the confidence interval $I_{n,d}(x)$ as

$$I_{n,d}(x) = [m_n(x) - d, \ m_n(x) + d] \, .$$

Also, define $n(d)=n(\alpha, d, x)$ as the smallest integer $n \geqq 1$ such that $(D^2 B)^{-1} d^2 n h_n^p \geqq v(x)/f(x) > 0$.

Let $\sigma^2(x) = Bv(x)/f(x)$ with B being given in (3.1).

The following lemma states the asymptotic properties of $N(d)$.

LEMMA 3.1. *Assume* $\mathrm{E}\,[Y^4] < \infty$. *Let* (H1), (H3) *and* (H4) *be satisfied. Consider a point* $x \in C(f) \cap C(q) \cap C(g)$ *with* $v(x) > 0$. *If either* $x \in C(\psi)$ *or* $\|\psi\|_\infty < \infty$ *holds then we have*

$$\mathrm{P}\,\{N(d) < +\infty\} = 1 \quad \textit{for each } d > 0 \, , \quad \mathrm{P}\,\{N(d) \uparrow \infty \textit{ as } d \to 0\} = 1 \, ,$$
$$N(d) h_{N(d)}^p d^2 / (D^2 \sigma^2(x)) \to 1 \quad \textit{as } d \to 0 \textit{ w.p.1.}, \textit{ and}$$
$$N(d) h_{N(d)}^p / (n(d) h_{n(d)}^p) \to 1 \quad \textit{as } d \to 0 \textit{ w.p.1.}.$$

PROOF. From the definition of $N(d)$ we get

(3.2) $\quad N(d) = $ smallest integer $n \geqq 1$ such that $b(n)/t \geqq y_n > 0$, where

$$y_n=\frac{v_n(x)/f_n(x)}{v(x)/f(x)}, \quad b(n)=\frac{nh_n^p}{v(x)/f(x)} \quad \text{and} \quad t=D^2Bd^{-2}.$$

Clearly,

(3.3) $\quad b(n)>0$ for all $n\geqq 1$, $\lim_{n\to\infty} b(n)=\infty$ and $\lim_{n\to\infty} b(n)/b(n-1)=1$.

We shall show that

(3.4) $$v_n(x)\geqq 0 \quad \text{for all } n\geqq 1 \text{ on } \Omega$$

and that for any fixed $\omega\in\Omega$

(3.5) $\quad v_n(x)>0$ for all $n\geqq m$ if $v_m(x)>0$ for some $m=m(\omega)\geqq 1$.

For simplicity we omit ω. By the definition of $v_n(x)$ it suffices to consider the case where $f_n(x)>0$.

That $v_n(x)\geqq$(resp. $>$) 0 for all $n\geqq 1$ is equivalent to

(3.6) $$g_n(x)f_n(x)-q_n^2(x)\geqq(\text{resp. }>)\ 0 \quad \text{for all } n\geqq 1.$$

Let any $\omega\in\Omega$ be fixed. First we shall prove (3.4). By the definitions of $f_n(x)$, $q_n(x)$ and $g_n(x)$ we have

(3.7) $$A_{n+1}=(1-a_n)^2A_n+a_n(1-a_n)D_n \quad \text{for each } n\geqq 1,$$

where

$$A_n=g_{n-1}(x)f_{n-1}(x)-q_{n-1}^2(x) \quad \text{and}$$

$$D_n=G_n(x, z_n)f_{n-1}(x)+K_n(x, X_n)g_{n-1}(x)-2Q_n(x, Z_n)q_{n-1}(x).$$

By Remark 2.1 we get

(3.8) $$D_n=\sum_{j=1}^{n-1} a_j\beta_{j\,n-1}\{G_n(x, Z_n)K_j(x, X_j)+K_n(x, X_n)G_j(x, Z_j)$$
$$-2Q_n(x, Z_n)Q_j(x, Z_j)\}+G_n(x, Z_n)\beta_{0\,n-1}c$$

for each $n\geqq 1$. From the definitions of $K_n(x, X_n)$, $Q_n(x, Z_n)$ and $G_n(x, Z_n)$ we have

$$G_n(x, Z_n)\beta_{0\,n-1}c\geqq 0 \quad \text{for each } n\geqq 1$$

and

$$G_n(x, Z_n)K_j(x, X_j)+K_n(x, X_n)G_j(x, Z_j)-2Q_n(x, Z_n)Q_j(x, Z_j)\geqq 0$$
$$\text{for each } j=1, \cdots, n-1 \text{ with } n\geqq 2,$$

which, together with (3.8), yields that

(3.9) $$D_n \geqq 0 \quad \text{for each } n \geqq 1 .$$

It follows from (3.7) and (3.9) that

(3.10) $$A_{n+1} \geqq (1-a_n)^2 A_n \quad \text{for each } n \geqq 1 .$$

As $A_1=0$, by (3.10) and induction we have $A_n \geqq 0$ for each $n \geqq 1$, which, together with (3.6), gives (3.4).

Next we shall prove (3.5). Assume that $v_m(x)>0$ for some $m \geqq 1$. From (3.6) we get $A_{m+1}>0$. Suppose that $A_n>0$ for some $n>m+1$. Then by (2.1) and (3.10) we have $A_{n+1}>0$. Hence by induction we obtain $A_n>0$ for all $n \geqq m+1$, which is equivalent to $v_n(x)>0$ for all $n \geqq m$. Thus (3.5) was proved.

Next we shall show that

(3.11) $$g_n(x) \to g(x) \quad \text{as } n \to \infty \text{ w.p.1.} .$$

Since Lemma 2.1, the definition of $G_n(x, Z_n)$ and $x \in C(g)$ give $\mathrm{E}\,[G_n(x, Z_n)] \to g(x)$ as $n \to \infty$, it follows from Remark 2.1 and the Toeplitz lemma (see Loève [7], page 238) that

(3.12) $$\mathrm{E}\,[g_n(x, Z_n)] \to g(x) \quad \text{as } n \to \infty .$$

Lemma 2.1 and (H4) give

$$\sum_{n=1}^{\infty} a_n^2\, \mathrm{E}\,[G_n^2(x, Z_n)] < \infty ,$$

which, together with Kolmogorov's convergence theorem, Kronecker's lemma, Remark 2.1 and (2.2), implies that

(3.13) $$g_n(x) - \mathrm{E}\,[g_n(x)] \to 0 \quad \text{as } n \to \infty \text{ w.p.1.} .$$

Thus, according to (3.12) and (3.13) we obtain (3.11). Lemma 2.2 and (3.11) give

(3.14) $$v_n(x) \to v(x) \quad \text{as } n \to \infty \text{ w.p.1.} .$$

From Lemma 2.2, (3.4), (3.5), (3.14) and the property of $f_n(x)$, there exists a null set A such that for each $\omega \in A^c$

$$y_n(\omega) \geqq 0 \quad \text{for all } n \geqq 1 , \quad \lim_{n\to\infty} y_n(\omega) = 1 \quad \text{and}$$

$$y_n(\omega) > 0 \quad \text{for all } n \geqq m \quad \text{if } y_m(\omega) > 0 \quad \text{for some } m = m(\omega) \geqq 1 ,$$

which, together with (3.2) and (3.3), permits us to apply Lemma 2.3 to obtain the first three assertions of Lemma 3.1. Replacing $N(d)$, $v_n(x)$ and $f_n(x)$ by $n(d)$, $v(x)$ and $f(x)$, respectively, we have that as $d \to 0$ $n(d) \to \infty$ and $n(d) h_{n(d)}^{p} d^2 / (D^2 \sigma^2(x)) \to 1$, which, together with the third assertion, implies the last assertion. This completes the proof.

Remark 3.1. By use of Lemmas 2.2 and 3.1, and Theorem 1 of Richter [11] we have that under all the conditions of Lemma 3.1

$$m_{N(d)}(x) \to m(x) \qquad \text{as } d \to 0 \text{ w.p.1.}.$$

The following theorem is one of main theorems.

THEOREM 3.1. *Assume* $\mathrm{E}[Y^4]<\infty$. *Let* (K1), (K2) *and* (H1)~(H5) *be satisfied. Suppose that there exist bounded, continuous second partial derivatives* $\partial^2 f(x)/\partial x_i \partial x_j$ *and* $\partial^2 q(x)/\partial x_i \partial x_j$ *on* R^p *for* $i, j=1, \cdots, p$. *Consider a point* $x \in C(g)$ *with* $v(x)>0$. *Assume* $x \in C(\psi)$ *or* $\|\psi\|_\infty<\infty$. *If*

$$N(d)/n(d) \underset{P}{\to} 1 \qquad \text{as } d \to 0 \tag{3.15}$$

then we obtain

$$(N(d)h^p_{N(d)})^{1/2}(m_{N(d)}(x)-m(x)) \underset{L}{\to} N(0, \sigma^2(x)) \qquad \text{as } d \to 0\,.$$

PROOF. For simplicity put $N=N(d)$ and $n=n(d)$. It follows from Lemma 3.1 that $n \to \infty$ as $d \to 0$. First we shall show that

$$B_N \underset{L}{\to} N(0, \Gamma) \qquad \text{as } d \to 0\,, \tag{3.16}$$

where Γ is given in Lemma 2.4. From Lemma 2.4 and the Cramér-Wold theorem (see Billingsley [2], page 49) we get

$$D'B_n \underset{L}{\to} N(0, D'\Gamma D) \qquad \text{as } d \to 0 \text{ for any } D' \in R^2\,. \tag{3.17}$$

Since

$$D'B_N = D'B_n + (D'B_N - D'B_n) \qquad \text{for any } D' \in R^2\,,$$

to prove (3.16) it suffices from (3.17) and the Cramér-Wold theorem to show that

$$D'B_N - D'B_n \underset{P}{\to} 0 \qquad \text{as } d \to 0 \text{ for any } D' \in R^2\,. \tag{3.18}$$

Let any $D'=(d_0, d_1) \in R^2$ be fixed. For $i \geqq 1$ set

$$S_i^{(t)} = \sum_{j=1}^{i} a_j \beta_{ji} U_j^{(t)} \qquad \text{for } t=0, 1\,.$$

It is clear that

$$D'B_N - D'B_n = \sum_{t=0}^{1} d_t (Nh_N^p)^{1/2}(S_N^{(t)} - S_n^{(t)}) + \{(Nh_N^p/(nh_n^p))^{1/2} - 1\} D'B_n\,. \tag{3.19}$$

Put

$$\xi^{(t)}(x) = (f(x))^{1-t}(g(x))^t \qquad \text{for } t=0, 1\,.$$

It follows from assumption and Lemma 2.1 that for $t=0, 1$

$$h_i^p \mathrm{E}\,[(U_i^{(t)})^2] \leqq \int_{R^p} h_i^{-p} K^2((x-u)/h_i)\xi^{(t)}(u)du \leqq C_1 \qquad \text{for all } i \geqq 1 .$$

Thus by use of Lemma 2.5 we have

$$\sum_{t=0}^{1} d_t (Nh_N^p)^{1/2}(S_N^{(t)} - S_n^{(t)}) \underset{P}{\rightarrow} 0 \qquad \text{as } d \rightarrow 0 . \tag{3.20}$$

Let any ε (>0) be fixed. From (3.15) and (H3) we get that for δ in (H3)

$$\mathrm{P}\,\{|h_N/h_n - 1| \geqq \varepsilon\} \leqq \mathrm{P}\,\{|N/n - 1| \geqq \delta\} \rightarrow 0 \qquad \text{as } d \rightarrow 0 ,$$

which implies that

$$h_N/h_n \underset{P}{\rightarrow} 1 \qquad \text{as } d \rightarrow 0 .$$

Therefore, (3.15) gives

$$\{Nh_N^p/(nh_n^p)\}^{1/2} - 1 \underset{P}{\rightarrow} 0 \qquad \text{as } d \rightarrow 0 ,$$

which, together with (3.17), yields that

$$\{(Nh_N^p/(nh_n^p))^{1/2} - 1\} D'B_n \underset{P}{\rightarrow} 0 \qquad \text{as } d \rightarrow 0 . \tag{3.21}$$

From (3.19), (3.20) and (3.21) we obtain (3.18). In the proof of Theorem 4.1 of [6] it was proved, under the assumptions of Theorem 3.1, that $\|B_i^* - B_i\|_2 \rightarrow 0$ as $i \rightarrow \infty$ on Ω. Hence by Lemma 3.1 we get $\|B_N^* - B_N\|_2 \rightarrow 0$ as $d \rightarrow 0$ w.p.1., which, together with (3.16), implies that

$$B_N^* \underset{L}{\rightarrow} N(0, \Gamma) \qquad \text{as } d \rightarrow 0 . \tag{3.22}$$

Define a function $T(u, v)$ on R^2 as

$$T(u, v) = \begin{cases} v/u & \text{if } u \neq 0 \\ 0 & \text{otherwise} . \end{cases}$$

Let $L' = (-q(x)/f^2(x),\ f^{-1}(x))$. By the Taylor theorem we get

$$(Nh_N^p)^{1/2}\{T(f_N(x), q_N(x)) - T(f(x), q(x))\} = L'B_N^* + \varepsilon_N \|B_N^*\|_2 \qquad \text{on } [N < +\infty] , \tag{3.23}$$

where

$$\varepsilon_i \rightarrow 0 \qquad \text{if } \|(f_i(x), q_i(x))' - (f(x), q(x))'\|_2 \rightarrow 0 .$$

According to Lemma 2.1 of [6] we have $m_N(x) = q_N(x)/f_N(x)$ on $[N < +\infty]$. Hence by the definitions of $T(u, v)$ and N we obtain

$$(3.24) \qquad T(f_N(x), q_N(x)) = m_N(x) \qquad \text{on } [N < +\infty]\,.$$

Since $m(x) = q(x)/f(x)$ it follows from (3.23) and (3.24) that

$$(3.25) \qquad (Nh_N^p)^{1/2}(m_N(x) - m(x)) = L'B_N^* + \varepsilon_N \|B_N^*\|_2 \qquad \text{on } [N < +\infty]\,.$$

Lemma 2.2 gives $\varepsilon_i \to 0$ as $i \to \infty$ w.p.1., which, together with Lemma 3.1, yields that

$$(3.26) \qquad \varepsilon_N \to 0 \qquad \text{as } d \to 0 \text{ w.p.1.}\,.$$

Combining (3.22) and (3.26) we get

$$(3.27) \qquad \varepsilon_N \|B_N^*\|_2 \underset{P}{\to} 0 \qquad \text{as } d \to 0\,.$$

Therefore, according to (3.22), (3.25), (3.27) and Lemma 3.1 we obtain

$$(Nh_N^p)^{1/2}(m_N(x) - m(x)) \underset{L}{\to} N(0, L'\Gamma L) \qquad \text{as } d \to 0\,,$$

which concludes the proof of Theorem 3.1.

We are now in the position to give our main result.

THEOREM 3.2. *Under all the conditions of Theorem* 3.1 *we have*

$$\mathrm{P}\{m(x) \in I_{N(d),d}(x)\} \to \alpha \qquad \text{as } d \to 0\,.$$

PROOF. Put $N = N(d)$. By Lemma 3.1 and Theorem 3.1 we have

$$\begin{aligned} &Dd^{-1}(m_N(x) - m(x)) \\ &\quad = (D^2\sigma^2(x)/(Nh_N^p d^2))^{1/2}(Nh_N^p/\sigma^2(x))^{1/2}(m_N(x) - m(x)) \underset{L}{\to} N(0, 1) \\ &\qquad\qquad \text{as } d \to 0\,. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \mathrm{P}\{m(x) \in I_{N,d}(x)\} = \mathrm{P}\{|Dd^{-1}(m_N(x) - m(x))| \leqq D\} \to \Phi(D) - \Phi(-D) = \alpha \\ \text{as } d \to 0\,. \end{aligned}$$

This completes the proof.

COROLLARY 3.1. *Assume* $\mathrm{E}[Y^4] < \infty$. *Let* (K1) *and* (K2) *be satisfied, and let* $\|\psi\|_\infty < \infty$. *Suppose that there exist bounded, continuous second partial derivatives* $\partial^2 f(x)/\partial x_i \partial x_j$ *and* $\partial^2 q(x)/\partial x_i \partial x_j$ *on* R^p *for* $i, j = 1, \cdots, p$ *and that* $g(x)$ *is continuous on* R^p. *Set*

$$h_n = n^{-r/p} \qquad \text{with } p/(p+4) < r < 1\,.$$

Let a *in* (2.1) *satisfy* $1 \geqq a > (1-r)/2$. *Then, for each point* x *with* $v(x) > 0$ *we obtain*

$$\mathrm{P}\{m(x) \in I_{N(d),d}(x)\} \to \alpha \qquad \text{as } d \to 0\,.$$

PROOF. We can easily verify (H1)~(H5) with $\beta=(2a+r-1)^{-1}$. Lemma 3.1 gives (3.15). Thus, since all the conditions of Theorem 3.1 are fulfilled, we obtain Corollary 3.1 by Theorem 3.2. This completes the proof.

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