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BHATTACHARYYA BOUND OF VARIANCES OF UNBIASED ESTIMATORS IN NONREGULAR CASES

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Summary

Bhattacharyya bound is generalized to nonregular cases when the support of the density depends on the parameter, while it is differentiable several times with respect to the parameter within the support. Some example is discussed, where it is shown that the bound is sharp.

1. Introduction

It is well known that the Cramér-Rao and the Bhattacharyya bounds are most important and very useful for the variances of unbiased estimators. They are, however, not applicable to the non-regular cases when the support of the distribution is dependent on the parameter. Same is true about more general and simpler bounds by Hammersley [6], Chapman and Robbins [2], Kiefer [8], Fraser and Guttman [5], Fend [4] and Chatterji [3], among others. (For an exposition of some of this work along with extensions in different directions, see Polfeldt [10], [11] and the recent papers of Víncze [15], Khatri [7] and Móri [9], among others). In his paper, Polfeldt [10] discussed the lower bound of the variances of the unbiased estimators when the class of probability measures is one-sided, that is, when P_{θ_1} is absolutely continuous with respect to P_{θ_2} (symbolically, $P_{\theta_1} \ll P_{\theta_2}$) when $\theta_1 < \theta_2$ or $\theta_1 > \theta_2$. In this note, our main interest is to obtain the Bhattacharyya bound when for any θ_1, θ_2, with $\theta_1 \neq \theta_2$, neither $P_{\theta_1} \ll P_{\theta_2}$ nor $P_{\theta_2} \ll P_{\theta_1}$.

2. Results

Let $\mathcal{X}$ be an abstract sample space with x as its generic point, $\mathcal{A}$ a σ-field of subsets of $\mathcal{X}$, and let Θ be a parameter space assumed to be an open set in the real line. Let $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ be a class of prob-

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ability measures on $(\mathcal{X}, \mathcal{A})$. We assume that for each $\theta \in \Theta$, $P_\theta(\cdot)$ is absolutely continuous with respect to a σ-finite measure μ. We denote $dP_\theta/d\mu$ by $f(x, \theta)$. For each $\theta \in \Theta$, we denote by $A(\theta)$ the set of points in $\mathcal{X}$ for which $f(x, \theta)>0$.

We shall consider the Bhattacharyya bound of variances of unbiased estimators at some specified point θ_0 in Θ. We make the following assumptions:

(A.1)
$$\mu((\bigcap_{c>0} (\bigcup_{|h|<c} A(\theta_0+h)))\triangle A(\theta_0))=0 ,$$

where $E\triangle F$ denotes the symmetric difference of two sets E and F.

(A.2) For every $\theta_0 \in \Theta$ there exists a positive number ε and a positive function $\rho(x)$ such that for every $x \in A(\theta)$ and every $\theta \in (\theta_0-\varepsilon, \theta_0+\varepsilon)$, $\rho(x)>f(x, \theta)$, and for every $\theta \in (\theta_0-\varepsilon, \theta_0+\varepsilon)$,

$$\int_{A(\theta)} |\gamma(x)| f(x, \theta) d\mu<\infty \text{ implies } \int_{\bigcup_{\theta\in(\theta_0-\varepsilon,\theta_0+\varepsilon)} A(\theta)} |\gamma(x)|\rho(x)d\mu<\infty .$$

(A.3) For some positive integer k

$$\varlimsup_{h\to 0} \sup_{x \in \bigcup_{j=1}^{i} A(\theta_0+jh)-A(\theta_0)} \frac{\left|\sum_{j=1}^{i} (-1)^j \binom{i}{j} f(x, \theta_0+jh)\right|}{|h|^i \rho(x)}<\infty , \quad i=1,\cdots, k .$$

First we prove the following lemma.

LEMMA. *Assume that* (A.1), (A.2) *and* (A.3) *hold. If* $\phi(x)$ *is any measurable function for which* $\int_{\mathcal{X}} |\phi(x)|\rho(x)d\mu<\infty$, *then*

$$\lim_{h\to 0} \frac{1}{h^i} \int_{\bigcup_{j=1}^{i} A(\theta_0+jh)-A(\theta_0)} \sum_{j=1}^{i} (-1)^j \binom{i}{j} \phi(x) f(x, \theta_0+jh) d\mu=0 .$$

PROOF. By (A.2) and (A.3) it follows that for every $i=1,\cdots,k$ and every $\theta_0 \in \Theta$, there exist positive numbers ε and K_i such that

$$\frac{1}{|h|^i}\left|\sum_{j=1}^{i} (-1)^j \binom{i}{j} f(x, \theta_0+jh)\right| <K_i\rho(x) \quad \text{for } |ih|<\varepsilon \text{ and } x \in A(\theta_0)^c ,$$

where $A(\theta_0)^c$ denotes the complement of the set $A(\theta_0)$. Also, it follows from (A.1) that for every $j=1,\cdots, i$, there exist a sequence $\{\varepsilon_{jn}\}$ of positive numbers converging to zero as $n\to\infty$ and a monotone non-increasing sequence $\{S_{jn}\}$ of measurable sets such that $|jh|<\varepsilon_{jn}$ implies

$A(\theta_0+jh)-A(\theta_0)\subset S_{jn}$ and $\mu\left(\bigcap_{n=1}^{\infty} S_{jn}\right)=0$. If for each $j=1,\cdots,i$, $|jh|<\varepsilon_{jn}$, then

$$\left|\frac{1}{h^i}\int_{\bigcup_{j=1}^{i} A(\theta_0+jh)-A(\theta_0)} \sum_{j=1}^{i}(-1)^j\binom{i}{j}\phi(x)f(x,\theta_0+jh)d\mu\right|$$
$$\leqq \int_{\bigcup_{j=1}^{i} A(\theta_0+jh)-A(\theta_0)} K_i|\phi(x)|\rho(x)d\mu \leqq \int_{\bigcup_{j=1}^{i} S_{jn}} K_i|\phi(x)|\rho(x)d\mu$$
$$\leqq \sum_{j=1}^{i}\int_{S_{jn}} K_i|\phi(x)|\rho(x)d\mu\ ,$$

which tends to zero as $n\to\infty$. The proof follows.

Remark. The assumption (A.2) together with the condition in the above Lemma is satisfied with $p(x)=\sum_{i=1}^{\infty} c_i f(x,\theta_i)$ when the following holds: For each θ_0 and $\varepsilon>0$, there exist countable points $\theta_1,\theta_2,\cdots$ and positive constants $c_1,c_2,\cdots$ such that $\bigcup_{i=1}^{\infty} A(\theta_i)\supset A(\theta)$ for all $\theta\in(\theta_0-\varepsilon,\theta_0+\varepsilon)$ and that $\sum_{i=1}^{\infty} c_i<\infty$ and $\sum_{i=1}^{\infty} c_i f(x,\theta_i)>f(x,\theta)$ for all $\theta\in(\theta_0-\varepsilon,\theta_0+\varepsilon)$ and almost all $x[\mu]$.

We assume the following:

(A.4) For each $x\in A(\theta_0)$, $f(x,\theta)$ is k-times continuously differentiable in θ at $\theta=\theta_0$.

(A.5) For each $i=1,\cdots,k$,

$$\overline{\lim_{h\to 0}}\sup_{x\in A(\theta_0)} \frac{\left|\frac{\partial^i}{\partial\theta^i}f(x,\theta_0+h)\right|}{\rho(x)}<\infty\ , \qquad \text{where } \rho(x) \text{ is defined in (A.2)}.$$

We now prove our main theorem on the Bhattacharyya bound of the variances of unbiased estimators.

THEOREM 2.1. *Assume that* (A.1) *to* (A.5) *hold. Let* $g(\theta)$ *be an estimable function which is k-times differentiable over* $\boldsymbol{\Theta}$. *Let* $\hat{g}(x)$ *be an unbiased estimator of* $g(\theta)$ *satisfying*

$$\int_{A(\theta_0)} |\hat{g}(x)|\rho(x)d\mu<\infty\ .$$

Further, let Λ *be a* $k\times k$ *non-negative definite matrix whose elements are*

$$\lambda_{ij}=\int_{A(\theta_0)} \frac{1}{f(x,\theta_0)}\left\{\frac{\partial^i f(x,\theta_0)}{\partial\theta^i}\,\frac{\partial^j f(x,\theta_0)}{\partial\theta^j}\right\}d\mu\ ,\qquad i,j=1,\cdots,k\ .$$

Assume that λ_{ii}, $i=1,\cdots,k$ are finite. If Λ is nonsingular at θ_0, then

$$\text{(2.1)}\qquad \operatorname*{Var}_{\theta_0}(\hat{g})\geqq(g^{(1)}(\theta_0),\cdots,g^{(k)}(\theta_0))\Lambda^{-1}(g^{(1)}(\theta_0),\cdots,g^{(k)}(\theta_0))'\ ,$$

where $g^{(i)}(\theta)$ is the i-th order derivative of $g(\theta)$.

PROOF. Denote

$$g_0(\theta)=\int_{A(\theta_0)} \hat{g}(x)f(x,\theta)d\mu\quad\text{and}\quad \Delta_h f(x,\theta)=f(x,\theta+h)-f(x,\theta)\ .$$

Then, by (A.5), we have

$$\begin{aligned}\text{(2.2)}\qquad \left[\frac{\partial^i}{\partial\theta^i}g_0(\theta)\right]_{\theta=\theta_0}&=\lim_{h\to 0}\frac{1}{h^i}\Delta_h^i g_0(\theta_0)=\lim_{h\to 0}\frac{1}{h^i}\int_{A(\theta_0)}\hat{g}(x)\Delta_h^i f(x,\theta_0)d\mu\\
&=\lim_{h\to 0}\int_{A(\theta_0)}\hat{g}(x)\frac{1}{h^i}\Delta_h^i f(x,\theta_0)d\mu\\
&=\lim_{h\to 0}\int_{A(\theta_0)}\hat{g}(x)\left\{\frac{\partial^i}{\partial\theta^i}f(x,\theta_0+\xi h)\right\}d\mu\\
&=\int_{A(\theta_0)}\hat{g}(x)\left[\frac{\partial^i}{\partial\theta^i}f(x,\theta)\right]_{\theta=\theta_0}d\mu\ ,\end{aligned}$$

where $\Delta_h^i g(\theta)=\Delta_h^{i-1}(\Delta_h g(\theta))$, $i=1,\cdots,k$ and $0<\xi<1$. Since $g(\theta)=\int_{A(\theta)}\hat{g}(x)\cdot f(x,\theta)d\mu$, we obtain for each $i=1,\cdots,k$

$$\begin{aligned}\text{(2.3)}\qquad &\Delta_h^i(g(\theta_0)-g_0(\theta_0))\\
&=\sum_{j=1}^{i}(-1)^j\binom{i}{j}\{g(\theta_0+jh)-g_0(\theta_0+jh)\}\\
&=\sum_{j=1}^{i}(-1)^j\binom{i}{j}\left\{\int_{A(\theta_0+jh)}\hat{g}(x)f(x,\theta_0+jh)d\mu\right.\\
&\qquad\left.-\int_{A(\theta_0)}\hat{g}(x)f(x,\theta_0+jh)d\mu\right\}\\
&=\sum_{j=1}^{i}(-1)^j\binom{i}{j}\left\{\left(\int_{A(\theta_0+jh)}-\int_{A(\theta_0)\cap A(\theta_0+jh)}-\int_{A(\theta_0)-A(\theta_0+jh)}\right)\right.\\
&\qquad\left.\cdot\,\hat{g}(x)f(x,\theta_0+jh)d\mu\right\}\\
&=\sum_{j=1}^{i}(-1)^j\binom{i}{j}\left\{\left(\int_{A(\theta_0+jh)}-\int_{A(\theta_0)\cap A(\theta_0+jh)}\right)\hat{g}(x)f(x,\theta_0+jh)d\mu\right\}\\
&=\sum_{j=1}^{i}(-1)^j\binom{i}{j}\int_{A(\theta_0+jh)-A(\theta_0)}\hat{g}(x)f(x,\theta_0+jh)d\mu\end{aligned}$$

$$=\sum_{j=1}^{i}(-1)^j\binom{i}{j}\int_{\bigcup_{k=1}^{i}A(\theta_0+kh)-A(\theta_0)}\hat{g}(x)f(x,\theta_0+jh)d\mu$$

$$=\int_{\bigcup_{k=1}^{i}A(\theta_0+kh)-A(\theta_0)}\sum_{j=1}^{i}(-1)^j\binom{i}{j}\hat{g}(x)f(x,\theta_0+jh)d\mu\ .$$

By (2.3) and Lemma, we have for each $i=1,\cdots,k$,

$$\left[\frac{\partial^i}{\partial\theta^i}g(\theta)\right]_{\theta=\theta_0}=\lim_{h\to0}\frac{1}{h^i}\Delta_h^i g(\theta_0) \tag{2.4}$$

$$=\lim_{h\to0}\frac{1}{h^i}\Delta_h^i(g(\theta_0)-g_0(\theta_0))+\lim_{h\to0}\frac{1}{h^i}\Delta_h^i g(\theta_0)$$

$$=\lim_{h\to0}\frac{1}{h^i}\Delta_h^i g(\theta_0)=\left[\frac{\partial^i}{\partial\theta^i}g_0(\theta)\right]_{\theta=\theta_0}.$$

From (2.2) and (2.4) we obtain for each $i=1,\cdots,k$,

$$\left[\frac{\partial^i}{\partial\theta^i}g(\theta)\right]_{\theta=\theta_0}=\int_{A(\theta_0)}\hat{g}(x)\left[\frac{\partial^i}{\partial\theta^i}f(x,\theta)\right]_{\theta=\theta_0}d\mu\ . \tag{2.5}$$

Proceeding now as in the regular case (see e.g. Zacks [16], page 190), one can show that the Bhattacharyya bound of the variances of the unbiased estimators of $g(\theta)$ is given by (2.1).

3. Example

We consider the location parameter case, i.e., $f(x,\theta)=f(x-\theta)$, and unbiased estimators of θ.

We assume that for any $p\geqq1$, the density function $f(x)$ is given by

$$f(x)=\begin{cases}c(1-x^2)^{p-1} & \text{if } |x|<1\\ 0 & \text{if } |x|\geqq1\end{cases}$$

where $c=1/B(1/2,p)$ with $B(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$ $(\alpha>0,\ \beta>0)$.

Case (i): Let $p=1$. Then the distribution is uniform, and it is easy to check that $\min_{\hat{\theta}:\text{ unbiased}}\operatorname{Var}_{\theta_0}(\theta)=0$ for any specific value θ_0. (See Takeuchi [14]).

Case (ii): Let $p=2$. In this case, it is easy to check that the Fisher information $\int_{-1}^{1}[f'(x)/f(x)]^2f(x)dx=\infty$.

For any $\varepsilon>0$, we define an estimator $\hat{\theta}_\varepsilon$ which satisfies

$$\hat{\theta}_\varepsilon(x)=\begin{cases} -c_\varepsilon f'(x)/f(x) & \text{if } |x|\leqq 1-\varepsilon\,, \\ 0 & \text{if } 1-\varepsilon<|x|\leqq 1\,, \end{cases}$$

where c_ε is a constant determined from the equations

$$\int_{-1}^{1}\hat{\theta}_\varepsilon(x)f(x)dx=0 \quad \text{and} \quad \int_{-1}^{1}\hat{\theta}_\varepsilon(x)f'(x)dx=-1\,. \tag{3.1}$$

We shall determine $\hat{\theta}_\varepsilon(x)$ for x outside the interval $[-1, 1]$ from the unbiasedness condition

$$\int_{-1+\theta}^{1+\theta}\hat{\theta}_\varepsilon(x)f(x-\theta)dx=\theta\,. \tag{3.2}$$

First consider the case $0<\theta\leqq 1$, and define

$$g(\theta)=\int_{-1+\theta}^{1}\hat{\theta}_\varepsilon(x)f(x-\theta)dx\,. \tag{3.3}$$

Since $\hat{\theta}_\varepsilon(x)$ and $f'(x)$ are bounded, $g(\theta)$ is differentiable and $g'(\theta)=-\int_{-1+\theta}^{1}\hat{\theta}_\varepsilon(x)f'(x-\theta)dx$.

If we assume that $\hat{\theta}_\varepsilon(x)$ is bounded for $1<x\leqq 2$, the right hand side of (3.3) is also differentiable, and we have by (3.2) and (3.3)

$$1-g'(\theta)=-\int_{1}^{1+\theta}\hat{\theta}_\varepsilon(x)f'(x-\theta)dx\,. \tag{3.4}$$

Differentiating (3.4) again, and noting that $\lim_{x\to 1-0} f'(x)=-3/2$, we have

$$g''(\theta)=-\frac{3}{2}\hat{\theta}_\varepsilon(1+\theta)-\int_{1}^{1+\theta}\hat{\theta}_\varepsilon(x)f''(x-\theta)dx\,. \tag{3.5}$$

If $\hat{\theta}_\varepsilon$ satisfies (3.5), then it also satisfies (3.4) since $\lim_{\theta\to 0} g'(\theta)=-1$; it also satisfies (3.3) since $\lim_{\theta\to 0} g(\theta)=0$ by (3.1).

Since the integral equation (3.5) is of Volterra's second type, it follows that the solution $\hat{\theta}_\varepsilon(x)$ exists and is bounded. Repeating the same process, we have the solution $\hat{\theta}_\varepsilon(x)$ for all $x>1$. Similarly, we can construct $\hat{\theta}_\varepsilon(x)$ for $x<-1$. On the other hand,

$$\mathrm{Var}_{\theta_0}(\hat{\theta}_\varepsilon)=c_\varepsilon^2\int_{-1+\varepsilon}^{1-\varepsilon}[f'(x)/f(x)]^2 f(x)dx\,, \tag{3.6}$$

while from (3.1), we have

$$c_\varepsilon\int_{-1+\varepsilon}^{1-\varepsilon}f'(x)dx=0 \quad \text{and} \quad c_\varepsilon\int_{-1+\varepsilon}^{1-\varepsilon}[f'(x)/f(x)]^2 f(x)dx=1\,. \tag{3.7}$$

Hence $\mathrm{Var}_{\theta_0}(\hat{\theta}_\varepsilon)=c_\varepsilon$.

Now, since $\lim_{\epsilon\to 0}\int_{-1+\epsilon}^{1-\epsilon}[f'(x)/f(x)]^2 f(x)dx=\infty$, we have from (3.7), $\lim_{\epsilon\to 0} c_\epsilon=0$. Consequently, $\inf_{\hat{\theta}:\text{ unbiased}} \mathrm{Var}_{\theta_0}(\hat{\theta})=0$ for any specified value θ_0.

Before proceeding on to the next cases, we note the following:

If $k<p/2$, then λ_{ii} $(i=1,\cdots,k)$ given in Theorem 2.1 are finite, since $\int_{-1}^{1}(1-x^2)^{p-2k-1}dx<\infty$.

Also,

$$\lambda_{ij}=\int_{\theta-1}^{\theta+1}\frac{1}{f(x-\theta)}\frac{\partial^i f(x-\theta)}{\partial\theta^i}\frac{\partial^j f(x-\theta)}{\partial\theta^j}dx \tag{3.8}$$
$$=\int_{-1}^{1}(-1)^{i+j}\frac{1}{f(x)}f^{(i)}(x)f^{(j)}(x)dx\,; \qquad i,j=1,\cdots,k\,,$$

We also obtain for $|x|<1$,

$$f^{(1)}(x)=-2c(p-1)x(1-x^2)^{p-2}\,;$$
$$f^{(2)}(x)=-2c(p-1)\{(1-x^2)^{p-2}-2(p-2)x^2(1-x^2)^{p-3}\}\,; \tag{3.9}$$
$$f^{(3)}(x)=4c(p-1)(p-2)\{3x(1-x^2)^{p-3}-2(p-3)x^3(1-x^2)^{p-4}\}\,;\cdots.$$

If $i+j$ is an odd number, if follows by (3.8) and (3.9) that $\lambda_{ij}=0$ since $f^{(i)}(x)f^{(j)}(x)$ is an odd function.

From (3.8) and (3.9), we have

$$\lambda_{11}=4c(p-1)^2B\left(\frac{3}{2},\,p-2\right)\,;$$

$$\lambda_{13}=8c(p-1)^2(p-2)\left\{2(p-3)B\left(\frac{5}{2},\,p-4\right)-3B\left(\frac{3}{2},\,p-3\right)\right\}\,;$$

$$\lambda_{22}=4c(p-1)^2\left\{B\left(\frac{1}{2},\,p-2\right)-4(p-2)B\left(\frac{3}{2},\,p-3\right)\right.$$
$$\left.+4(p-2)^2B\left(\frac{5}{2},\,p-4\right)\right\}\,;$$

$$\lambda_{33}=16c(p-1)^2(p-2)^2\left\{9B\left(\frac{3}{2},\,p-4\right)-12(p-3)B\left(\frac{5}{2},\,p-5\right)\right.$$
$$\left.+4(p-3)^2B\left(\frac{7}{2},\,p-6\right)\right\}\,;\cdots.$$

Case (iii): Let $p=3,4$. Then, we have, for any unbiased estimator $\hat{\theta}(X)$ of θ,

$$\int_{-1}^{1}\hat{\theta}(x)f(x)dx=0 \tag{3.10}$$

$$\int_{-1}^{1} \hat{\theta}(x) f'(x) dx = -1 \tag{3.11}$$

$$\int_{-1}^{1} \hat{\theta}(x) f^{(k)}(x) dx = 0 , \qquad k=2, \cdots, p-1 . \tag{3.12}$$

Noting that $\int_{-1}^{1} \left[\left\{ \sum_{k=1}^{p-1} c_k f^{(k)}(x) \right\}^2 \Big/ f(x) \right] dx < \infty$ implies $c_2 = \cdots = c_{p-1} = 0$, we have by Takeuchi and Akahira [13], that the infimum of $\mathrm{Var}_0(\hat{\theta}) = \int_{-1}^{1} \hat{\theta}^2(x) f(x) dx$ under (3.10), (3.11) and (3.12) is given by $\inf_{\hat{\theta}: (3.10) \sim (3.12)} \mathrm{Var}_0(\hat{\theta}) = 1/\lambda_{11}$, where $\lambda_{11} = \int_{-\infty}^{\infty} [f'(x)/f(x)]^2 f(x) dx = (p-1)(2p-1)/(p-2)$ and for any $\varepsilon > 0$ there exists $\hat{\theta}_\varepsilon(x)$ in $(-1, 1)$ satisfying (3.10), (3.11) and (3.12), and $\int_{-1}^{1} \hat{\theta}_\varepsilon^2(x) f(x) dx < (1/\lambda_{11}) + \varepsilon$. We can extend $\hat{\theta}_\varepsilon(x)$ for x outside $(-1, 1)$ from the unbiasedness condition $\int_{-1+\theta}^{1+\theta} \hat{\theta}_\varepsilon(x) f(x-\theta) dx = \theta$. First, we consider the case when $0 \leqq \theta < 1$, and define $g(\theta) = \int_{-1+\theta}^{1} \hat{\theta}_\varepsilon(x) f(x-\theta) dx$. In a similar way to the case (ii), we have for $k=2, \cdots, p-1$,

$$(-1)^{k+1} g^{(k+1)}(\theta) = B_k \hat{\theta}_\varepsilon(1+\theta) - \int_{1}^{1+\theta} \hat{\theta}_\varepsilon(x) f^{(k+1)}(x-\theta) dx , \tag{3.13}$$

$$B_k = \lim_{x \to 1-0} f^{(k)}(x) .$$

Since the integral equation (3.13) is again of Volterra's second type, it follows that the solution $\hat{\theta}_\varepsilon(x)$ exists. Repeating the process, we can construct an unbiased estimator $\hat{\theta}_\varepsilon(x)$ for all values of x. Then, it follows that $\inf_{\hat{\theta}: \text{unbiased}} \mathrm{Var}_{\theta_0}(\hat{\theta}) = 1/\lambda_{11}$ for any specified value θ_0.

Case (iv): Let $p=5, 6$. Note that

$$\int_{-1}^{1} [f'(x) f''(x)/f(x)] dx = 0 ,$$

$$\int_{-1}^{1} [f''(x)/f(x)]^2 f(x) dx = \frac{(p-1)(2p-1)(2p-3)}{(p-2)(p-3)(p-4)} (2p^2 - 7p + 8)$$
$$= \lambda_{22} \qquad \text{(say)} ,$$

and

$$\int_{-1}^{1} \left\{ \sum_{k=1}^{p-1} c_k f^{(k)}(x)/f(x) \right\}^2 f(x) dx < \infty$$

imply $c_3 = \cdots = c_{p-1} = 0$. Then we see that

$$\mathrm{Var}_{\theta_0}(\hat{\theta}) \geqq (1, 0) \begin{pmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\lambda_{11}}$$

for any specified θ_0, where λ_{11} is defined above (3.8). Here again, as in the previous case $\inf_{\hat{\theta}:\text{ unbiased}} \mathrm{Var}_{\theta_0}(\hat{\theta})=1/\lambda_{11}$ for any specific θ_0.

Case (v): Let $p=7$. In this case, we see that $k=3$. Using Theorem 2.1, (3.8) and (3.9), we obtain

$$\mathrm{Var}_{\theta_0}(\hat{\theta}) \geqq \frac{1}{|\Lambda|}\begin{vmatrix} \lambda_{22} & 0 \\ 0 & \lambda_{33} \end{vmatrix} = \left[\lambda_{11}\left(1-\frac{\lambda_{13}^2}{\lambda_{11}\lambda_{33}}\right)\right]^{-1}$$

where

$$\Lambda=\begin{pmatrix} \lambda_{11} & 0 & \lambda_{13} \\ 0 & \lambda_{22} & 0 \\ \lambda_{13} & 0 & \lambda_{33} \end{pmatrix}$$

with

$$\lambda_{11}=144cB\left(\frac{3}{2}, 5\right) ;$$

$$\lambda_{13}=1440c\left\{8B\left(\frac{5}{2}, 3\right)-3B\left(\frac{3}{2}, 4\right)\right\} ;$$

$$\lambda_{22}=144c\left\{B\left(\frac{1}{2}, 5\right)-20B\left(\frac{3}{2}, 4\right)+100B\left(\frac{5}{2}, 3\right)\right\} ;$$

$$\lambda_{33}=14400c\left\{9B\left(\frac{3}{2}, 3\right)-48B\left(\frac{5}{2}, 2\right)+64B\left(\frac{7}{2}, 1\right)\right\} .$$

We also obtain

$$\frac{\lambda_{13}^2}{\lambda_{11}\lambda_{33}}=\frac{\{8B(5/2, 3)-3B(3/2, 4)\}^2}{B(3/2, 5)\{9B(3/2, 3)-48B(5/2, 2)+64B(7/2, 1)\}} .$$

Here again $\inf_{\hat{\theta}:\text{ unbiased}} \mathrm{Var}_{\theta_0}(\hat{\theta})=[\lambda_{11}(1-(\lambda_{13}^2/\lambda_{11}\lambda_{13})]^{-1}$ for any specific θ_0, i.e. we have a sharp bound.

Case (vi): For $p\geqq 8$ we can continue in a similar manner by choosing $k=[(p-1)/2]$, where $[s]$ denotes the largest integer less than or equal to s.

The above discussion establishes that here the bound is sharp but generally it is not attainable.

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