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SOME TEST STATISTICS BASED ON THE MARTINGALE TERM OF THE EMPIRICAL DISTRIBUTION FUNCTION

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Summary

It is proved that the martingale term of the empirical distribution function converges weakly to a Gaussian process in $D[0, 1]$. Some statistics for goodness-of-fit tests based on the martingale term of the empirical distribution function are proposed. Asymptotic distributions of these statistics under the null hypothesis are given. The approximate Bahadur efficiencies of the statistics to the Kolmogorov-Smirnov statistic and to the Cramér-von Mises statistic are also calculated.

1. Introduction

Suppose $X_1, X_2, \cdots, X_n$ are independent and have a common distribution function $F(t)$ on $[0, \infty)$. $F_n(t)$ denotes the empirical distribution function of the variables $X_1, X_2, \cdots, X_n$. Then $nF_n(t)$ can be regarded as an example of the counting process (cf. e.g. Aalen [1] and Jacobsen [11]). Khmaladze [12] proposed that the martingale term of the empirical distribution function can be available for construction of test statistics for goodness of fit. In particular, he asserted that the limit process of the estimated empirical process is a diffusion process and its martingale term, which is a Wiener process, can be approximated by the sample so that the approximated martingale term converges weakly to the Wiener process in $L_2[0, 1]$.

The purpose of this paper is to construct some test statistics based on the martingale term of the empirical distribution function and to investigate properties of those statistics. In this paper, however, we treat only goodness-of-fit tests for simple hypotheses for two reasons; the one is that the estimated case is too complicated to be treated in our framework for the present and the other is that as far as we define test statistics for simple and composite hypotheses by a function

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of the martingale term of the empirical process and by the same function of the martingale term of the estimated empirical process, respectively, the asymptotic distributions of two statistics are the same, since the martingale terms of the limit processes of the empirical process and of the estimated empirical process are the same, which are both Wiener processes (cf. Khmaladze [12]).

Suppose $U_1, U_2, \cdots, U_n$ are independent uniformly distributed random variables on $[0, 1]$. Let $\Gamma_n(t)$ be the empirical distribution function corresponding to $U_1, U_2, \cdots, U_n$. We shall prove in Section 2 that the stochastic process

$$W_n(t)=\sqrt{n}\left(\Gamma_n(t)-\int_0^t \frac{1-\Gamma_n(s)}{1-s}\,ds\right) \tag{1.1}$$

is a martingale and converges weakly to a standard Wiener process in $D[0, 1]$, which is the space of functions on $[0, 1]$ that are right continuous and have left-hand limits endowed with the Skorohod topology (cf. [4], [14]). We need this theorem for proving weak convergence of some test statistics such as $\sup_t |W_n(t)|$ etc., although the weak convergence of (1.1) in $L_2[0, 1]$ was proved by Khmaladze [12].

In Section 3 we shall study asymptotic distributions of linear functionals of (1.1) and some related topics. Neyman's 'smooth' test will be shown to be written as a function of (1.1).

We shall propose in Section 4 the test statistic $T_n^s=\sup_t |W_n(t)|$ and give its exact and asymptotic distributions. Moreover, we shall give a condition under which the sequence of the test statistics $\{T_n^s\}$ has the approximate Bahadur slope. The approximate Bahadur efficiency of T_n^s to the Kolmogorov-Smirnov statistic will be also given.

In the last section Cramér-von Mises type statistics based on (1.1) will be investigated. The asymptotic distributions and the approximate Bahadur slopes under the same setup of Section 4 will be obtained.

2. Some limit theorems on empirical distribution functions

Let $U_1, U_2, \cdots, U_n$ be independent uniformly distributed random variables on $[0, 1]$. Let $U_{(1)}\leqq U_{(2)}\leqq \cdots \leqq U_{(n)}$ be the order statistics corresponding to $U_1, U_2, \cdots, U_n$. $\Gamma_n(t)$ denotes the empirical distribution function of $U_1, U_2, \cdots, U_n$. Consider the stochastic process

$$W_n(t)=\sqrt{n}\left(\Gamma_n(t)-\int_0^t \frac{1-\Gamma_n(s)}{1-s}\,ds\right), \qquad 0\leqq t\leqq 1. \tag{2.1}$$

THEOREM 2.1. *$W_n(t)$ converges weakly in $D[0, 1]$ to $W(t)$, which is a standard Wiener process on $D[0, 1]$.*

Remark 2.1. The weak convergence in $L_2[0, 1]$ of the process (2.1) was proved by Khmaladze [12]. Jacobsen [11] suggested that the theorem can be proved by using a central limit theorem for local martingales. Recently, Al-Hussaini and Elliott ([2], Lemma 5.4) proved this fact briefly by applying the result of Liptser and Shiryaev [15]. However, for our later use, we will give another rigorous proof based on the theory for counting processes.

For proving Theorem 2.1, we use a theorem and two lemmas. The next theorem, which is due to Rebolledo [18] (cf. also [19]), is Theorem 5.1.3 of Jacobsen [11].

THEOREM 2.2. *(Rebolledo) Let $\{M_n(t);\ 0 \leqq t < \infty\}_{n\geqq 1}$ be a sequence of locally square-integrable martingales in $D[0, \infty)$. Suppose that for every $t \geqq 0$ and $\varepsilon > 0$,*

$$\mathrm{E}\sum_{s\leqq t} (\Delta M_n(s))^2 1_{(|\Delta M_n(s)|>\varepsilon)} \to 0 \qquad as\ n\to\infty ,$$

where we write $\Delta f(t) = f(t) - f(t-)$, and suppose also that there exists a non-decreasing continuous function $\Phi\colon [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that for every $t \geqq 0$ and $\varepsilon > 0$,

$$\mathrm{P}\{|\langle M_n\rangle_t - \Phi(t)| > \varepsilon\} \to 0 \qquad as\ n\to\infty ,$$

where $\langle M_n\rangle_t$ means the quadratic variational process of M_n, i.e. the unique predictable, increasing, right continuous, left limit process such that $M_n^2 - \langle M_n\rangle$ is a martingale. Then M_n converges weakly to $\{B(\Phi(t));\ 0 \leqq t < \infty\}$ in $D[0, \infty)$, where $\{B(t);\ 0 \leqq t < \infty\}$ is a standard Brownian motion.

LEMMA 2.1. *$n\Gamma_n(t)$ is a counting process and has the integrated intensity process $n\Lambda_t$, where*

$$\Lambda_t = \begin{cases} \displaystyle\int_0^t \frac{1-\Gamma_n(s)}{1-s}\,ds , & if\ t \leqq 1 , \\[2ex] \displaystyle\int_0^1 \frac{1-\Gamma_n(s)}{1-s}\,ds , & if\ t > 1 . \end{cases}$$

PROOF. It was shown in Example 1.2.6 of Jacobsen [11] that $n\Gamma_n(t)$ is a counting process. Example 1.3.8 of Jacobsen [11] gives the conditional probability

$$G_{mt_1\cdots t_m}(t) := \mathrm{P}\,(U_{(m+1)} > t \mid U_{(1)} = t_1, \cdots, U_{(m)} = t_m) = \left(\frac{1-t}{1-t_m}\right)^{n-m} .$$

Then from Problem 1.9 of Jacobsen [11], it is easily checked that the integrated intensity of the counting process $n\Gamma_n(t)$ is $n\Lambda_t$. This completes the proof.

LEMMA 2.2. $\sup_{0\leq t\leq 1}|\Gamma_n(t)-t|\times(-\log(1-U_{(n)}))\to 0$ *in probability.*

PROOF. It is well known that $n(1-U_{(n)})$ converges in distribution to the exponential distribution and hence $-\log(1-U_{(n)})=O_p(\log n)$. Since

$$\sqrt{n}\sup_{0\leq t\leq 1}|\Gamma_n(t)-t|\to\sup_{0\leq t\leq 1}|\beta(t)|$$

in distribution, where $\{\beta(t);\ 0\leqq t\leqq 1\}$ is a Brownian bridge,

$$\sup_{0\leq t\leq 1}|\Gamma_n(t)-t|=O_p(n^{-1/2})\ .$$

Then the lemma holds.

PROOF OF THEOREM 2.1. Let $Y_n(t)=\sqrt{n}(\Gamma_n(t)-\Lambda_t)$, $0\leqq t<\infty$. Then $W_n(t)$ is the restriction of $Y_n(t)$ to $0\leqq t\leqq 1$. Now we shall show that $Y_n(t)$ converges weakly to $B(\Phi(t))$, where

$$\Phi(t)=\begin{cases} t\,, & 0\leqq t\leqq 1\,,\\ 1\,, & 1<t\,,\end{cases}$$

and $\{B(t);\ 0\leqq t<\infty\}$ is a standard Brownian motion on $D[0,\infty)$. For that, we shall check the conditions of Theorem 2.2.

First, we shall show that $Y_n(t)$ is a square integrable martingale. Lemma 2.1 and Theorem 1.5.1 of Jacobsen [11] immediately imply that $Y_n(t)$ is a martingale and that $Y_n(t)$ has quadratic variational process $\langle Y_n\rangle_t=\Lambda_t$. We show the square integrability of $Y_n(t)$. For each $t\in[0,\infty)$, we note that

$$|Y_n(t)|\leqq\sqrt{n}\left(1+\int_0^1\frac{1-\Gamma_n(s)}{1-s}\,ds\right)\leqq\sqrt{n}(1-\log(1-U_{(n)}))\ .$$

Hence we have

$$\mathrm{E}\{Y_n(t)\}^2\leqq n\int_0^1\{1-\log(1-x)\}^2nx^{n-1}dx<n^2\int_0^1\{1-\log(1-x)\}^2dx<\infty\ .$$

Secondly, we shall show that for every $t\in[0,\infty)$ and $\varepsilon>0$,

$$\mathrm{E}\{\sum_{s\leq t}(\Delta Y_n(s))^2 1_{(|\Delta Y_n(s)|>\varepsilon)}\}\to 0\qquad\text{as } n\to\infty\ . \tag{2.2}$$

From the definition of $Y_n(t)$, $Y_n(t)$ is discontinuous only at $U_{(1)}, U_{(2)},\cdots, U_{(n)}$ and the size of the jump of $Y_n(t)$ at $U_{(i)}$ is $1/\sqrt{n}$. Then for all $n\geqq[1/\varepsilon^2]+1$,

$$\sum_{s\leq t}(\Delta Y_n(s))^2 1_{(|\Delta Y_n(s)|>\varepsilon)}=0\ ,$$

which implies (2.2).

Finally, we check that for every $t\in[0,\infty)$ and $\varepsilon>0$,

$$\text{(2.3)} \qquad \mathrm{P}\left(|\langle Y_n\rangle_t - \Phi(t)| > \varepsilon\right) \to 0 \quad \text{as } n \to \infty .$$

From Lemma 2.1, we can easily see that $\langle Y_n\rangle_t = A_t$. Then, for every $t \in [0, \infty)$,

$$\langle Y_n\rangle_t - \Phi(t) = \begin{cases} \displaystyle\int_0^t \frac{s - \Gamma_n(s)}{1-s}\, ds , & \text{if } t \leqq 1 , \\ \displaystyle\int_0^1 \frac{s - \Gamma_n(s)}{1-s}\, ds , & \text{if } t > 1 . \end{cases}$$

Hence it holds that for all $t \in [0, \infty)$,

$$|\langle Y_n\rangle_t - \Phi(t)| \leqq \sup_{0 \leqq t \leqq 1} |\Gamma_n(t) - t| \times (-\log(1 - U_{(n)})) + (1 - U_{(n)}) .$$

Since it is easily seen that $1 - U_{(n)} \to 0$ in probability as $n \to \infty$, (2.3) is shown from Lemma 2.2.

Because all conditions of Theorem 2.2 have been checked, we have that $Y_n(t)$ converges to $B(\Phi(t))$ in $D[0, \infty)$. Since $W_n(t)$ is the restriction of $Y_n(t)$ to $[0, 1]$ and $W(t)$ is the restriction of $B(\Phi(t))$, it can be seen by using Theorem 3′ of Lindvall [14] that $W_n(t)$ converges weakly to $W(t)$ in $D[0, 1]$. This completes the proof of Theorem 2.1.

THEOREM 2.3. *Suppose $X_1, X_2, \cdots, X_n$ are independent and have a common distribution function $F(t)$ over $[0, 1]$. Let $F_n(t)$ be the empirical distribution function corresponding to $X_1, X_2, \cdots, X_n$. Let $Y_n(t) = \sqrt{n}\left(F_n(t) - \int_0^t \frac{1 - F_n(s)}{1 - F(s)}\, dF(s)\right)$. Then $Y_n(t)$ converges weakly in $D[0, 1]$ to $W(F(t))$, where $W(t)$ is a standard Wiener process on $D[0, 1]$.*

PROOF. The theorem can be proved similarly as Theorem 16.4 of Billingsley [4]. We define an inverse to F by $F^{-1}(s) = \inf\{t;\ s \leqq F(t)\}$. If $\eta_1, \eta_2, \cdots, \eta_n$ are independent uniformly distributed random variables over $[0, 1]$, then

$$\mathrm{P}(F^{-1}(\eta_i) \leqq t) = F(t) , \qquad i = 1, 2, \cdots, n .$$

Since the theorem states only about the distributions of the processes, we may write $X_i = F^{-1}(\eta_i)$ $i = 1, 2, \cdots, n$. Then we have that $Y_n(t) = Z_n(F(t))$, where

$$Z_n(t) = \sqrt{n}\left(\Gamma_n(t) - \int_0^t \frac{1 - \Gamma_n(s)}{1 - s}\, ds\right)$$

and $\Gamma_n(t)$ is the empirical distribution function for $\eta_1, \eta_2, \cdots, \eta_n$. Define $\psi : D[0, 1] \to D[0, 1]$ by $(\psi x)(t) = x(F(t))$. Let $x, x_1, x_2, \cdots$ be elements of $D[0, 1]$. If x_n converges to x in the Skorohod topology and $x \in C[0, 1]$, then ψx_n converges to ψx in the Skorohod topology. Therefore, Theorem

2.1 and Theorem 5.1 of Billingsley [4] imply that $Y_n(t)$ converges to $W(F(t))$, which completes the proof.

3. Linear functionals of W_n and Neyman's 'smooth' test

Let $X_1, X_2, \cdots, X_n$ be independent random variables having a common distribution function $F(t)$ over $[0, 1]$. In this section we study the testing problem whether $F(t)=t$ on $[0, 1]$. $F_n(t)$ denotes the empirical distribution function of $X_1, X_2, \cdots, X_n$. Let $W_n(t)=\sqrt{n}\left(F_n(t)-\int_0^t \frac{1-F_n(s)}{1-s}\cdot ds\right)$.

First, we consider linear functionals of $W_n(t)$ as the test statistics. The asymptotic distributions of the statistics are obtained under the null hypothesis.

THEOREM 3.1. *Suppose $X_1, X_2, \cdots, X_n$ are independent uniformly distributed random variables on $[0, 1]$. Let h be a continuously differentiable function on $[0, 1]$. Then as $n\to\infty$, $T_n(h)=\int_0^1 h(t)dW_n(t)$ converges in distribution to the normal distribution with mean zero and variance $\int_0^1 h^2(t)dt$.*

PROOF. Since h is continuously differentiable and $W_n(t)$ is of bounded variation,

$$\int_0^1 h(t)dW_n(t)=h(1)W_n(1)-h(0)W_n(0)-\int_0^1 h'(t)W_n(t)dt$$

holds from integration by parts of Stieltjes integral. Define $\psi: D[0, 1]\to R$ by $\psi(x)=h(1)x(1)-h(0)x(0)-\int_0^1 h'(t)x(t)dt$. Let x_n and x be elements of $D[0, 1]$. If x_n converges to x in the Skorohod topology and $x\in C[0, 1]$, then the convergence is uniform and hence $\psi(x_n)$ converges to $\psi(x)$. Since W has its support on $C[0, 1]$, Theorem 5.1 of Billingsley [4] and Theorem 2.2 imply that $T_n(h)$ converges in distribution to

$$h(1)W(1)-h(0)W(0)-\int_0^1 h'(t)W(t)dt\,. \tag{3.1}$$

But (3.1) is equal to $\int_0^1 h(t)dW(t)$, by using integration by parts of Wiener integral. It is well known that $\int_0^1 h(t)dW(t)$ is normally distributed with mean zero and variance $\int_0^1 h^2(t)dt$. This completes the proof.

PROPOSITION 3.1. *Suppose that a function h on $[0, 1]$ is continuous.*

Then $T_n(h)=\int_0^1 h(t)dW_n(t)$ *is written as*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(h(X_i)-H(X_i))\,, \qquad \textit{where } H(t)=\int_0^t \frac{h(s)}{1-s}ds\,.$$

PROOF. Let $X_{(1)}\leqq X_{(2)}\leqq\cdots\leqq X_{(n)}$ be the order statistics of $X_1, X_2, \cdots, X_n$. Then we have

$$\begin{aligned}\int_0^1 h(t)dW_n(t)&=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}h(X_i)-\sqrt{n}\sum_{i=1}^{n}\frac{n-i+1}{n}\int_{X_{(i-1)}}^{X_{(i)}}\frac{h(t)}{1-t}dt\\&=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}h(X_i)-\sqrt{n}\sum_{i=1}^{n}\frac{n-i+1}{n}(H(X_{(i)})-H(X_{(i-1)}))\\&=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(h(X_i)-H(X_i))\,.\end{aligned}$$

Thus the assertion is proved.

In particular, we let $h(t)=a_0+a_1t+\cdots+a_mt^m$. Then we have

$$H(t)=-\left(\sum_{j=0}^{m}a_j\right)\log(1-t)-\sum_{i=1}^{m}\frac{1}{i}\left(\sum_{j=i}^{m}a_j\right)t^i\,.$$

Set

$$g(t)=h(t)-H(t)=a_0+\sum_{i=1}^{m}\left(a_i+\frac{1}{i}\sum_{j=i}^{m}a_j\right)t^i+\left(\sum_{j=0}^{m}a_j\right)\log(1-t)\,.$$

Denote by $E_g(\theta\,;x)$ the one-parameter exponential family of probability density

$$C_g(\theta)\exp\{\theta g(x)\} \qquad \text{for } 0<x<1 \text{ and } \theta>-\frac{1}{\sum_{j=0}^{m}a_j}\,.$$

Then $T_n(h)$ is the test statistic of the uniformly most powerful unbiased test for $H\colon\theta=0$ vs. $K\colon\theta\neq 0$ in $\mathrm{E}_g(\theta\,;x)$ (cf. Lehmann [13], page 126).

Next, we shall show that Neyman's 'smooth' test statistics can be represented as functionals of $W_n(t)$. Neyman's 'smooth' test was defined in [17] by

$$T_n^N=\frac{1}{n}\sum_{j=1}^{k}\left(\sum_{i=1}^{n}\pi_j(X_i)\right)^2, \qquad \text{for some } k\,,$$

where $\pi_1, \pi_2, \cdots, \pi_k$ are orthonormal polynomials on [0, 1]. If X's are uniformly distributed over [0, 1], then T_n^N converges in distribution to the chi-square distribution with k degrees of freedom.

PROPOSITION 3.2. *Neyman's 'smooth' test statistics are written as functionals of* $W_n(t)$, *i.e.*

$$T_n^N=\sum_{j=1}^{k}\left(\int_0^1 \frac{1}{1-t}\left(\int_0^t \pi_j'(s)(1-s)ds\right)dW_n(t)+\sqrt{n}\,\pi_j(0)\right)^2 .$$

PROOF. For each $j=1, 2, \cdots, k$, we set

$$p_j^0(t)=\frac{1}{1-t}\int_0^t \pi_j'(s)(1-s)ds .$$

Then it holds

$$p_j^0(t)-\int_0^t \frac{p_j^0(s)}{1-s}ds=\pi_j(t)-\pi_j(0) .$$

Hence, by Proposition 3.1 we have, for each $j=1, 2, \cdots, k$,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\pi_j(X_i)=\int_0^1 p_j^0(t)dW_n(t)+\sqrt{n}\,\pi_j(0) .$$

Thus the proposition is now proved.

4. Supremum of absolute value of $W_n(t)$

Let $\boldsymbol{F}$ be the totality of continuous distribution functions over [0, 1]. Every continuous distribution on $(-\infty, \infty)$ can be arranged to be in $\boldsymbol{F}$ by a monotone transformation. We consider a goodness-of-fit test for distributions in $\boldsymbol{F}$, since the test statistic to be treated in this section is invariant for such a transformation.

Let $H\in\boldsymbol{F}$ be fixed as the null hypothesis. Let $X_1, X_2, \cdots, X_n$ be a random sample from a distribution $F\in\boldsymbol{F}$. F_n denotes the empirical distribution function for $X_1, X_2, \cdots, X_n$. We shall use

$$T_n=\sqrt{n}\sup_{0\le t\le 1}\left|F_n(t)-\int_0^t \frac{1-F_n(s)}{1-H(s)}dH(s)\right|$$

for testing the hypothesis $F=H$. We could, however, define

$$T_n=\sqrt{n}\sup_{0\le t\le 1}\left|F_n(H^{-1}(t))-\int_0^t \frac{1-F_n(H^{-1}(u))}{1-u}du\right| ,$$

which shows that the testing problem is equivalent to testing whether $H(X_1), H(X_2), \cdots, H(X_n)$ are uniformly distributed over $(0, 1)$ by using the corresponding test statistic. Therefore, without loss of generality, we assume that H is the uniform distribution over $(0, 1)$.

Now we shall use

$$T_n^s=\sqrt{n}\sup_{0\le t\le 1}\left|F_n(t)-\int_0^t \frac{1-F_n(s)}{1-s}ds\right|$$

as the test statistic for testing the hypothesis $F(t)=t$ on [0, 1]. The asymptotic distribution of T_n under the null hypothesis is given by the next theorem.

THEOREM 4.1. *Suppose $X_1, X_2, \cdots, X_n$ are independent uniformly distributed random variables over* $[0, 1]$. *Then T_n^s converges in distribution to* $\sup_{0\leq t\leq 1} |W(t)|$, *where* $\{W(t);\ 0\leqq t\leqq 1\}$ *is a standard Wiener process in* $D[0, 1]$.

PROOF. Under the null hypothesis, $T_n^s = \sup_{0\leq t\leq 1} |W_n(t)|$, where $W_n(t)$ was defined by (2.1). We define $\psi : D[0, 1] \to R$ by $\psi(x) = \sup_{0\leq t\leq 1} |x(t)|$. Then the functional ψ is continuous on $C[0, 1]$. Since the Wiener measure has its support in $C[0, 1]$, Theorem 2.1 and Theorem 5.1 of Billingsley [4] immediately imply the weak convergence of T_n^s, which completes the proof.

Remark 4.1. The distribution function G of $\sup_{0\leq t\leq 1} |W(t)|$ is continuous and it is known to be

$$G(u) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left\{-\frac{\pi^2(2k+1)^2}{8u^2}\right\}$$

(cf. e.g. Feller [7]). It is known also that the distribution is the limit distribution of the statistic for testing symmetry of the underlying distribution proposed by Butler [5].

Now we shall investigate the approximate Bahadur efficiency of the statistic to the Kolmogorov-Smirnov statistic. Following Bahadur [3] we briefly mention some definitions about efficiency of tests. Problems related to efficiency of tests of fit were discussed by several authors (e.g. Wieand [21], Gregory [8] and [9]).

DEFINITION 4.1. (Bahadur [3]) Let $\{P_\theta,\ \theta \in \Theta\}$ be a set of probability measures on a measurable space $(S, \mathcal{S})$. Let Θ_0 be some subset of Θ and let H be the hypothesis that $\theta \in \Theta_0$. Suppose we are given a sequence of real-valued statistics $\{T_n\}$ defined on $(S, \mathcal{S})$ based on a sample of size n. We say that $\{T_n\}$ is a standard sequence if the following three conditions are satisfied:

I. There exists a continuous probability distribution function G such that, for each $\theta \in \Theta_0$,

$$\lim_{n\to\infty} P(T_n < x) = G(x) \qquad \text{for every } x\,.$$

II. There exists a constant a, $0 < a < \infty$, such that

$$\log(1 - G(x)) = -\frac{ax^2}{2}[1 + o(1)]\,,$$

where $o(1)\to 0$ as $x\to\infty$.

III. There exists a real valued function $b(\theta)$ on $\Theta\backslash\Theta_0$, with $0<b(\theta)<\infty$, such that, for each $\theta\in\Theta\backslash\Theta_0$,

$$\lim_{n\to\infty} \mathrm{P}\left(\left|\frac{T_n}{\sqrt{n}}-b(\theta)\right|>x\right)=0 \qquad \text{for every } x .$$

For every standard sequence $\{T_n\}$, $c(\theta)=ab^2(\theta)$ is called the approximate Bahadur slope. For two standard sequences $\{T_n^{(1)}\}$ and $\{T_n^{(2)}\}$, the approximate Bahadur efficiency of $\{T_n^{(1)}\}$ to $\{T_n^{(2)}\}$ is defined by the ratio of the approximate Bahadur slopes, $c_1(\theta)/c_2(\theta)$.

In our framework, we assume that $S=[0, 1]$, $\{\mathrm{P}_\theta,\ \theta\in\Theta\}$ is a subset of $\boldsymbol{F}$, $\Theta_0=\{\theta_0\}$ and P_{θ_0} is the uniform distribution over $[0, 1]$.

Now we define two conditions on the distribution function F.

CONDITION A. Let $Y_1, Y_2,\cdots, Y_n$ be independent random variables having the common distribution function $F(1-e^{-z})$, $z>0$. Let $Y_{(n)}=\max\{Y_1,\cdots, Y_n\}$. Then there exist two sequences $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ and a distribution Δ such that, as $n\to\infty$, $\sqrt{n}\,a_n\to\infty$, $b_n/(\sqrt{n}\,a_n)\to 0$ and $a_nY_{(n)}+b_n$ converges to Δ in distribution.

CONDITION B. $\displaystyle\int_0^1 \frac{1-F(t)}{1-t}\,dt<\infty.$

Remark 4.2. If the conditions in Condition A are satisfied, then Δ must be one of three types of limiting distributions (cf. e.g. David [6]).

We can note that even when there exist two sequences $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ and a distribution Δ such that $a_nY_{(n)}+b_n\to\Delta$ in distribution, the conditions $\sqrt{n}\,a_n\to\infty$ and $b_n/\sqrt{n}\,a_n\to 0$ are not necessarily satisfied. For example, we let $F(x)=(2/\pi)\tan^{-1}(-\log(1-x))$. Suppose $X_1, X_2,\cdots, X_n$ are independent random variables having common distribution function F. Let $Y_i=-\log(1-X_i)$, $i=1, 2,\cdots, n$. Then $Y_1, Y_2,\cdots, Y_n$ are independent random variables having the common distribution function $G(y)$ $(=F(1-e^{-y}))=(2/\pi)\tan^{-1}(y)$, $y>0$. Setting $a_n=1/n$ and $b_n=0$, we have

$$\mathrm{P}(a_nY_{(n)}+b_n\leq y)=\left\{\left(1-\frac{1}{ny}ny(1-G(ny))\right)\right\}^n\to\exp\left(-\frac{2}{\pi y}\right), \qquad y>0 ,$$

But $\sqrt{n}\,a_n\to 0\neq\infty$.

Remark 4.3. Every distribution on $[0, 1]$ can not satisfy Condition B. It is clear that Condition B does not hold if F is the distribution of the point mass at 1. Even if F is a continuous distribution on $(0, 1)$, it does not necessarily satisfy Condition B. Consider

$$F(t)=\begin{cases} 1+\dfrac{1}{\log(1-t)}\,, & \text{if } 1-1/e<t<1\,, \\ 0\,, & \text{if } 0<t\leqq 1-1/e\,. \end{cases}$$

Then it does not satisfy Condition B.

LEMMA 4.1. *Condition B is equivalent to the condition*

$$\sup_{0\leqq t\leqq 1}\left|F(t)-\int_0^t \frac{1-F(s)}{1-s}\,ds\right|<\infty\,. \tag{4.1}$$

The proof is easy and so we omit it.

Now we fix a testing problem as we previously described. We denote by F_θ the distribution function over $[0, 1]$ corresponding to P_θ, $\theta\in\Theta$.

THEOREM 4.2. *Suppose that, for every $\theta\in\Theta$, F_θ satisfies Conditions A and B. Then the sequence of test statistics $\{T_n^s\}$ is a standard sequence.*

Before proving the theorem, we prove the next lemma.

LEMMA 4.2. *Suppose a distribution function F satisfies Condition A. Then $(1/\sqrt{n})(-\log(1-X_{(n)}))$ converges to zero in probability, where $X_{(n)}$ is the maximum among independent random variables $X_1, X_2, \cdots, X_n$ having the common distribution function F.*

PROOF. Set $Y_i=-\log(1-X_i)$ for each $i=1, 2, \cdots, n$. Then $Y_1, Y_2, \cdots, Y_n$ are independent random variables having the common distribution function $F(1-e^{-y})$. If we define $Y_{(n)}=\max\{Y_1, Y_2, \cdots, Y_n\}$, then we have $Y_{(n)}=-\log(1-X_{(n)})$. From Condition A, there exist two sequences $\{a_n\}$, $\{b_n\}$ and a distribution function Δ such that $\sqrt{n}\,a_n\to\infty$, $b_n/\sqrt{n}\,a_n\to 0$ and $a_nY_{(n)}+b_n\to\Delta$ in distribution. Then the result is obvious.

PROOF OF THEOREM 4.2. We shall check the conditions in Definition 4.1.

From Theorem 4.1 and Remark 4.1 we can see that $\{T_n^s\}$ satisfies Condition I in Definition 4.1. We denote by G the distribution function of $\sup\limits_{0\leqq t\leqq 1}|W(t)|$.

Next we show that G satisfies Condition II in Definition 4.1, i.e. there exists a constant a, $0<a<\infty$, such that

$$\log(1-G(x))=-\frac{ax^2}{2}[1+o(1)]\,,$$

where $o(1)\to 0$ as $x\to\infty$.

Noting that

$$1-G(z)=\mathrm{P}(\sup_{0\le t\le 1}|W(t)|\geqq z)=\mathrm{P}(\{\sup_{0\le t\le 1}W(t)\geqq z\}\cup\{\sup_{0\le t\le 1}(-W(t))\geqq z\})\,.$$

We have

$$\begin{aligned}\mathrm{P}(\sup_{0\le t\le 1}W(t)\geqq z)&\leqq\mathrm{P}(\{\sup_{0\le t\le 1}|W(t)|\geqq z\})\\&\leqq\mathrm{P}(\{\sup_{0\le t\le 1}W(t)\geqq z\})+\mathrm{P}(\{\sup_{0\le t\le 1}(-W(t))\geqq z\})\,.\end{aligned}$$

Since Brownian motion is symmetrically distributed, we see that

$$\mathrm{P}(\{\sup_{0\le t\le 1}(-W(t))\geqq z\})=\mathrm{P}(\{\sup_{0\le t\le 1}W(t)\geqq z\})$$

and hence we have

$$M(z)\leqq 1-G(z)\leqq 2M(z)\,,$$

where $M(z)=\mathrm{P}(\{\sup_{0\le t\le 1}W(t)\geqq z\})$.

By Proposition 2.9 of Hida [10], it holds

$$M(z)=\frac{2}{\sqrt{2\pi}}\int_z^\infty\exp\left[-\frac{x^2}{2}\right]dx\,.$$

Here we note that, if $w>0$, the next inequalities hold

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\left\{\frac{1}{w}-\frac{1}{w^3}\right\}\exp\left[-\frac{w^2}{2}\right]&<\frac{1}{\sqrt{2\pi}}\int_w^\infty\exp\left[-\frac{x^2}{2}\right]dx\\&<\frac{1}{\sqrt{2\pi}}\,\frac{1}{w}\exp\left[-\frac{w^2}{2}\right]\,.\end{aligned}$$

Thus we have

$$\frac{2}{\sqrt{2\pi}}\left\{-\frac{1}{z}-\frac{1}{z^3}\right\}\exp\left[-\frac{z^2}{2}\right]\leqq 1-G(z)\leqq\frac{4}{\sqrt{2\pi}}\,\frac{1}{z}\exp\left[-\frac{z^2}{2}\right]\,.$$

Hence,

$$\log(1-G(x))=-\frac{x^2}{2}(1+o(1))\qquad\text{as } x\to\infty\,.$$

Therefore Condition II in Definition 4.1 is now satisfied for $a=1$.

Finally we shall check Condition III in Definition 4.1, i.e. there exists a real valued function $b(\theta)$ on $\Theta\backslash\Theta_0$, with $0<b(\theta)<\infty$, such that, for each $\theta\in\Theta\backslash\Theta_0$,

$$\lim_{n\to\infty}\mathrm{P}\left(\left|\frac{T_n^s}{\sqrt{n}}-b(\theta)\right|>x\right)=0\qquad\text{for every } x\,.$$

Let

$$b(\theta)=\sup_{0\le t\le 1}\left|F_\theta(t)-\int_0^t\frac{1-F_\theta(s)}{1-s}ds\right|\,.$$

We shall note that, if $\theta\in\Theta\backslash\Theta_0$, then $0<b(\theta)<\infty$. The definition $b(\theta)$ and Lemma 4.1 imply that $0\leqq b(\theta)<\infty$. We show that if $b(\theta)=0$, then $\theta\in\Theta_0$. Suppose that $b(\theta)=0$ for some $\theta\in\Theta$. Then we can write

$$F_\theta(t)=\int_0^t \frac{1-F_\theta(s)}{1-s}ds\,, \qquad 0\leqq t\leqq 1\,.$$

Since F_θ is continuous on $[0,1]$, $F_\theta(t)$ is continuously differentiable on $(0,1)$ and hence $F_\theta(t)$ satisfies the differential equation

$$\frac{d}{dt}(F_\theta(t))+\frac{F_\theta(t)}{1-t}=\frac{1}{1-t}\,, \qquad F_\theta(0)=0\,.$$

Then we have $F_\theta(t)=t$ and hence $\theta=\theta_0$. Now we shall prove that $T_n^s/\sqrt{n}$ converges to $b(\theta)$ in probability for every $\theta\in\Theta\backslash\Theta_0$. The triangle inequality implies that

$$\left|\frac{T_n^s}{\sqrt{n}}-b(\theta)\right|\leqq \sup_{0\leq t\leq 1}|F_n(t)-F_\theta(t)|+\int_0^1\frac{|F_n(s)-F_\theta(s)|}{1-s}ds\,.$$

We can easily see that

$$\int_0^1\frac{|F_n(s)-F_\theta(s)|}{1-s}ds\leqq \sup_{0\leq t\leq 1}|F_n(t)-F_\theta(t)|\times(-\log(1-X_{(n)}))$$
$$+\int_{X_{(n)}}^1\frac{1-F_\theta(s)}{1-s}ds\,.$$

Since $X_{(n)}$ converges to $\inf\{t;\ F_\theta(t)=1\}$ in probability, $\int_{X_{(n)}}^1\frac{1-F_\theta(s)}{1-s}ds$ converges in probability to zero by Condition B. Then Glivenko-Cantelli's theorem and Lemma 4.2 imply that $|T_n^s/\sqrt{n}-b(\theta)|$ converges to zero in probability.

Therefore all conditions in Definition 4.1 are checked and this completes the proof.

The following result is an immediate consequence of Theorem 4.2.

COROLLARY. *If F_θ satisfies Conditions A and B for every $\theta\in\Theta$, then the approximate Bahadur slope of T_n^s is given by*

$$c^s(\theta)=\left\{\sup_{0\leq t\leq 1}\left|F_\theta(t)-\int_0^t\frac{1-F_\theta(s)}{1-s}ds\right|\right\}^2\,.$$

Remark 4.4. The approximate Bahadur efficiencies of T_n^s to other well-known test statistics immediately can be given by using Table 1 of Wieand [21].

Now we consider a concrete example of the testing problem in which we can calculate the approximate Bahadur slope of T_n^s numerically.

Example 4.1. Let $\boldsymbol{\Theta}=(0, \infty)$. Define, for every $\theta \in \boldsymbol{\Theta}$, $F_\theta(t)=t^\theta$, $0 \leqq t \leqq 1$. We test the hypothesis that $\theta=1$ by the test statistic T_n^s. Then the following propositions show that the test statistic T_n^s is asymptotically rather good for $\theta \geqq 1$.

PROPOSITION 4.1. *In the problem of Example* 4.1, $\{T_n^s\}$ *is a standard sequence.*

PROOF. From Theorem 4.2, it suffices that we check that Conditions A and B hold for Example 4.1. Since for every $\theta>0$,

$$\int_0^1 \frac{1-s^\theta}{1-s} ds = \gamma + \psi(\theta+1) < \infty ,$$

Condition B is satisfied, where γ is Euler's constant and $\psi(z)$ is the digamma function. It is easy to see that Condition A is satisfied with $a_n=1$ and $b_n=-\log(n\theta)$, which completes the proof.

PROPOSITION 4.2. *In Example* 4.1, *the approximate Bahadur slope for* T_n^s *is given by*

$$c^s(\theta) = (\psi(\theta+1)+\gamma-1)^2 .$$

PROOF. From Proposition 4.1, $\{T_n^s\}$ in Example 4.1 is a standard sequence. Then the corollary of Theorem 4.2 shows that the approximate Bahadur slope is given by

$$c^s(\theta) = \{\sup_{0 \leq t \leq 1} |\varphi_\theta(t)|\}^2 , \qquad \text{where } \varphi_\theta(t) = t^\theta - \int_0^t \frac{1-s^\theta}{1-s} ds ,$$

It is not so difficult to see that

$$\sup_{0 \leq t \leq 1} |\varphi_\theta(t)| = \varphi_\theta(1) .$$

Then the desired result is now proved.

PROPOSITION 4.3. *In the setup of Example* 4.1 *we consider the Kolmogorov-Smirnov statistic* $T_n^{KS} = \sqrt{n} \sup_{0 \leq t \leq 1} |F_n(t)-t|$ *as another test statistic. Then the approximate Bahadur efficiency of* $\{T_n^s\}$ *to* $\{T_n^{KS}\}$ *is written as*

$$\frac{(\psi(\theta+1)+\gamma-1)^2}{4\left(\left(\frac{1}{\theta}\right)^{\theta/(\theta-1)}(1-\theta)\right)^2} .$$

PROOF. The approximate Bahadur slope of the Kolmogorov-Smirnov statistic was given by Bahadur [3] as

$$c^{KS}(\theta)=4\sup_{0\leq t\leq 1}|F_\theta(t)-t| .$$

For $F_\theta(t)=t^\theta$, it is easily seen that

$$c^{KS}(\theta)=4\left\{\left(\frac{1}{\theta}\right)^{\theta/(\theta-1)}(1-\theta)\right\}^2 .$$

Then the result is proved from Proposition 4.2.

We denote by $E_1(\theta)$ the approximate Bahadur efficiency of $\{T_n^s\}$ to $\{T_n^{KS}\}$ at θ. The values of $E_1(\theta)$ for some θ are given in Table 1. The table shows that, if $\theta>2$, then the approximate Bahadur efficiency of $\{T_n^s\}$ to $\{T_n^{KS}\}$ at θ is greater than 1. We, however, are also interested in the value of $E_1(\theta)$ when $\theta\to 1$. The next result can be proved from Proposition 4.3.

PROPOSITION 4.4. $\lim_{\theta\to 1} E_1(\theta)=\frac{e^2}{4}\left(\frac{\pi^2}{6}-1\right)^2$.

Now we shall give the exact distribution of T_n^s *under the null hypothesis.*

THEOREM 4.3. *Suppose* $X_1, X_2, \cdots, X_n$ *are independent uniformly distributed random variables over* $[0, 1]$. *Then the exact distribution function of* T_n^s *is represented as*

$$P(T_n^s\leq x)$$
$$=\begin{cases} \displaystyle\int_{0\vee(1-\sqrt{n}x)}^{\sqrt{n}x} dt_1 \int_{t_1\vee(2-\sqrt{n}x)}^{1+\sqrt{n}x} dt_2 \cdots \int_{t_{n-1}\vee(n-\sqrt{n}x)}^{n-1+\sqrt{n}x} \exp(-t_n)dt_n , & \text{if } x>1/(2\sqrt{n}) , \\ 0 , & \text{if } x\leq 1/(2\sqrt{n}) , \end{cases}$$

where $a\vee b$ *denotes the maximum of* a *and* b.

PROOF. $T_n^s/\sqrt{n}$ is written as $\sup_{0\leq t\leq X_{(n)}}(1/n)|nF_n(t)-nA_t|$. Since $nF_n(t)$ is a counting process with integrated intensity nA_t by Lemma 2.1, $P_0(s)=nF_n(A_s^{-1})$ is a Poisson process with intensity 1 (cf. e.g. Proposition 1.4.6 of Jacobsen [11]). Hence we have

$$\frac{T_n^s}{\sqrt{n}}=\sup_{0\leq s\leq \tau_n}\frac{1}{n}|P_0(s)-s| ,$$

where τ_n is the nth jump time of the Poisson process P_0. Let $\boldsymbol{R}$ be the domain between two lines $y=t\pm nd$. Then,

$$\left\{\frac{T_n^s}{\sqrt{n}}\leq d\right\}=\{(\tau_1, 0)\in \boldsymbol{R}, (\tau_1, 1)\in \boldsymbol{R}, (\tau_2, 1)\in \boldsymbol{R}, (\tau_2, 2)\in \boldsymbol{R},$$
$$\cdots, (\tau_n, n, -1)\in \boldsymbol{R} \text{ and } (\tau_n, n)\in \boldsymbol{R}\} .$$

Since P_0 is a Poisson process with intensity 1, the density function of τ_1 and the conditional density function of τ_{r+1} given $\tau_1, \cdots, \tau_r$ are written, respectively, as

$$p(t_1)=\exp(-t_1) \qquad \text{and} \qquad p(t_{r+1}|\tau_1, \cdots, \tau_r)=\exp(-(t_{r+1}-\tau_r)) .$$

Note that, if $d \leqq 1/(2n)$, then either $(\tau_1, 0) \notin \boldsymbol{R}$ or $(\tau_1, 1) \notin \boldsymbol{R}$ holds and hence $P(T_n^s \leqq d)=0$. Then we assume that $d>1/(2n)$. Suppose that $(\tau_i, i-1) \in \boldsymbol{R}$ and $(\tau_i, i) \in \boldsymbol{R}$ for every $1, 2, \cdots, r$. Then $(\tau_{r+1}, r) \in \boldsymbol{R}$ and $(\tau_{r+1}, r+1) \in \boldsymbol{R}$ hold if and only if τ_{r+1} exists in the interval $A_r=(\tau_r \vee (r-nd+1), r+nd)$. Therefore we have, if $d>1/(2n)$,

$$P\left(\frac{T_n^s}{\sqrt{n}} \leqq d\right)=\int_{A_0} dt_1 p(t_1) \int_{A_1} dt_2 p(t_2|t_1) \int_{A_2} dt_3 p(t_3|t_1, t_2) \cdots \int_{A_{n-1}} dt_n p(t_n|t_1, \cdots, t_{n-1}) .$$

This proves the theorem.

5. Cramér-von Mises type statistic of the martingale term of the empirical process

In this section we also consider the same testing problem as in Section 4, i.e. testing the hypothesis that the observations are uniformly distributed over $[0, 1]$. Let $\{F_\theta;\ \theta \in \Theta\}$ be a subset of $\boldsymbol{F}$. We assume that there exists $\theta_0 \in \Theta$ such that F_{θ_0} is the uniform distribution over $[0, 1]$. Suppose that $X_1, X_2, \cdots, X_n$ are independent random variables having a common distribution function F_θ. We denote by F_n the empirical distribution function correponding to $X_1, X_2, \cdots, X_n$. Let

$$W_n(t)=\sqrt{n}\left(F_n(t)-\int_0^t \frac{1-F_n(s)}{1-s} ds\right) .$$

We shall consider testing hypothesis that $\theta=\theta_0$ by the test statistic

$$T_n^Q=\int_0^1 W_n^2(t)dt .$$

We can also consider as a test statistic

$$\int_0^1 W_n^2(t)\psi(t)dt \qquad \text{for some weight function } \psi .$$

THEOREM 5.1. *Suppose that a function* ψ *on* $[0, 1]$ *is integrable. Then* $\int_0^1 \psi(t)W_n^2(t)dt$ *converges to* $\int_0^1 \psi(t)W^2(t)dt$ *in distribution.*

PROOF. Define $\Phi: D[0, 1] \to R$ by $\Phi(x)=\int_0^1 \psi(t)(x(t))^2 dt$. If x_n converges to x in the Skorohod topology in $D[0, 1]$ and $x \in C[0, 1]$, then the convergence is uniform and hence $\Phi(x_n)$ converges to $\Phi(x)$ since the next inequality holds,

$$|\Phi(x_n)-\Phi(x)|$$
$$\leqq (\sup_{0\leqq t\leqq 1} |x_n(t)-x(t)|)(\sup_{0\leqq t\leqq 1} |x_n(t)-x(t)|+2\sup_{0\leqq t\leqq 1} |x(t)|)\times\int_0^1 |\psi(t)|dt\ .$$

Thus Φ is continuous on $C[0, 1]$. Then Theorem 5.1 of Billingsley [4] and Theorem 2.2 imply the desired result. This completes the proof.

Remark 5.1. The distribution function G of $\int_0^1 W^2(t)dt$ was given by Rothman and Woodroofe [20] as

$$G(x)=2^{3/2}\sum_{j=0}^{\infty}\binom{-1/2}{j}\left\{1-\Phi\left(\frac{4j+1}{2\sqrt{x}}\right)\right\}, \qquad x>0,$$

where Φ denotes the standard normal distribution function. Further, MacNeill [16] obtained the characteristic function of $\int_0^1 \psi(t)W^2(t)dt$ and he tabulated selected percentage points for the random variable for some weight functions. They investigated the distributions as limit distributions of some test statistics which are different from our statistics.

Now we shall calculate the approximate Bahadur efficiency of T_n^Q to the Cramér-von Mises statistic.

THEOREM 5.2. *Suppose that, for every $\theta \in \Theta$, F_θ satisfies Conditions A and B. Then the sequence of test statistics $\{(T_n^Q)^{1/2}\}$ is a standard sequence.*

PROOF. From Theorem 5.1, T_n^Q converges to $\int_0^1 W^2(t)dt$ in distribution. Then $(T_n^Q)^{1/2}$ converges to $\left\{\int_0^1 W^2(t)dt\right\}^{1/2}$ and Condition I of Definition 4.1 holds. It is well known that $\eta:=\int_0^1 W^2(t)dt$ can be written as $\sum_{i=1}^{\infty}\lambda_i\xi_i^2$, where $\lambda_i=\dfrac{4}{((2i-1)\pi)^2}$, $i=1, 2, \cdots$, and $\xi_1, \xi_2, \cdots$ is a sequence of i.i.d. random variables with standard normal distribution (cf. e.g. Rothman and Woodroofe [20]). Then, from the result obtained by Zolotarev [20], we have

$$\log P(\sqrt{\eta}>x)=\log P(\eta>x^2)=-\frac{x^2}{2\lambda_1}(1+o(1))\ ,$$

where $o(1)\to 0$ as $x\to\infty$.

Thus Condition II in Definition 4.1 is satisfied for $a=1/\lambda_1=\pi^2/4$. Now we shall check Condition III in Definition 4.1. Let

$$b(\theta)=\left\{\int_0^1\left(F_\theta(t)-\int_0^t\frac{1-F_\theta(s)}{1-s}ds\right)^2dt\right\}^{1/2}.$$

Similarly as in the proof of Theorem 4.2, we have that, if $\theta\in\Theta_0$, then $0<b(\theta)<\infty$. We prove that T_n^Q/n converges to $b^2(\theta)$ in probability for every $\theta\in\Theta\backslash\Theta_0$. It is easily seen that

$$\left|\int_0^1\left\{\left(F_n(t)-\int_0^t\frac{1-F_n(s)}{1-s}ds\right)^2-\left(F_\theta(t)-\int_0^1\frac{1-F_\theta(s)}{1-s}ds\right)^2\right\}dt\right| \tag{5.1}$$

$$\leqq\left(\sup_{0\leq t\leq 1}|F_n(t)-F_\theta(t)|+\sup_{0\leq t\leq 1}\left|\int_0^t\frac{F_n(s)-F_\theta(s)}{1-s}ds\right|\right)$$
$$\times\left\{2+\int_0^1\frac{1-F_n(s)}{1-s}ds+\int_0^1\left(\int_0^t\frac{1-F_\theta(s)}{1-s}ds\right)dt\right\}.$$

Noting that

$$\int_0^1\left(\int_0^t\frac{1-F_\theta(s)}{1-s}ds\right)dt\leqq\int_0^1\frac{1-F_\theta(s)}{1-s}ds\,,$$

we have that (5.1) is not greater than

$$\left(\sup_{0\leq t\leq 1}|F_n(t)-F_\theta(t)|+\sup_{0\leq t\leq 1}\left|\int_0^t\frac{F_n(s)-F_\theta(s)}{1-s}ds\right|\right)$$
$$\times\left(2+2\int_0^1\frac{1-F_\theta(s)}{1-s}ds+\left|\int_0^1\frac{F_\theta(s)-F_n(s)}{1-s}ds\right|\right).$$

It is easily seen that

$$\int_0^1\frac{|F_n(s)-F_\theta(s)|}{1-s}ds\leqq\sup_{0\leq t\leq 1}|F_n(t)-F_\theta(t)|\times(-\log(1-X_{(n)}))$$
$$+\int_{X_{(n)}}^1\frac{1-F_\theta(s)}{1-s}ds\,.$$

By Lemma 4.2, we have

$$\sup_{0\leq t\leq 1}|F_n(t)-F_\theta(t)|\times(-\log(1-X_{(n)}))\to 0\qquad\text{in probability}\,.$$

Since $X_{(n)}$ converges to $\inf\{t;\ F_\theta(t)=1\}$ in probability,

$$\int_{X_{(n)}}^1\frac{1-F_\theta(s)}{1-s}ds$$

converges in probability to zero by Condition B.

Consequently, (5.1) converges to zero in probability and this completes the proof.

COROLLARY. *If F_θ satisfies Conditions A and B for every $\theta \in \Theta$, then the approximate Bahadur slope of $(T_n^Q)^{1/2}$ is written as*

$$c^Q(\theta)=\frac{\pi^2}{4}\int_0^1\left(F_\theta(t)-\int_0^t\frac{1-F_\theta(s)}{1-s}ds\right)^2 dt .$$

Now we calculate the approximate Bahadur slope in the setup described in Example 4.1.

Example 5.1. We consider the same situation as Example 4.1, i.e. $\Theta=(0, \infty)$ and $F_\theta(t)=t^\theta$, $0\leqq t\leqq 1$ for every $\theta\in\Theta$. We take $(T_n^Q)^{1/2}$ as a test statistic. From the corollary of Theorem 5.2, the approximate Bahadur slope of $(T_n^Q)^{1/2}$ is given by

$$c^Q(\theta)=\frac{\pi^2}{4}\int_0^1\left(t^\theta-\int_0^t\frac{1-s^\theta}{1-s}ds\right)^2 dt .$$

If θ is equal to a positive integer m, we can calculate it further. By integration by parts, we can easily obtain that

$$\begin{aligned}(5.2)\quad &\int_0^1\left(t^m-\int_0^t\frac{1-s^m}{1-s}ds\right)^2 dt\\ &=\left(\sum_{i=2}^m\frac{1}{i}\right)^2+\left(\sum_{i=2}^m\frac{1}{i}\right)\left(\frac{2(m-1)}{m+1}-2\sum_{i=2}^m\frac{1}{i}\right)+\frac{2m^2}{(m+1)(2m+1)}\\ &-2\sum_{i=2}^{m+1}\frac{1}{i}+2\sum_{i=m+2}^{2m+1}\frac{1}{i}-\frac{2m}{m+1}\left(\sum_{i=m+2}^{2m+1}\frac{1}{i}\right)\\ &+2\sum_{i=1}^m\frac{1}{i+1}\left(\sum_{j=i+2}^{m+i+1}\frac{1}{j}\right).\end{aligned}$$

If we consider in this example the Cramér-von Mises statistic

$$T_n^{CM}=\int_0^1 n(F_n(t)-t)^2 dt$$

as another test statistic, the approximate Bahadur slope $c^{CM}(\theta)$ of $(T_n^{CM})^{1/2}$ is equal to $\dfrac{2\pi^2(\theta-1)^2}{3(\theta+2)(2\theta+1)}$, since it is generally written as $\pi^2\int_0^1(F_\theta(t)-t)^2dt$ (cf. e.g. Wieand [21]).

Values listed in Table 1 are as follows:
$c^s(\theta)$, $c^{KS}(\theta)$, $c^Q(\theta)$ and $c^{CM}(\theta)$ are the approximate Bahadur slopes of T_n^s, T_n^{KS}, T_n^Q and T_n^{CM}, respectively,

$E_1(\theta)=\dfrac{c^s(\theta)}{c^{KS}(\theta)}$ is the approximate Bahadur efficiency of T_n^s to T_n^{KS},

and

$E_2(\theta)=\dfrac{c^Q(\theta)}{c^{CM}(\theta)}$ is the approximate Bahadur efficiency of T_n^Q to T_n^{CM}.

Table 1.

θ	$c^s(\theta)$	$c^{KS}(\theta)$	$c^Q(\theta)$	$c^{CM}(\theta)$	$E_1(\theta)$	$E_2(\theta)$
2	0.2500	0.2500	0.3290	0.3290	1.0000	1.0000
4	1.1736	0.8929	1.1867	1.0966	1.3144	1.0821
6	2.1025	1.3566	1.7933	1.5817	1.5499	1.1338
8	2.9510	1.6906	2.2208	1.8965	1.7455	1.1710
10	3.7209	1.9423	2.5369	2.1149	1.9157	1.1995
12	4.4235	2.1393	2.7807	2.2747	2.0677	1.2224
14	5.0695	2.2981	2.9748	2.3965	2.2060	1.2413
16	5.6679	2.4291	3.1335	2.4923	2.3333	1.2573
18	6.2256	2.5394	3.2659	2.5697	2.4516	1.2709
20	6.7483	2.6336	3.3782	2.6334	2.5623	1.2828

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