Ann. Inst. Statist. Math.
37 (1985), Part A, 557-566

SOME E AND MV-OPTIMAL DESIGNS FOR THE TWO-WAY ELIMINATION OF HETEROGENEITY

MIKE JACROUX

(Received Apr. 26, 1984; revised Feb. 21, 1985)

Summary

It is well known that in experimental settings where v treatments are being tested in b blocks of size k, a group divisible design having parameters $\lambda_2=\lambda_1+1$ and whose corresponding C-matrix has maximal trace is both E and MV-optimal among all possible competing designs. In this paper, we show that under certain conditions, the E and MV-optimal group divisible block designs mentioned in the previous sentence can be used to construct E and MV-optimal row-column designs to handle experimental situations in which heterogeneity is to be eliminated in two directions and where v treatments are being tested in b columns and k rows. Examples are given to illustrate how the results obtained can be applied.

1. Introduction

Block designs are used for experiments where it is important to eliminate heterogeneity in one direction. However, in many experimental situations, the position that an experimental unit occupies within a block can also affect observed responses. When this happens, row-column designs can often be used to eliminate heterogeneity in two directions. The row-column designs considered here have bk experimental units arranged in a rectangular array of b columns and k rows such that each unit receives only one of the v treatments being studied. This paper gives some optimal row-column designs which can be constructed from some well known optimal block designs.

Let d denote an arbitrary row-column design such as described above. Observations y_{mnp} obtained after applying the mth treatment to the unit occurring in the nth column and pth row are assumed to follow the usual three-way additive model, i.e.

Key words: Row-column design, E-optimality, MV-optimality, group divisible design.
Research sponsored in part by National Science Foundation Grant No. DMS-8401943.

$$y_{mnp}=\alpha_m+\beta_n+\gamma_p+e_{mnp}\ , \qquad 1\leq m\leq v,\ 1\leq n\leq b,\ 1\leq p\leq k\ ,$$

where α_m=the effect of the mth treatment, β_n=the nth column effect, γ_p=the pth row effect and e_{mnp} is a random variable with zero expectation. All observations are assumed to be uncorrelated and have the same variance σ^2. $M_d=(m_{dij})$ and $N_d=(n_{dij})$ are used to denote the treatment-row and treatment-column incidence matrices, i.e., m_{dij} gives the number of times treatment i occurs in row j and n_{dij} gives the number of times treatment i occurs in the jth column. The coefficient matrix of the reduced normal equations for estimating the treatment effects in d can be written as

$$C_d=\text{diag}\,(r_{d1},\cdots,r_{dv})-(1/k)N_dN_d'-(1/b)M_dM_d'+(1/bk)R_dR_d' \tag{1.1}$$

where A' denotes the transpose of a matrix A, r_{di} represents the number of times treatment i is replicated in d, $R_d=(r_{d1},\cdots,r_{dv})'$, and diag $(r_{d1},\cdots,r_{dv})$ is a $v\times v$ diagonal matrix. C_d, the C-matrix of the design, is known to be positive semi-definite with zero row sums.

In a row-column design a linear combination $\sum_{i=1}^{v} c_i\alpha_i$ of the treatment effects is said to be estimable provided there exists a linear combination $\sum_i\sum_j\sum_k a_{ijk}y_{ijk}$ of the observations such that $\text{E}\,(\sum_i\sum_j\sum_k a_{ijk}y_{ijk})=\sum_{i=1}^{v} c_i\alpha_i$. A treatment contrast is any linear combination $\sum_{i=1}^{v} c_i\alpha_i$ of the treatment effects where $\sum_{i=1}^{v} c_i=0$ and it is well known that an estimable function of the treatment effects in a row-column design must be a contrast. Any row-column design in which all treatment contrasts are estimable is said to be treatment connected, e.g. see Eccleston and Russell [6]. Alternatively, a row-column design is treatment connected if and only if its C-matrix has rank $v-1$. Henceforth, $D(v, b, k)$ denotes the class of all treatment connected designs having v treatments arranged in b columns and k rows.

With each row-column design $d\in D(v, b, k)$ we associate the block designs d_N and d_M with incidence matrices N_d and M_d, respectively, i.e. d_N is that block design which can be obtained from d by treating the columns of d as blocks and ignoring the row effects whereas d_M is that block design which can be obtained from d by treating the rows of d as blocks and ignoring the column effects. When all the entries of N_d are 0 or 1, we say d_N is a binary design. N_dN_d' is called the concurrence matrix of d_N and its entries are denoted by λ_{dij}^N. If d_N has N_dN_d' with all of its diagonal entries equal to one value and all of its off-diagonal entries equal to another value, then d_N is called a balanced block design (BBD). A binary BBD where $v>k$ is called a balanced incomplete block design (BIBD). We call d_N a group divisible (GD) design with two associate classes and parameters p^N, q^N, λ_1^N, and λ_2^N if the treatments of d_N can be partitioned into p^N disjoint sets of size q^N such that if treatments i and j occur in the same group, $\lambda_{dij}^N=\lambda_1^N$, whereas if treatments i and j occur in different groups, $\lambda_{dij}^N=\lambda_2^N$. If d_N has all of its treatments replicated bk/v times, then the dual of d_N, denoted by $\tilde{d}_N$, is the design whose incidence matrix is $N_{\tilde{d}_N}=N_d'$. While the definitions given in this paragraph have been presented in terms of d_N, parallel definitions do exist of course in terms of d_M. The coefficient matrices of the reduced normal equations for estimating the treatment effects in d_N and d_M are, under the appropriate two-way model,

$$C_d^N=\operatorname{diag}(r_{d1},\cdots,r_{dv})-(1/k)N_dN_d' \quad \text{and} \quad C_d^M=\operatorname{diag}(r_{d1},\cdots,r_{dv})-(1/b)M_dM_d' . \tag{1.2}$$

These matrices are called the C-matrices of d_N and d_M, respectively, and possess the same properties as C_d. $D_N(v, b, k)$ ($D_M(v, b, k)$) will henceforth denote the class of all $d_N(d_M)$ corresponding to $d \in D(v, b, k)$ and consists of all connected block designs having v treatments arranged in $b(k)$ blocks of size $k(b)$. For each $d_N \in D_N(v, b, k)$ ($d_M \in D_M(v, b, k)$) there are many corresponding designs in $D(v, b, k)$ as there are many orders in which treatments can occur in blocks. Using $[\cdot]$ to denote the greatest integer function and $\operatorname{tr} A$ to denote the trace of a matrix A, we also employ the following notation throughout the sequel:

For $x>0$ a positive number,

$$\begin{aligned} &R^N(x)=(x-b[x/v])([x/v]+1)^2+(b-x+b[x/v])([x/v])^2\\ &R^M(x)=(x-k[x/v])([x/v]+1)^2+(b-x+k[x/v])([x/v])^2\\ &r=[bk/v]\\ &\lambda^N=[(rk-R^N(r))/(v-1)]\\ &\lambda^M=[(rk-R^M(r))/(v-1)] \end{aligned} \tag{1.3}$$

$$\bar{D}_N(v, b, k)=\{d_N \in D_N(v, b, k) \text{ such that } \operatorname{tr} C_d^N \text{ is maximal}\}$$

$$\bar{D}_M(v, b, k)=\{d_M \in D_M(v, b, k) \text{ such that } \operatorname{tr} C_d^M \text{ is maximal}\} .$$

With regard to $\bar{D}_N(v, b, k)$ and $\bar{D}_M(v, b, k)$, it follows from the results of Jacroux and Seely [13] that

$$\bar{D}_N(v, b, k)=\{d_N \in D_N(v, b, k) \text{ such that } n_{dij}=[k/v] \text{ or } [k/v]+1 \text{ for all } i, j\}$$

$$\bar{D}_M(v, b, k) = \{d_M \in D_M(v, b, k) \text{ such that } m_{dij} = [b/v] \text{ or } [b/v]+1 \text{ for all } i, j\} .$$

For $d \in D(v, b, k)$, let $z_{d0}=0<z_{d1}\leqq z_{d2}\leqq\cdots\leqq z_{d,v-1}$ denote the eigenvalues of C_d, $z^N_{d0}=0<z^N_{d1}\leqq\cdots\leqq z^N_{d,v-1}$ the eigenvalues of C^N_d, and $z^M_{d0}=0<z^M_{d1}\leqq\cdots\leqq z^M_{d,v-1}$ the eigenvalues of C^M_d. Following Cheng [1] or Magda [15], we can write

$$\begin{aligned}(1.4)\qquad C_d &= C^N_d-(1/b)M_d(I_k-(1/k)J_{kk})M'_d \\ &= C^M_d-(1/k)N_d(I_b-(1/b)J_{bb})N'_d \\ &= C^N_d+C^M_d-\operatorname{diag}(r_{d1},\cdots, r_{dv})+(1/bk)R_dR'_d\end{aligned}$$

where I_n denotes the $n\times n$ identity matrix and J_{mn} the $m\times n$ matrix of ones. Since $M_d(I_k-(1/k)J_{kk})M'_d$ and $N_d(I_b-(1/b)J_{bb})N'_d$ are positive semi-definite,

$$(1.5)\quad z_{di}\leqq z^N_{di} \text{ for } i=1,\cdots, v-1 \quad\text{and}\quad z_{di}\leqq z^M_{di} \text{ for } i=1,\cdots, v-1 .$$

The optimality criteria considered here for selecting optimal designs in $D(v, b, k)$ are the E and MV-optimality criteria. The E-optimality criterion was introduced by Ehrenfeld [7]. This criterion chooses those designs $d \in D(v, b, k)$ such that z_{d1} is maximal and is equivalent to finding those designs which minimize the maximum variance among all least squares estimates obtained for treatment contrasts of the form $\sum_{i=1}^{v} c_i\alpha_i$ where $\sum_{i=1}^{v} c_i^2=1$. The E-optimality criterion is appropriate to use in experimental settings where it is desired to estimate all treatment contrasts with as much precision as possible. However, in many experimental situations, the primary interest of the experimenter is not to optimally estimate arbitrary treatment contrasts, but rather to optimally estimate the differences in the effects that the treatments under study have on the various experimental units. In terms of estimable functions, this leads to the need to be able to estimate all treatment contrasts of the form $\alpha_i-\alpha_j$ with as much precision as possible. Contrasts of the form $\alpha_i-\alpha_j$ are called elementary treatment contrasts or elementary treatment differences. We shall call $d \in D(v, b, k)$ a minimum variance design and say d is MV-optimal in $D(v, b, k)$ if the maximal variance with which it estimates elementary treatment differences $\alpha_i-\alpha_j$ is minimal among all designs in $D(v, b, k)$.

A number of results are already known concerning the E and MV-optimality of designs in classes like $D_N(v, b, k)$, e.g., see Takeuchi [14], [20], Cheng [2], Constantine [3], [4] and Jacroux [10]–[12]. Here we consider the problem of determining E and MV-optimal designs in classes $D(v, b, k)$. The only results known to the author concerning this problem are those which can be obtained from theorems proven in Kiefer

[14], Cheng [1] and Jacroux [12]. A generalized Youden design (GYD) $d \in D(v, b, k)$ is a row-column design in which $d_N \in \bar{D}_N(v, b, k)$ and $d_M \in \bar{D}_M(v, b, k)$ are both BBD's. Using the results of Kiefer [14], it is easily seen that a GYD is both E and MV-optimal in $D(v, b, k)$. Cheng [1] proved that if $d^* \in D(v, b, k)$ is such that d_N^* is E-optimal in $D_N(v, b, k)$ and each treatment of d^* occurs in each row of d^* exactly b/v times, then $C_{d^*} = C_{d^*}^N$ and d^* is E-optimal in $D(v, b, k)$. Jacroux [12] showed that by augmenting certain types of designs such as those proven E-optimal by Kiefer [14] and Cheng [1] with additional columns, some E-optimal row-column designs having treatments unequally replicated can be obtained. In this paper we derive some further results concerning the E and MV-optimality of designs in $D(v, b, k)$. In particular, it is well known that if $d_N \in \bar{D}_N(v, b, k)$ is a GD design having $\lambda_2^N = \lambda_1^N + 1$, then d_N is E and MV-optimal in $D_N(v, b, k)$. (See Takeuchi [18]). Here we give some methods for constructing row-column designs which are E and MV-optimal in $D(v, b, k)$ from designs $d_N \in D_N(v, b, k)$ which are GD designs of the form mentioned in the preceding sentence.

2. Main results

In this section, we give our main results concerning E and MV-optimality. In particular, we give some sufficient conditions for a design d^* to be E and MV-optimal in $D(v, b, k)$ whose corresponding design d_N^* is a GD design having $\lambda_2^N = \lambda_1^N + 1$. We note, however, that in any of the results given in this section, the rolls of d_N^* and d_M^* could be interchanged. We begin by proving a lemma which is useful in deriving our main theorems.

LEMMA 2.1. *Let $d \in D(v, b, k)$ be arbitrary and let $c'\alpha = \sum_{i=1}^{v} c_i \alpha_i$ be a contrast which is estimable in both d and d_N. If $\mathrm{var}_d(c'\hat{\alpha})$ and $\mathrm{var}_{d_N}(c'\hat{\alpha})$ denote the variances of the least squares estimators of $c'\alpha$ obtained from d and d_N, respectively, then*

$$\mathrm{var}_d(c'\hat{\alpha}) \geqq \mathrm{var}_{d_N}(c'\hat{\alpha}) .$$

PROOF. For $d \in D(v, b, k)$, it is easily seen that if $c'\alpha = \sum_{i=1}^{v} c_i \alpha_i$ is an estimable function of the treatment effects, then $\mathrm{var}_d(c'\hat{\alpha}) = c' C_d^- c$ where C_d^- is any reflexive generalized inverse of C_d, i.e., a generalized inverse satisfying $C_d^- C_d C_d^- = C_d^-$. Similarly, if $c'\alpha$ is estimable in d_N, then $\mathrm{var}_{d_N}(c'\hat{\alpha}) = c' C_d^{N-} c$ where C_d^{N-} is any reflexive generalized inverse of C_d^N. Now, since $d \in D(v, b, k)$, d is connected, C_d has rank $v-1$ and one reflexive generalized inverse of C_d is given by

$$C_d^- = \begin{bmatrix} C_{dvv}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

where C_{dvv} is the principal minor obtained by eliminating the last row and column of C_d. Also, because d is connected, it follows that d_N is connected, i.e., all treatment contrasts $c'\alpha$ are estimable in d_N. Thus C_d^N has rank $v-1$ and a reflexive generalized inverse for C_d^N is given by

$$C_d^{N-} = \begin{bmatrix} (C_{dvv}^N)^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

where C_{dvv}^N is the principal minor obtained by eliminating the last row and last column of C_d^N. Now, from (1.4), it follows that $C_d^N - C_d$ is positive semi-definite, hence that $C_{dvv}^N - C_{dvv}$ is positive semi-definite and $C_{dvv}^{-1} - (C_{dvv}^N)^{-1}$ is positive semi-definite. The result now follows since for any vector c,

$$c'C_{dvv}^{-1}c \geqq c'(C_{dvv}^N)^{-1}c \ .$$

We now prove a result which is similar to Theorem 3.1 of Cheng [1].

THEOREM 2.2. *Let $d^* \in D(v, b, k)$ be such that d_N^* is E-optimal in $D_N(v, b, k)$. If $z_{d^*1} = z_{d^*1}^N$, then d^* is E-optimal in $D(v, b, k)$.*

PROOF. If d_N^* is E-optimal in $D_N(v, b, k)$, then for each $d_N \in D_N(v, b, k)$, $z_{d1}^N \leqq z_{d^*1}^N$. Now, by (1.4), it follows that for any $d \in D(v, b, k)$

$$z_{d1} \leqq z_{d1} \leqq z_{d^*1}^N = z_{d_1^*} \ ,$$

thus establishing the E-optimality of d^* in $D(v, b, k)$.

COROLLARY 2.3. *Let $d^* \in D(v, b, k)$ be such that $d_N^* \in \bar{D}_N(v, b, k)$ is a GD design with parameters p^N, q^N and $\lambda_2^N = \lambda_1^N + 1$. Also assume that*

i) *$d_M^* \in \bar{D}_M(v, b, k)$ is a BBD or*

ii) *$d_M^* \in D_M(v, b, k)$ is a GD design with parameters p^M, $q^M = lq^N$ $(l \geqq 1)$, such that the following conditions hold;*

   a) $\lambda_1^M = \lambda_{d^*11}^M = \cdots = \lambda_{d^*vv}^M$,

   b) *treatments which are first associates in d_N^* are also first associates in d_M^*,*

   c) $z_{d^*,v-1}^N - (q^M\lambda_1^M - q^M\lambda_2^M)/b \geqq z_{d^*1}^N$.

Then d^* is E-optimal in $D(v, b, k)$.

PROOF. By the results of Takeuchi [19], d_N^* is E-optimal in $D_N(v, b, k)$. If d_M^* is BBD, then each treatment must occur in each row of d^* the same number of times, and the result follows by Theorem 3.1 of Cheng [1]. If d_M^* is a GD design such as described, it is easy to verify that the eigenvalues of d^* are

$z^N_{d^*1}=(rk-R^N(r)+\lambda^N_1)/k$ (occurring with multiplicity $p^N(q^N-1)$)

$z^N_{d^*,v-1}-(q^M\lambda^M_1-q^M\lambda^M_2)/b$

(occurring with multiplicity $v-p^M(q^N-1)-p^M(l-1)-1$)

$z^N_{d^*,v-1}=(rk-R^N(r)-(q^N-1)\lambda^N_1+q^N\lambda^N)/k$

(occurring with multiplicity $p^M(l-1)$) .

The condition that $z^N_{d^*,v-1}-(q^M\lambda^M_1-q^M\lambda^M_2)/b\geqq z^N_{d^*1}$ then guarantees that $z_{d^*1}=z^N_{d^*1}$ and the result follows from Theorem 2.2.

THEOREM 2.4. *Suppose $d^*\in D(v,b,k)$ is such that d^*_N is MV-optimal in $D_N(v,b,k)$ and $\max_{i\neq j}\text{var}_d(\hat{\alpha}_i-\hat{\alpha}_j)=\max_{i\neq j}\text{var}_{d^*_N}(\hat{\alpha}_i-\hat{\alpha}_j)$. Then d^* is MV-optimal in $D(v,b,k)$.*

PROOF. Let $d\in D(v,b,k)$ be arbitrary. Since d^*_N is MV-optimal in $D_N(v,b,k)$, it follows by the assumptions of the Theorem and Lemma 2.1 that

$$\max_{i\neq j}\text{var}_d(\hat{\alpha}_i-\hat{\alpha}_j)\geqq\max_{i\neq j}\text{var}_{d_N}(\hat{\alpha}_i-\hat{\alpha}_j)\geqq\max_{i\neq j}\text{var}_{d^*_N}(\hat{\alpha}_i-\hat{\alpha}_j)=\max_{i\neq j}\text{var}_d(\hat{\alpha}_i-\hat{\alpha}_j)\ ,$$

hence that d^* is MV-optimal in $D(v,b,k)$.

COROLLARY 2.5. *Suppose $d^*\in D(v,b,k)$ is such that d^*_N satisfies the conditions of Corollary 2.3. Also assume that*

i) *$d^*_M\in\bar{D}_M(v,b,k)$ is a BBD or*

ii) *$d^*_M\in D_M(v,b,k)$ is a GD design satisfying the conditions of Corollary 2.3. Then d^* is MV-optimal in $D(v,b,k)$.*

PROOF. If d^*_M is a BBD, it follows from the results of Cheng [1] that $C_{d^*}=C^N_{d^*}$, hence that $z_{d^*1}=z^N_{d^*1}$. On the other hand, if d^*_M satisfies the conditions of Corollary 2.3, from the proof of the corollary, it is seen that $z_{d^*1}=z^N_{d^*1}$. So in either case, we have that

$$z_{d^*1}=z^N_{d^*1}=(rk-R^N(r)+\lambda^N_1)/k\ .$$

From Raghavarao [17], the previous equality and Lemma 2.1, it follows that since d^*_N is a GD design with parameters $\lambda^N_2=\lambda^N_1+1$,

$$2/z^N_{d^*1}=\max_{i\neq j}\text{var}_{d^*_N}(\hat{\alpha}_i-\hat{\alpha}_j)\leqq\max_{i\neq j}\text{var}_{d^*}(\hat{\alpha}_i-\hat{\alpha}_j)\leqq 2/z_{d^*1}\ .$$

Now, by the results of Takeuchi [19], d^*_N is uniquely MV-optimal in $D_N(v,b,k)$. Thus d^* satisfies the conditions of Theorem 2.4 and the corollary follows.

We now give several examples to illustrate the results given in

Corollaries 2.3 and 2.5.

Example 2.6. Consider the class of designs $D(6, 9, 4)$ and consider $d^* \in D(6, 9, 4)$ having treatments assigned to rows and columns as follows:

$$d^* = \begin{bmatrix} 1 & 1 & 2 & 3 & 3 & 5 & 2 & 4 & 6 \\ 2 & 2 & 1 & 1 & 4 & 3 & 3 & 6 & 5 \\ 4 & 6 & 5 & 4 & 1 & 6 & 5 & 3 & 2 \\ 5 & 4 & 6 & 5 & 6 & 1 & 4 & 2 & 3 \end{bmatrix}.$$

Then d_N^* is a GD design having parameters $p^N=2$, $q^N=3$, $\lambda_1^N=3$ and $\lambda_2^N=4$ and d_M^* is a GD design having parameters $p^M=2$, $q^M=3$, $\lambda_1^M=10$ and $\lambda_2^M=8$. It is also easy to verify that $z_{d^*1}=z_{d^*1}^N=21/4$ and that $z_{d^*,v-1}^N-(q^M\lambda_1^M-q^M\lambda_2^M)/b=(24/4)-(3(10)-3(8))/9=16/3>21/4$. Thus d^* is E and MV-optimal in $D(6, 9, 4)$ by Corollaries 2.3 and 2.5.

Comment. The process of constructing designs which satisfy the conditions of Corollaries 2.3 and 2.5 is relatively simple. To begin with, let $d_N^* \in \bar{D}_N(v, b, k)$ be a GD design having parameters $\lambda_2^N=\lambda_1^N+1$. Now, arrange treatments within the blocks of d_N^* so that all treatments which are first associates occur in the same rows of d^* the same number of times and such that the resulting GD design d_M^* satisfies the conditions of Corollaries 2.3 and 2.5. For example, if b is a multiple of v, then arrange the treatments within the blocks of d_N^* so that each treatment occurs within each row of d^* exactly (b/v) times. That treatments can be arranged in blocks in this manner can be proven using arguments similar to those given by Constantine [5]. Using this technique, the author has been able to find row-column designs satisfying Corollaries 2.3 and 2.5 for a number of parameter sets v, b and k. Given below are parameter sets for which the author has constructed designs satisfying Corollaries 2.3 and 2.5 for values of v, b and k where b is not a multiple of v:

v	b	k	v	b	k	v	b	k	v	b	k
4	10	2	6	69	2	9	21	3	12	9	8
4	22	2	6	99	2	10	25	2	14	49	2
4	34	2	6	14	3	12	54	2	15	25	3
4	46	2	6	9	4	12	16	3	16	36	4
4	58	2	8	18	4	12	20	3	18	81	2
4	70	2	8	44	2	12	40	3	18	48	3
6	9	2	8	52	2	12	9	4	20	100	2
6	39	2	8	12	4	12	30	4	20	25	4

v	b	k	v	b	k	v	b	k	v	b	k
20	16	5	30	25	6	40	72	5	56	64	7
21	49	3	35	49	5	42	49	6	56	49	8
24	80	3	36	81	4	48	64	6	63	81	7
28	49	4	40	100	4	54	81	6			

Comment. Hoblyn, et al. [9] and Freeman [8] considered some types of row-column designs of which those satisfying Corollaries 2.3 and 2.5 are special cases. Using the notation of Hoblyn, et al. [9], $d \in D(v, b, k)$ is an $X:YZ$ design where X, Y and Z may be any of the letters O-standing for orthogonal, T-standing for totally balanced or P-standing for partially balanced, and X describes the type of block design obtained by considering rows as treatments arranged within columns whereas Y and Z describe the types of block designs that d_N and d_M are, respectively. Thus the designs satisfying Corollaries 2.3 and 2.5 and those given in the previous table are $O:PP$ designs. However, Hoblyn, et al. [9] and Freeman [8] did not investigate any of the designs which they considered in terms of optimality. Freeman [8] gives a catalogue of some useful $O:PP$ designs, some of which are E and MV-optimal by the results given here. The reader is referred to Freeman [8] for additional information on other types of $O:PP$ designs.

An interesting aspect of the E and MV-optimal row-column designs described in Corollaries 2.3 and 2.5 is that while a GD design $d_N^* \in \bar{D}_N(v, b, k)$ is uniquely E and MV-optimal in $D_N(v, b, k)$ (see Takeuchi [18]), there may exist more than one design in $D(v, b, k)$ which satisfies both of Corollaries 2.3 and 2.5 as the next example illustrates.

Example 2.7. Consider the class of designs $D(4, 16, 2)$ and consider $d_1^* \in D(4, 16, 2)$ and $d_2^* \in D(4, 16, 2)$ having treatments assigned to rows and columns as follows:

$$d_1^* = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 4 & 3 & 4 & 2 & 2 & 2 & 2 & 3 & 4 & 3 & 4 \\ 2 & 3 & 4 & 2 & 1 & 1 & 1 & 1 & 3 & 4 & 3 & 4 & 2 & 2 & 4 & 3 \end{bmatrix}$$

$$d_2^* = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 4 & 3 & 4 & 2 & 2 & 2 & 2 & 3 & 2 & 3 & 4 \\ 2 & 3 & 4 & 2 & 3 & 1 & 1 & 1 & 3 & 4 & 3 & 4 & 2 & 4 & 4 & 3 \end{bmatrix}.$$

Then d_{N1}^* and d_{N2}^* are both GD designs having parameters $p^N=2$, $q^N=2$, $\lambda_1^N=2$ and $\lambda_2^N=3$. However, $d_{M1}^* \in \bar{D}_M(4, 16, 2)$ is a BBD and $d_{M2}^* \in D_M(4, 16, 2)$ is a GD design having parameters $p^M=2$, $q^M=2$, $\lambda_1^M=34$ and $\lambda_2^M=30$. It is also easy to verify that d_1^* and d_2^* have minimal nonzero eigenvalues equal to 5 and that

$$z^N_{d^*2, v-1} - (q^M \lambda_1^M - q^M \lambda_2^M)/b = 6 - (2(34) - 2(30))/16 = 11/2 > 5 .$$

Thus both d_1^* and d_2^* are E and MV-optimal in $D(4, 16, 2)$.

Comment. While both the designs d_1^* and d_2^* given in the previous example are E and MV-optimal by Corollaries 2.3 and 2.5, d_1^* is probably the one that would be used in practice since it can be shown that it is better than d_2^* under a number of other optimality criteria such as the A and D-optimality criteria. So if more than one design exists in a given class $D(v, b, k)$ which satisfy Corollaries 2.3 and 2.5, other optimality criteria can be used to select a single design for usage from among the competing E and MV-optimal designs.

WASHINGTON STATE UNIVERSITY

REFERENCES

[1] Cheng, C. S. (1978). Optimal designs for the elimination of multi-way heterogeneity, *Ann. Statist.*, **6**, 1202-1273.

[2] Cheng, C. S. (1980). On the E-optimality of some block designs, *J. Roy. Statist. Soc.*, B, **42**, 199-204.

[3] Constantine, G. M. (1981). Some E-optimal block designs, *Ann. Statist.*, **9**, 886-892.

[4] Constantine, G. M. (1982). On the E-optimality of PBIB designs with a small number of blocks, *Ann. Statist.*, **10**, 1027-1031.

[5] Constantine, G. M. (1983). On the trace efficiency for control of reinforced BIB designs, *J. R. Statist. Soc.*, B, **45**, 31-36.

[6] Eccleston, J. and Russell, K. (1975). Connectedness and orthogonality in multifactor designs, *Biometrika*, **62**, 341-345.

[7] Ehrenfeld, S. (1955). On the efficiency of experimental designs, *Ann. Math. Statist.*, **26**, 247-255.

[8] Freeman, G. H. (1958). Families of designs for two successive experiments, *Ann. Math. Statist.*, **29**, 1063-1078.

[9] Hoblyn, T. N., Pearce, S. C. and Freeman, G. H. (1954). Some considerations in the design of successive experiments in fruit plantations, *Biometrics*, **10**, 503-515.

[10] Jacroux, M. (1980a). On the E-optimality of regular graph designs, *J. Roy. Statist. Soc.*, B, **42**, 205-209.

[11] Jacroux, M. (1980b). On the determination and construction of E-optimal block designs with unequal numbers of replicates, *Biometrika*, **67**, 661-667.

[12] Jacroux, M. (1982). Some E-optimal designs for the one-way and two-way elimination of heterogeneity, *J. Roy. Statist. Soc.*, B, **44**, 253-261.

[13] Jacroux, M. and Seeley, J. (1978). Some sufficient conditions for (M, S) optimality, *J. Statist. Plann. Inf.*, **4**, 3-11.

[14] Kiefer, J. (1975). Construction and optimality of generalized Youden designs, *in a Survey of Statistical Designs and Linear Models* (ed. J. N. Srivastava), 333-353, Amsterdam, North Holland.

[15] Magda, C. G. (1980). Circular balanced repeated measurements designs, *Commun. Statist. Theor. Meth.*, **A9**, 1901-1918.

[16] Pearce, S. C. (1975). Row and column designs, *Appl. Statist.*, **24**, 60-74.

[17] Raghavarao, D. (1971). *Constructions and Combinatorial Problems in Design of Experiments*, Wiley, New York.

[18] Shah, K. R., Raghavarao, D. and Khatri, C. G. (1976). Optimality of two and three factor designs, *Ann. Statist.*, **4**, 429-432.

[19] Takeuchi, K. (1961). On the optimality of certain type of PBIB designs, *Rep. Statist. Appl. Res.*, J.U.S.E., **8**, 140-145.

[20] Takeuchi, K. (1963). A remark added to "On the optimality of certain type of PBIB designs", *Rep. Statist. Appl. Res.*, J.U.S.E., **10**, 47.