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MEASURES OF LOCATION IN THE PLANE

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Summary

There are many possible candidates for measures of location of asymmetric probability distributions. This difficulty is compounded for multivariate distributions. It is the purpose of this paper to characterize the set of all possible measures of location for a given bivariate probability distribution. A closed, convex region in the plane will be constructed, any point of which is a reasonable measure of location. Reasonable here refers to the invariance of the region under certain transformations and order relations. The size of this region can be used to characterize the degree of asymmetry that a distribution possesses.

1. Introduction

1.1. *Review of the univariate case*

A random variable X is said to be symmetric about a point $\theta \in \boldsymbol{R}$ if $X-\theta \sim -X+\theta$. If X has distribution function F, this is equivalent to $F(x+\theta)=1-F(-x+\theta)$, $\forall x \in \boldsymbol{R}$. It is clear that in this case, the only reasonable measure of location for X is θ. In the case X is not symmetric, many points may serve as location parameters. Doksum [2] characterizes this set of location parameters and establishes some important results concerning it. He uses three different methods to construct a set, any point of which can serve as a measure of where a given distribution is located on the line.

Doksum's first method is to approximate a given distribution function F as closely as possible from above and below with a symmetric distribution function. The respective points of symmetry $\underline{\theta}_F$ and $\overline{\theta}_F$ of these two approximating distributions are then used to define the location interval $[\underline{\theta}_F, \overline{\theta}_F]$.

The second method is to consider the set of location parameters

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for F; that is, the points which satisfy the four location axioms given by Bickel and Lehmann [1].

Let X be a random variable with distribution function F. Then $\mu(F)$ (or $\mu(X)$) will denote a measure of location for F (or X) if $\mu(F)$ satisfies the following four axioms:

($u1$) $F(x) \geqq G(x)$, $\forall x$ implies $\mu(F) \leqq \mu(G)$

($u2$) $\mu(aX) = a\mu(X)$, $\forall a > 0$

($u3$) $\mu(X+b) = \mu(X) + b$, $\forall b \in R$

($u4$) $\mu(-X) = -\mu(X)$.

Doksum defines a location parameter to be the real-valued location functional $\theta.$ (defined on $\mathscr{F}$, the class of all distribution functions) satisfying axioms ($u1$)–($u4$) for all F with finite support. Let D denote the set of all real-valued location parameters $\theta.$. Then for a given distribution function F,

$$L_F = \{\theta_F \colon \theta. \in D,\ \theta_F \text{ exists}\}$$

is taken to be the location set for F.

The third method involves the use of the function of symmetry. In the case of symmetry, there exists a unique constant θ such that $X \sim -X + 2\theta$. This property can be generalized to any distribution (not necessarily symmetric) by letting θ be a function for each F. If $\bar{F}$ denotes the distribution function of $-X$, then $X \sim -X + 2\theta$ implies $F(x+2\theta) = \bar{F}(x)$, $\forall x$. In general, take $\theta_F(x)$ such that $\bar{F}(x) = F(x + 2\theta_F(x))$, or more precisely, $2\theta_F(x) = \sup\{\theta\colon F(x) \leqq \bar{F}(x-2\theta)\}$, for x in the support of F. We can solve this last equation for $\theta_F(x)$ to obtain the function of symmetry

$$\theta_F(x) = \frac{1}{2}[x - \bar{F}^{-1}(F(x))] ,$$

for x in the support of F. The range of $\theta_F(x)$, $R(\theta_F) = \{\theta_F(x)$ for x in the support of $F\}$ can then be taken as a location set for F.

Probably the most significant result of Doksum's work is the equivalence of these three methods of constructing a location set. He proves the following theorem:

THEOREM. (a) *If F is increasing and continuous on its support, then the closure of the location set L_F equals the location interval* $[\underline{\theta}_F, \bar{\theta}_F]$. (b) *If F is increasing and continuous on its support, then the closure of the range of $\theta_F(\cdot)$ equals the location interval* $[\underline{\theta}_F, \bar{\theta}_F]$.

1.2. *Statement of problem and summary of results*

It is the purpose of this paper to extend Doksum's results to multivariate distribution functions. The bivariate case will be emphasized due to the ease of geometric interpretation. Multivariate generalizations are then easily obtained.

By an approach analogous to that used in the univariate case, a well-defined, closed, convex location set will be obtained, any point of which would be an appropriate measure of where the given bivariate distribution is located in the plane. It will be shown that this region depends on the distribution function only through its marginal distribution so that the bivariate case can actually be reduced to the univariate problem.

Section 3 briefly describes a possible application of the location region. The size of the region can be used to characterize the degree of asymmetry of a distribution. A measure of asymmetry for standardized random vectors is given.

2. Construction of the bivariate location region

2.1. *Method* I: *Approximation by a symmetric distribution*

Given a bivariate distribution function F, we wish to construct a symmetric bivariate distribution function H, and then use H to approximate F from above and below. It is thus apparent that an ordering of distribution functions is needed. This can be done as follows:

DEFINITION. Let $\boldsymbol{X}=(X_1, X_2)'$ and $\boldsymbol{Y}=(Y_1, Y_2)'$, where "prime" denotes transpose, be two random vectors with distribution functions F and G respectively. We say $\boldsymbol{X}$ is stochastically smaller than $\boldsymbol{Y}$ iff

$$F(\boldsymbol{x}) \geqq G(\boldsymbol{x}) , \qquad \forall \boldsymbol{x} \in R^2 .$$

While it is possible to construct H by methods somewhat analogous to those used in the univariate case, this is not necessary. Rather than extend the univariate results to the bivariate case, it is possible to actually reduce the bivariate problem to a univariate one. This will be done by way of the marginal distribution functions. It will be shown that the boundaries of the location region depend on F only through its marginals.

Let $\boldsymbol{X}=(X_1, X_2)'$ be a random vector with distribution function F, and let $S(F)=\{\boldsymbol{x}=(x_1, x_2)' \colon 0<F(x_1, x_2)<1\}$ be the support of F. Assume F is continuous and strictly increasing in each argument. Denote the marginal distribution functions of F by

$$F_1(x_1)=F(x_1, +\infty) , \qquad F_2(x_2)=F(+\infty, x_2) .$$

Consider the following result:

THEOREM 1. *Suppose F and H are continuous bivariate distribu-*

tion functions which have the same support. Let

$$B=\{\boldsymbol{\theta}=(\theta_1, \theta_2)': H(x_1-\theta_1, x_2-\theta_2)\geqq F(x_1, x_2), \ \forall x_1, x_2\}$$

$$B_1=\{\theta_1: \boldsymbol{\theta}=(\theta_1, \theta_2)' \in B\}$$

$$U_1=\{\theta_1: H_1(x_1-\theta_1)\geqq F_1(x_1), \ \forall x_1\} . \qquad \textit{Then } U_1=B_1 .$$

PROOF. Let $\boldsymbol{\theta}=(\theta_1, \theta_2)' \in B$. Then

$$H(x_1-\theta_1, x_2-\theta_2)\geqq F(x_1, x_2) , \qquad \forall x_1, x_2 .$$

In particular, for $x_2=+\infty$, we have

$$H_1(x_1-\theta_1)\geqq F_1(x_1) , \qquad \forall x_1$$

$$\Longrightarrow \theta_1 \in U_1$$

$$\Longrightarrow B_1\subset U_1 .$$

We now wish to show $U_1\subset B_1$. Assume not. Let $\theta_1 \in U_1$, but $\theta_1 \notin B_1$. Then for each value of θ_2, there exist x_1, x_2 such that

$$(*) \qquad H(x_1-\theta_1, x_2-\theta_2)<F(x_1, x_2) .$$

Case (1). Suppose the support of F and H is the entire plane. Then (*) implies there exist x_1, x_2 such that

$$H_1(x_1-\theta_1)=H(x_1-\theta_1, \infty)\leqq F(x_1, x_2)<F_1(x_1) ,$$

contradicting $\theta_1 \in U_1$. This establishes $U_1\subset B_1$ and the theorem.

Case (2). Suppose the support of F and H is bounded (that is, contained in a sphere of finite radius). Then there exists $y^*<\infty$ such that $H(x_1, y^*)=H_1(x_1)$, $\forall x_1$.

We can choose θ_2 such that for corresponding value(s) of x_2 (and x_1) which satisfy (*), x_2 also satisfies

$$(**) \qquad x_2\geqq \theta_2+y^*$$

$$\Longleftrightarrow x_2-\theta_2\geqq y^*$$

$$\Longrightarrow H_1(x_1-\theta_1)=H(x_1-\theta_1, x_2-\theta_2)<F(x_1, x_2)\leqq F_1(x_1) ,$$

again contradicting $\theta_1 \in U_1$ and thus establishing the theorem. It remains only to show that for some θ_2, there exists x_2 (and x_1) satisfying (**) as well as (*). Suppose the contrary. Assume that for every θ_2 the corresponding x_2 satisfying (*) is always such that

$$x_2<\theta_2+y^* \qquad \Longrightarrow \text{as } \theta_2\rightarrow-\infty, \text{ so does } x_2 .$$

So take θ_2 so small that the corresponding x_2 is no longer in the support. Then for this θ_2, x_1, and x_2,

$$F(x_1, x_2)=0 \quad \Longrightarrow H(x_1-\theta_1, x_2-\theta_2)<0, \text{ a contradiction}.$$

If $S=\{\boldsymbol{\theta}=(\theta_1, \theta_2)': H(x_1-\theta_1, x_2-\theta_2)\leqq F(x_1, x_2),\ \forall x_1, x_2\}$,

$$S_1=\{\theta_1: \boldsymbol{\theta}=(\theta_1, \theta_2)' \in S\},$$

$$S_2=\{\theta_2: \boldsymbol{\theta}=(\theta_1, \theta_2)' \in S\},$$

$$B_2=\{\theta_2: \boldsymbol{\theta}=(\theta_1, \theta_2)' \in B\},$$

$$L_1=\{\theta_1: H_1(x_1-\theta_1)\leqq F_1(x_1),\ \forall x_1\},$$

$$U_2=\{\theta_2: H_2(x_2-\theta_2)\geqq F_2(x_2),\ \forall x_2\},$$

and $$L_2=\{\theta_2: H_2(x_2-\theta_2)\leqq F_2(x_2),\ \forall x_2\},$$

a similar argument yields

$$B_2=U_2, \quad S_1=L_1 \quad \text{and} \quad S_2=L_2.$$

Let $\boldsymbol{\theta}_F^*=\sup_{\boldsymbol{\theta}} B$, $\boldsymbol{\theta}_F^{**}=\inf_{\boldsymbol{\theta}} S$ using the component-wise ordering of points in the plane. Take A to be the rectangular region

$$A=\{\boldsymbol{x}: \boldsymbol{\theta}_F^*\leqq \boldsymbol{x}\leqq \boldsymbol{\theta}_F^{**}\}.$$

Then in the spirit of Doksum's construction for the univariate case, A is a location set for F. Some comments are in order:

(1) By the theorem, A is simply the Cartesian product of the following two intervals:

$$I_1=[\underline{\theta}_{F_1}, \bar{\theta}_{F_1}] \quad \text{and} \quad I_2=[\underline{\theta}_{F_2}, \bar{\theta}_{F_2}]$$

where
$$\underline{\theta}_{F_k}=\sup_{\theta}\{\theta: H_k(x-\theta)\geqq F_k(x),\ \forall x\}$$
$$\bar{\theta}_{F_k}=\inf_{\theta}\{\theta: H_k(x-\theta)\leqq F_k(x),\ \forall x\} \qquad \text{for } k=1, 2.$$

Thus, this theorem results in a reduction of the bivariate problem to a univariate problem by way of the two marginal distributions F_1 and F_2.

(2) Clearly, it is desirable to make A as small as possible; i.e., approximate F as closely as possible with some H. Since $A=I_1\times I_2$, taking H_1 and H_2 s.t.

$$H_k^{-1}(u)=\frac{1}{2}[F_k^{-1}(u)+\bar{F}_k^{-1}(u)], \qquad u\in(0, 1),$$

for $k=1, 2$, respectively yields the smallest I_1 and I_2 and thus minimizes the size of A. Hence any bivariate distribution function H with marginals H_1 and H_2 as defined above is appropriate.

(3) Distribution functions are based on "infinite rectangles" with sides parallel to the coordinate axes. It is natural to wish the location region independent of the orientation of the axes. To this end, let R_α denote the linear transformation representing a counterclockwise rotation of the plane through an angle α. Let F_α denote the distribution function of $Y=R_\alpha(X)$ with corresponding marginals $F_{1\alpha}$ and $\bar{F}_{2\alpha}$. $F_{1\alpha}$ and $\bar{F}_{2\alpha}$ will denote the distribution functions of $-Y_1$ and $-Y_2$ respectively. For each $\alpha \in (0, 2\pi]$, let $H_{1\alpha}$ and $H_{2\alpha}$ be distribution functions whose inverses are given by

$$H_{k\alpha}^{-1}(u)=\frac{1}{2}[F_{k\alpha}^{-1}(u)+\bar{F}_{k\alpha}^{-1}(u)], \qquad u \in (0, 1)$$

for $k=1, 2$. For each $\alpha \in (0, 2\pi]$ construct the region A_α as in Theorem 1. (See Figure A) Then the region given by

$$A=\bigcap_{\alpha \in (0,2\pi]} R_{-\alpha}(A_\alpha) \text{ is a location region for } F.$$

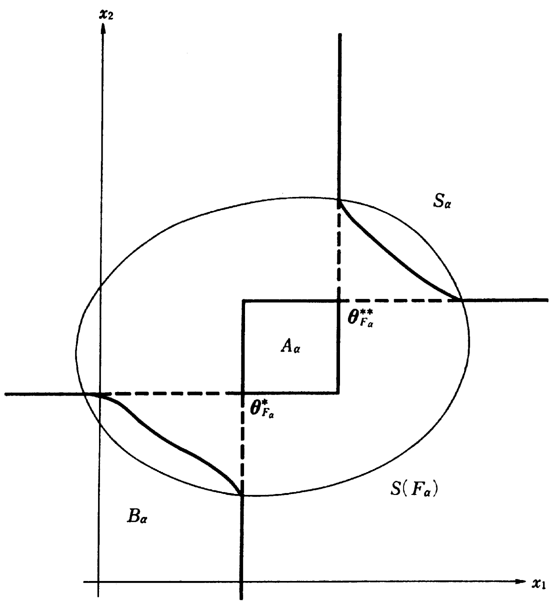

Fig. A. Diagram of a typical location region A_α for the distribution function F_α for $\alpha \in (0, 2\pi]$.

2.2. *Method* II: *Axioms of location*

Let $\theta_X=\theta_F$ be a candidate for a measure of location for F. Desired properties of θ_F are:

(B1) $\theta_X \leqq \theta_Y$ whenever X is stochastically smaller than Y. (The component-wise ordering of points in the plane is being used here to define $\boldsymbol{\theta}_X \leqq \boldsymbol{\theta}_Y$).

(B2) $\boldsymbol{\theta}_{OX} = O\boldsymbol{\theta}_X$, $\forall$ orthogonal transformations O.

(B3) $\boldsymbol{\theta}_{X+\boldsymbol{a}} = \boldsymbol{\theta}_X + \boldsymbol{a}$, $\forall \boldsymbol{a} \in R^2$.

(B4) $\boldsymbol{\theta}_{SX} = S\boldsymbol{\theta}_X$, $\forall$ positive definite symmetric transformations S.

We will call $\boldsymbol{\theta}.$ a bivariate location parameter if it satisfies (B1)–(B4) for all F with finite support. Let D denote the collection of all location parameters $\boldsymbol{\theta}.$. For given F, define the location set $L_F = \{\boldsymbol{\theta}_F : \boldsymbol{\theta}. \in D, \boldsymbol{\theta}_F \text{ exists}\}$.

2.3. *Method* III: *Function of symmetry*

A distribution function F is symmetric about a point $\boldsymbol{\theta}$ if $(X_1 - 2\theta_1, X_2 - 2\theta_2)' \sim (-X_1, -X_2)'$. Thus, we can extend this idea to nonsymmetric distributions by paralleling the univariate approach marginally. Hence, the bivariate function of symmetry in direction α is given by

$$\boldsymbol{\theta}_{F_\alpha}(\boldsymbol{x}) = \begin{pmatrix} (1/2)[x_1 - \bar{F}_{1\alpha}^{-1}(F_{1\alpha}(x_1))] \\ (1/2)[x_2 - \bar{F}_{2\alpha}^{-1}(F_{2\alpha}(x_2))] \end{pmatrix}, \qquad \alpha \in (0, 2\pi] .$$

Now $\boldsymbol{\theta}_{F_\alpha}(\boldsymbol{x}) = \boldsymbol{\theta}_{F_\alpha}(x_1, x_2)'$ and if we let $x_1 = F_{1\alpha}^{-1}(u)$, $x_2 = F_{2\alpha}^{-1}(v)$ for $u, v \in (0, 1/2]$, we have

$$\boldsymbol{\theta}_{F_\alpha}(\boldsymbol{x}) = \boldsymbol{\theta}_{F_\alpha}\begin{pmatrix} F_{1\alpha}^{-1}(u) \\ F_{2\alpha}^{-1}(v) \end{pmatrix} = \begin{pmatrix} (1/2)[F_{1\alpha}^{-1}(u) + F_{1\alpha}^{-1}(1-u)] \\ (1/2)[F_{2\alpha}^{-1}(v) + F_{2\alpha}^{-1}(1-v)] \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} m_{F_{1\alpha}}(u) \\ m_{F_{2\alpha}}(v) \end{pmatrix},$$

using Doksum's notation.

Define $\theta_{F_{1\alpha}}^* = \inf\{m_{F_{1\alpha}}(u) : 0 < u \leqq 1/2\}$,

$\theta_{F_{1\alpha}}^{**} = \sup\{m_{F_{1\alpha}}(u) : 0 < u \leqq 1/2\}$,

$\theta_{F_{2\alpha}}^* = \inf\{m_{F_{2\alpha}}(v) : 0 < v \leqq 1/2\}$,

$\theta_{F_{2\alpha}}^{**} = \sup\{m_{F_{2\alpha}}(v) : 0 < v \leqq 1/2\}$.

Then for $B_\alpha = \left\{\boldsymbol{x} : \begin{pmatrix} \theta_{F_{1\alpha}}^* \\ \theta_{F_{2\alpha}}^* \end{pmatrix} \leqq \boldsymbol{x} \leqq \begin{pmatrix} \theta_{F_{1\alpha}}^{**} \\ \theta_{F_{2\alpha}}^{**} \end{pmatrix}\right\}$, the location region for F is $B = \bigcap_{\alpha \in (0, 2\pi]} R_{-\alpha}(B_\alpha)$.

THEOREM 2. *Let F be continuous and strictly increasing in each argument. Let A be the location region for F obtained via Method* I, *B the location region for F obtained via Method* III. *Then $A = B$.*

PROOF. Using the previous notation, for each $\alpha \in (0, 2\pi]$ we have

$$\begin{aligned} \boldsymbol{\theta}_{F_\alpha}^* &= \sup B_\alpha \\ &= (\sup B_{1\alpha}, \sup B_{2\alpha})' \end{aligned}$$

$$=(\sup U_{1\alpha}, \sup U_{2\alpha})' \qquad \text{(by Theorem 1)}$$
$$=(\sup\{\theta_1 : H_{1\alpha}(x_1-\theta_1)\geqq F_{1\alpha}(x_1),\ \forall x_1\},$$
$$\sup\{\theta_2 : H_{2\alpha}(x_2-\theta_2)\geqq F_{2\alpha}(x_2),\ \forall x_2\})'$$
$$=(\inf\{m_{F_{1\alpha}}(u) : 0<u\leqq 1/2\},\ \inf\{m_{F_{2\alpha}}(v) : 0<v\leqq 1/2\})$$
$$\text{(by a result due to Doksum for the univariate case)}$$
$$=(\theta^*_{F_{1\alpha}}, \theta^*_{F_{2\alpha}})' .$$

Similarly, $\boldsymbol{\theta}^{**}_{F_\alpha}=(\theta^{**}_{F_{1\alpha}}, \theta^{**}_{F_{2\alpha}})'$. Hence, $A_\alpha=B_\alpha$, for each $\alpha\in(0, 2\pi]$ implying $A=B$.

THEOREM 3. *If F is continuous and strictly increasing in each argument, then the closure of the location set L_F equals the location region A obtained via method* I.

PROOF. Let $\boldsymbol{\theta}.: F\to R^2$ be a functional satisfying (B1)–(B4). We will show $\boldsymbol{\theta}_F\in A$. For each $\alpha\in(0, 2\pi]$, $H_\alpha(\boldsymbol{x}-\boldsymbol{\theta})$ is symmetric about $\boldsymbol{\theta}$. Also,

$$H_\alpha(\boldsymbol{x}-\boldsymbol{\theta})\geqq F_\alpha(\boldsymbol{x}),\ \forall \boldsymbol{x} \text{ and } \forall \boldsymbol{\theta}\in B_\alpha$$
$$H_\alpha(\boldsymbol{x}-\boldsymbol{\theta})\leqq F_\alpha(\boldsymbol{x}),\ \forall \boldsymbol{x} \text{ and } \forall \boldsymbol{\theta}\in S_\alpha$$
$$\Longrightarrow \boldsymbol{\theta}_{F_\alpha}\leqq\boldsymbol{\theta},\ \forall\boldsymbol{\theta}\in B_\alpha \text{ and } \boldsymbol{\theta}_{F_\alpha}\geqq\boldsymbol{\theta},\ \forall\boldsymbol{\theta}\in S_\alpha. \quad \text{(Axiom (B1))}$$

Hence, $\boldsymbol{\theta}^*_{F_\alpha}\leqq\boldsymbol{\theta}_{F_\alpha}\leqq\boldsymbol{\theta}^{**}_{F_\alpha}$, implying $\boldsymbol{\theta}_{F_\alpha}\in A_\alpha$

$$\Longrightarrow \boldsymbol{\theta}_{F_\alpha}=\boldsymbol{\theta}_{R_\alpha X}=R_\alpha\boldsymbol{\theta}_X\in A_\alpha \quad \text{(axiom (B3))}$$
$$\Longrightarrow \boldsymbol{\theta}_F=\boldsymbol{\theta}_X\in R_{-\alpha}(A_\alpha),\ \forall\alpha\in(0, 2\pi].$$
$$\Longrightarrow \boldsymbol{\theta}_F\in\bigcap_{\alpha\in(0,2\pi]} R_{-\alpha}(A_\alpha)$$
$$\Longrightarrow L_F\subset A.$$

Conversely, we will now show for $\boldsymbol{\theta}\in A$, there exists a functional $\boldsymbol{\theta}.: F\to R^2$ satisfying (B1)–(B4) such that $\boldsymbol{\theta}=\boldsymbol{\theta}_F$. Let $\boldsymbol{\theta}\in A$ be given. $\Longrightarrow\boldsymbol{\theta}\in R_{-\alpha}(A_\alpha)$ for all $\alpha\in(0, 2\pi]$. Hence, Theorem 2 implies that for every $\alpha\in(0, 2\pi]$, there exists $u(\alpha)\in(0, 1/2]$ such that

$$\boldsymbol{\theta}=R_{-\alpha}\begin{pmatrix} m_{F_{1\alpha}}(u(\alpha)) \\ m_{F_{2\alpha}}(u(\alpha+3\pi/2))\end{pmatrix}, \qquad \forall\alpha\in(0, 2\pi].$$

This function u is called the direction function for F. Its existence allows us to characterize any point in A. If F is strictly increasing, u is unique. Clearly, u is periodic with period 2π. Hence, u can be assumed to be defined on all of R. Actually, the existence of u is all that is required in the following argument. This is guaranteed by Theorem 2.

Claim: $\boldsymbol{\theta}_F=\begin{pmatrix} m_{F_1}(u(2\pi)) \\ m_{F_2}(u(7\pi/2)) \end{pmatrix}$ is the desired functional. Clearly $\boldsymbol{\theta}=\boldsymbol{\theta}_F$ since $\boldsymbol{\theta}=R_{-\alpha}\begin{pmatrix} m_{F_1}(u(\alpha)) \\ m_{F_2}(u(\alpha+3\pi/2)) \end{pmatrix}$ for all $\alpha\in(0, 2\pi]$ and so in particular for $\alpha=2\pi$. We must show this $\boldsymbol{\theta}_F=\boldsymbol{\theta}_X$ satisfies axioms (B1)-(B4). First note that since any symmetric positive definite matrix S can be written as $S=O'DO$ where O is orthogonal and $D=\text{diag}\{d_1, d_2\}$ with $d_1>0$, $d_2>0$, axiom (B4) could, without loss of generality, be replaced by

$$\text{(B4}')\boldsymbol{\theta}_{DX}=D\boldsymbol{\theta}_X \qquad \text{for } D=\text{diag}\{d_1, d_2\},\ d_1>0,\ d_2>0\,.$$

That $\boldsymbol{\theta}_F$ satisfies (B1), (B2) and (B4′) follows immediately now from the univariate axioms, (as applied marginally to F), because these axioms represent component-wise operations.

Second, any orthogonal transformation of the plane is either a rotation of the plane, or a reflection of the plane about the x_2-axis followed by a rotation. But if T is a reflection of the plane about the x_2-axis, then, again $\boldsymbol{\theta}_{TX}=T\boldsymbol{\theta}_X$ by the univariate axiom (U4).

Hence, it is necessary to show only that

$$\boldsymbol{\theta}_{R_\beta X}=R_\beta\boldsymbol{\theta}_X$$

where R_β is a rotation of the plane through an angle β. Let G be the distribution function of $R_\beta \boldsymbol{X}$. The direction function v for G is then

$$v(\alpha)=u(\alpha+\beta)\ , \qquad \forall\,\alpha\in(0, 2\pi]$$

where u is the direction function of F.
Now $\boldsymbol{\theta}\in A_G$

$$\Longrightarrow \boldsymbol{\theta}=R_{-\alpha}\begin{pmatrix} m_{G_{1\alpha}}(v(\alpha)) \\ m_{G_{2\alpha}}(v(\alpha+3\pi/2)) \end{pmatrix}, \qquad \forall\,\alpha$$

$$\Longrightarrow \boldsymbol{\theta}=R_{-\alpha}\begin{pmatrix} m_{G_{1\alpha}}(u(\alpha+\beta)) \\ m_{G_{2\alpha}}(u(\alpha+\beta+3\pi/2)) \end{pmatrix}, \qquad \forall\,\alpha\,.$$

In particular,

$$\begin{aligned}\boldsymbol{\theta}_{R_\beta X}=\boldsymbol{\theta}&=\begin{pmatrix} m_{G_1}(u(\beta)) \\ m_{G_2}(u(\beta+3\pi/2)) \end{pmatrix}\\ &=\begin{pmatrix} m_{F_{1\beta}}(u(\beta)) \\ m_{F_{2\beta}}(u(\beta+3\pi/2)) \end{pmatrix}\\ &=R_\beta\boldsymbol{\theta}_X\,.\end{aligned}$$

Thus as in the univariate case, all three methods yield essentially the same location region in the plane.

3. Application—a measure of asymmetry

Intuitively, it is clear that the larger the location region, the more F deviates from symmetry. It is possible to apply a concept from convex analysis to characterize the degree of asymmetry in F.

To be specific, assume that F has bounded support and has been standardized so that each component random variable has unit variance. Let $\bar{L}_F$ denote the closure of the location set for F. Then $\bar{L}_F$ is a compact, convex set in the plane. Let $\boldsymbol{u}$ be a unit vector in R^2. Define the width of $\bar{L}_F$ in the direction perpendicular to $\boldsymbol{u}$ to be the distance between two parallel supporting lines of $\bar{L}_F$, both of which are perpendicular to $\boldsymbol{u}$. Denote this quantity by $W(\boldsymbol{u})$. Thus, $W(\cdot)$ is a mapping from unit sphere $S(0, 1)$, centered at the origin (in R^2 in this case) to the nonnegative real numbers. We have the following result in its multivariate generality:

THEOREM. *The width function for a nonempty, compact set S in R^n is continuous on $S(0, 1)$.*

COROLLARY. *If S is a nonempty compact set in R^n $(n>1)$, then its width function W assumes a minimal value in some direction $\boldsymbol{u}_1$ and a maximal value in some direction $\boldsymbol{u}_2$. If $W(\boldsymbol{u}_1)<W(\boldsymbol{u}_2)$, then W assumes every value intermediate to $W(\boldsymbol{u}_1)$ and $W(\boldsymbol{u}_2)$. If $n>2$, W assumes every value intermediate infinitely often and at least once in the plane of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.*

Thus, $W(\boldsymbol{u})$ is a measure of how far F deviates from symmetry in the direction perpendicular to $\boldsymbol{u}$. $W(\cdot)$ contains all the information necessary to describe the asymmetric nature of F. There are three cases;

(1) $W(\boldsymbol{u})=0$, $\forall \boldsymbol{u} \in S(0, 1)$ iff F is symmetric about a point.

(2) $W(\boldsymbol{u})=0$ for exactly two values $\boldsymbol{u}_1, \boldsymbol{u}_2 \in S(0, 1)$, $\boldsymbol{u}_2=-\boldsymbol{u}_1$ iff F is symmetric about a line perpendicular to $\boldsymbol{u}_1$.

(3) $W(\boldsymbol{u})>0$, $\forall \boldsymbol{u} \in S(0, 1)$ iff F is asymmetric in every direction.

Let $\mathscr{F}^*$ be the class of all standardized distribution functions on R^2 with bounded support, and let $C[S(0, 1)]$ be the class of continuous functions on the unit circle $S(0, 1)$. We then can define a mapping from $\mathscr{F}^*$ into $C[S(0, 1)]$ in which each distribution function F is mapped into its width function W_F. On $C[S(0, 1)]$ we can use the following metric: For $F, G \in \mathscr{F}^*$,

$$d(W_F, W_G)=\max\{|W_F(\boldsymbol{u})-W_G(\boldsymbol{u})|:\ \boldsymbol{u} \in S(0, 1)\}\ ,$$

Let $F^{\boldsymbol{\theta}} \in \mathscr{F}^*$ be any distribution symmetric about $\boldsymbol{\theta} \in R^2$. Then $W_{F^{\boldsymbol{\theta}}}(\boldsymbol{u})$

$\equiv 0$. For any $F \in \mathcal{F}^*$, we can thus measure "how far" F is from symmetry by

$$\begin{aligned} d(W_{F\theta}, W_F) &= \max\{|W_{F\theta}(\boldsymbol{u}) - W_F(\boldsymbol{u})| : \boldsymbol{u} \in S(0, 1)\} \\ &= \max\{W_F(\boldsymbol{u}) : \boldsymbol{u} \in S(0, 1)\} \\ &= \text{diameter of } \bar{L}_F . \end{aligned}$$

Thus, there is some justification in considering the diameter of $\bar{L}_F$ as the measure of asymmetry of F.

4. Conclusion

The construction of a location region in the plane has been accomplished by reducing a bivariate problem to a univariate problem and then applying the results Doksum has obtained for the univariate case. This may be done by considering the univariate marginal distributions in Section 2.

These same procedures can be applied to the general multivariate case. Thus, with the same definition of stochastic ordering extended to n-dimensional random vectors, hyperrectangles can be constructed by forming the Cartesian product of the ranges of the marginal symmetry functions. By then considering all orthogonal transformations of the random vector, an analogous averaging can be done to obtain the location region.

The extensions to higher dimensions of the width function and ordering of distributions are also apparent. It is clear that there is still much work to be done in both of these applications.

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