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THE DISTRIBUTION OF THE SUM OF INDEPENDENT GAMMA RANDOM VARIABLES

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Summary

The distribution of the sum of n independent gamma variates with different parameters is expressed as a single gamma-series whose coefficients are computed by simple recursive relations.

1. Introduction

The distribution of the sum of n independent gamma random variables has been investigated in a recent paper, Mathai [2]. This distribution has applications in queueing type problems. For example one is interested in the total waiting time $X_1+X_2+\cdots+X_n$ where the component times may be assumed independent exponential or more generally gamma distributed variables. It also has applications in engineering. For example the total excess water-flow into a dam is $X_1+X_2+\cdots+X_n$ where X_i represents the ith excess flow at occasion i, and the X_i's may be assumed independent gamma with distinct parameters.

Let $\{X_i\}$, $i=1,\cdots,n$ be a set of mutually independent gamma variates with parameters $\alpha_i>0$ and $\beta_i>0$. Then the density of X_i is given by

$$f_i(x_i)=x_i^{\alpha_i-1}e^{-x_i/\beta_i}/[\beta_i^{\alpha_i}\Gamma(\alpha_i)]\ , \qquad x_i>0 \tag{1.1}$$

and $f_i(x_i)=0$ elsewhere. Mathai [2] has given a number of expressions for the density of $Y=X_1+X_2+\cdots+X_n$, including: a) a finite sum representation by using a partial fraction technique when all the α_i's are integers, and b) a series in terms of zonal polynomials when all the α_i's are equal. In the general case in which the α_i's are distinct and the β_i's are also distinct, the density was expressed in terms of a confluent hypergeometric function in $n-1$ variables (see Mathai and Saxena [3], p. 163, for a definition of this function).

Key words: Gamma distribution, series representation, moment generating function.

In this note it is shown that a variation of Mathai's method of inverting the moment generating function, leads to a single gamma-series for the density and distribution function of Y. It will be seen in the next section that the new representation is very convenient for computational purposes since the coefficients are easily computed by simple recursive relations. Moreover a bound for the truncation error is readily obtainable.

2. The exact density of Y

Since the X_i's are independent, the m.g.f. (moment generating function) of Y is the product of the m.g.f.'s of the X_i's, i.e.

$$M(t)=\prod_{i=1}^{n}(1-\beta_i t)^{-\alpha_i}\,. \tag{2.1}$$

Without loss of generality, assume that $\beta_1=\min(\beta_i)$. Application of the identity

$$1-\beta_i t=(1-\beta_1 t)(\beta_i/\beta_1)[1-(1-\beta_1/\beta_i)/(1-\beta_1 t)] \tag{2.2}$$

to (2.1) gives,

$$\log M(t)=\log[C\cdot(1-\beta_1 t)^{-\rho}]+\sum_{k=1}^{\infty}\gamma_k(1-\beta_1 t)^{-k} \tag{2.3}$$

where

$$C=\prod_{i=1}^{n}(\beta_1/\beta_i)^{\alpha_i} \tag{2.4}$$

$$\gamma_k=\sum_{i=1}^{n}\alpha_i(1-\beta_1/\beta_i)^k/k\,,\qquad k=1,2,\cdots \tag{2.5}$$

$$\rho=\sum_{i=1}^{n}\alpha_i>0\,.$$

The expression is valid for all t such that $\max_i|(1-\beta_1/\beta_i)/(1-\beta_1 t)|<1$. Thus, $M(t)$ can be expressed as

$$M(t)=C(1-\beta_1 t)^{-\rho}\exp\left(\sum_{k=1}^{\infty}\gamma_k(1-\beta_1 t)^{-k}\right)\,. \tag{2.6}$$

We now let

$$\exp\left(\sum_{k=1}^{\infty}\gamma_k(1-\beta_1 t)^{-k}\right)=\sum_{k=0}^{\infty}\delta_k(1-\beta_1 t)^{-k}\,. \tag{2.7}$$

Upon differentiating (2.7) with respect to $(1-\beta_1 t)^{-1}$, it follows that the coefficients δ_k can be obtained recursively by the formula,

$$\delta_{k+1}=\frac{1}{k+1}\sum_{i=1}^{k+1} i\gamma_i\delta_{k+1-i}\ , \qquad k=0, 1, 2, \cdots \tag{2.8}$$

with $\delta_0=1$. Thus, on using (2.7) and inverting (2.6) term-by-term we can obtain a gamm-series representation for the density of Y.

THEOREM 1. *If* $\{X_i\}$, $i=1, \cdots, n$ *are independently distributed as in* (1.1), *then the density of* $Y=X_1+\cdots+X_n$ *can be expressed as*

$$g(y)=C\sum_{k=0}^{\infty} \delta_k y^{\rho+k-1}e^{-y/\beta_1}/[\Gamma(\rho+k)\beta_1^{\rho+k}]\ , \qquad y>0 \tag{2.9}$$

and 0 *elsewhere, where* $\rho=\sum_{i=1}^{n}\alpha_i$, C *is given in* (2.4) *and* δ_k *in* (2.8).

The distribution function $G(w)=\Pr(Y\leqq w)$ is readily available from (2.9) by term-by-term integration, i.e.

$$G(w)=C\sum_{k=0}^{\infty}\delta_k\int_0^w (y^{\rho+k-1}e^{-y/\beta_1}/[\Gamma(\rho+k)\beta_1^{\rho+k}])dy\ . \tag{2.10}$$

The interchange of the integration and summation above will be justified from the uniform convergence which we now establish.

For $i=1, 2, \cdots$, and $b=\max_{2\leqq j\leqq n}(1-\beta_1/\beta_j)$ we have

$$|\gamma_i|=\sum_{j=1}^{n}\alpha_j(1-\beta_1/\beta_j)^i/i\leqq\rho b^i/i\ .$$

Thus, from (2.8) we obtain

$$|\delta_{k+1}|\leqq(\rho/(k+1))\sum_{i=1}^{k+1} b^i|\delta_{k+1-i}|\ , \qquad k=0, 1, 2, \cdots$$

from which it follows by induction that

$$|\delta_{k+1}|\leqq b^{k+1}(\rho)_{k+1}/(k+1)!\ , \tag{2.11}$$

where $(\rho)_k=\rho(\rho+1)\cdots(\rho+k-1)$, $(\rho)_0=1$. Hence,

$$\begin{aligned} g(y)&=(C\beta_1^{-\rho}/\Gamma(\rho))y^{\rho-1}e^{-y/\beta_1}\sum_{k=0}^{\infty}(\delta_k/(\rho)_k)(y/\beta_1)^k \\ &\leqq(C\beta_1^{-\rho}/\Gamma(\rho))y^{\rho-1}e^{-y/\beta_1}\sum_{k=0}^{\infty}(by/\beta_1)^k/k! \\ &=(C\beta_1^{-\rho}/\Gamma(\rho))y^{\rho-1}e^{-y(1-b)/\beta_1} \end{aligned} \tag{2.12}$$

which proves the uniform convergence of (2.9) and justifies (2.10).

For practical purposes, one may use the first $m+1$, i.e. $k=m$, terms of the series (2.10) where m is such that the desired accuracy is attained. (Routines for the computation of the incomplete gamma integral are widely available, e.g. IMSL MDGAM.) A bound for the truncation error may be obtained conveniently by using (2.12) as

$$E_m(w)=(C\beta_1^{-\rho}/\Gamma(\rho))\int_0^w y^{\rho-1}e^{-y(1-b)/\beta_1}dy-G_m(w)$$

where $G_m(w)$ is the sum of the first $m+1$ terms of (2.10) for $k=0, 1, \cdots, m$.

It should be noted that the present method is also applicable to linear combinations of independent gammas or exponentials (by rescaling) and also linear combinations of independent central chi-squares; single-series representations for the last case are found in Ruben [4] and Kotz et al. [1].

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