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LOCAL POWERS OF TWO-SAMPLE AND MULTI-SAMPLE RANK TESTS FOR LEHMANN'S CONTAMINATED ALTERNATIVE

TAKA-AKI SHIRAISHI

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Summary

Local powers of two- and k-sample rank tests under alternatives of contaminated distributions are investigated. It is shown that the rank tests based on normal scores and Wilcoxon scores are superior to the t-test or the F-test for many choices of alternatives of contaminated distributions and that the values of the asymptotic relative efficiency of the rank test based on Wilcoxon scores with respect to the normal scores are about one in all the investigated cases.

1. Introduction

Let $\{X_{ij}: j=1,\cdots,n_i\}$ be a random sample from a population with unknown and absolutely continuous distribution function $F_i(t)$ for $i=1,2$. Let R_{ij} be the rank of X_{ij} among the overall observations $\{X_{ij}: j=1,\cdots,n_i,\ i=1,2\}$ and let $a_N(\cdot)$ be a scores function defined on $\{1,\cdots,N\}$, where $N=\sum_{i=1}^{2} n_i$. Our problem is to test the null hypothesis $H: F_1(t)=F_2(t)$, based on rank R_{ij} and scores function $a_N(\cdot)$. Then Hoeffding [3] considered a class of rank statistic

$$(1.1)\qquad S=\sum_{j=1}^{n_2} a_N(R_{2j})$$

against the one-sided location alternative and showed that the locally most powerful rank test exists in the class. Chernoff and Savage [1] derived the asymptotic powers of the tests based on S under a sequence of location alternatives converging to the null hypothesis. But as, in many cases, observations after receiving treatment may have a distorted distribution which cannot be represented by a simple location alternative, we consider, in this paper, the hypothesis $K: F_i(t)=(1-\theta_i)F(t)+$

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$\theta_i H(t)$ for $i=1, 2$ with $\theta_1=0$ and $\theta_2>0$, where $F(t)$ and $H(t)$ are assumed to be absolutely continuous distribution functions. This is an alternative of a contaminated distribution and is an extension of so-called Lehmann's alternative (Lehmann [4]). In order to simplify expressions of locally most powerful rank tests and asymptotic distributions, without loss of generality, we always assume that $H(t)=G(F(t))$ in this paper, where $G(u)$ is a distribution function on [0, 1] with density $g(u)$.

In Section 2, we derive locally most powerful rank tests against K defined above. In Section 3, we derive the powers of the rank tests based on S under the contiguous sequence of alternatives of contaminated distributions $K_N: F_i(t)=(1-\Delta_i/\sqrt{N})F(t)+(\Delta_i/\sqrt{N})G(F(t))$ for $i=1$, 2 with $\Delta_1=0$ and $\Delta_2>0$, and compare the tests with the most powerful test or the likelihood ratio t-test for the equality of means in two normal populations. We find that the asymptotic power of the test based on S given by the best scores function is close to that of the most powerful test if the ratio of first sample size n_1 to pooled sample size N is close to 1.

Further we investigate a multi-sample case, using the same notations as in the two-sample case. Let $\{X_{ij}: j=1,\cdots, n_i\}$ be a random sample from a population with unknown and absolutely continuous distribution function $F_i(t)$ for $i=1,\cdots, I$. Let R_{ij} be the rank of X_{ij} among the overall observations $\{X_{ij}: j=1,\cdots, n_i,\ i=1,\cdots, I\}$ and let $N=\sum_{i=1}^{I} n_i$. In order to test the null hypothesis $H: F_1(t)=\cdots=F_I(t)$, Puri [5] considered the quadratic rank test statistic

$$T=(N-1)\sum_{i=1}^{I} n_i\{\bar{a}_N(R_{i\cdot})-\bar{a}_N\}^2\Big/\left[\sum_{k=1}^{N}\{a_N(k)-\bar{a}_N\}^2\right], \tag{1.2}$$

where $\bar{a}_N(R_{i\cdot})=\sum_{j=1}^{n_i} a_N(R_{ij})/n_i$ and $\bar{a}_N=\sum_{k=1}^{N} a_N(k)/N$, and derived the limiting noncentral chi-square distribution of T under the sequence of location alternatives.

So in Section 4, we derive the asymptotic powers of the tests based on T under the contiguous sequence of alternatives $K_N': F_i(t)=(1-\Delta_i/\sqrt{N})F(t)+(\Delta_i/\sqrt{N})G(F(t))$, where $\Delta_i\geqq 0$ for all i and $\Delta_i\neq\Delta_j$ for some (i, j), and compare the tests with the likelihood ratio F-test.

Lastly in Section 5, using asymptotic relative efficiencies, the numerical comparisons of the rank tests with respect to the t-test or F-test are studied. It is shown that (i) the rank tests based on normal scores and Wilcoxon scores are superior to the t-test or the F-test for many choices of alternatives of contaminated distributions and (ii) the rank test based on sign scores is inferior to the t-test or the F-test for some distributions except for an exponential distribution. Further the asymptotic relative efficiencies of the rank test based on Wilcoxon scores with respect to the normal scores are nearly equal to one in all cases investigated.

2. Two-sample locally most powerful rank tests

In this section, we shall write the alternative hypothesis K: $F_1(t)=F(t)$ and $F_2(t)=(1-\theta)F(t)+\theta G(F(t))$ for $\theta>0$. Then the likelihood under K is given by

$$p_\theta(x)=\prod_{i=1}^{n_1} f(x_{1i}) \prod_{j=1}^{n_2} \{(1-\theta)f(x_{2j})+\theta g(F(x_{2j}))f(x_{2j})\} , \tag{2.1}$$

where $F'(t)=f(t)$ and $G'(u)=g(u)$. Then the probability of rank vector $\boldsymbol{R}=(R_{11},\cdots, R_{1n_1}, R_{21},\cdots, R_{2n_2})$ for a permutation $\boldsymbol{r}=(r_{11},\cdots, r_{1n_1}, r_{21},\cdots, r_{2n_2})$ of $(1, 2,\cdots, N)$ is expressed by

$$\begin{aligned}\Pr\{\boldsymbol{R}=\boldsymbol{r}\}&=\int\cdots\int_{\boldsymbol{R}=\boldsymbol{r}} p_\theta(x)dx \\ &=1/N!+\sum_{l=1}^{n_2}\int\cdots\int_{\boldsymbol{R}=\boldsymbol{r}} \prod_{i=1}^{n_1} f(x_{1i}) \prod_{j=1}^{l-1} f(x_{2j}) \\ &\quad\times[\{(1-\theta)f(x_{2l})+\theta g(F(x_{2l}))f(x_{2l})\}-f(x_{2l})] \\ &\quad\times\prod_{k=l+1}^{n_2} \{(1-\theta)f(x_{2k})+\theta g(F(x_{2k}))f(x_{2k})\}d\boldsymbol{x} .\end{aligned} \tag{2.2}$$

It follows that

$$\left.\frac{dP_\theta\{\boldsymbol{R}=\boldsymbol{r}\}}{d\theta}\right|_{\theta=0}=-n_2/N!+\sum_{l=1}^{n_2} \mathrm{E}\{g(U_N^{(r_{2l})})\} , \tag{2.3}$$

where $U_N^{(r)}$ is the rth order statistic in a sample of size N from the uniform distribution on $(0, 1)$. Hence the critical region $\sum_{j=1}^{n_2} \mathrm{E}\{g(U_N^{(R_{2j})})\} \geq s$ gives a locally most powerful rank test. Especially if $G(u)=u^2$ in alternative K, we get the two-sample Wilcoxon test which was studied by Lehmann [4].

3. Asymptotic property of S

In order to get an asymptotic property, we set the scores function $a_N(\cdot)$ the following

Assumption (1). The scores function $a_N(\cdot)$ satisfies

$$\lim_{N\to\infty}\int_0^1 \{a_N(1+[uN])-\phi(u)\}^2du=0 \tag{3.1}$$

for some square integrable function $\phi(u)$ with $\int_0^1 \left\{\phi(u)-\int_0^1 \phi(v)dv\right\}^2 du>0$, where $[v]$ denotes the largest integer not exceeding v.

The equation (3.1) is satisfied if we put $a_N(k)=\mathrm{E}\{\phi(U_N^{(k)})\}$, $N\int_{(k-1)/N}^{k/N}\phi(u)du$ or $\phi(k/(N+1))$ (see V.1.4 and 1.6 of Hájek and Šidák [2]). Then Hájek and Šidák [2] have shown that $\sqrt{N(N-1)}(S-n_2\bar{a}_N)\Big/\sqrt{n_1n_2\sum_{k=1}^{N}\{a_N(k)-\bar{a}_N\}^2}$ has asymptotically a standard normal distribution under H if $\lim_{N\to\infty}(n_2/N)=\alpha$ with $0<\alpha<1$. To get the asymptotic local power, we consider the sequence of density functions

$$p_\Delta(x)=\prod_{i=1}^{n_1} f(x_{1i})\prod_{j=1}^{n_2}\{(1-\Delta/\sqrt{N})f(x_{2j})+(\Delta/\sqrt{N})g(F(x_{2j}))f(x_{2j})\}\ , \tag{3.2}$$

where the null hypothesis H is specified by $\Delta=0$.

We can get the following

THEOREM 1. *Assume that density $g(u)=G'(u)$ is bounded and $\lim_{N\to\infty}(n_2/N)=\alpha$ with $0<\alpha<1$. If Assumption (1) is satisfied, the rank statistic $\sqrt{N(N-1)}(S-n_2\bar{a}_N)\sqrt{n_1n_2\sum_{k=1}^{N}\{a_N(k)-\bar{a}_N\}^2}$ has asymptotically a normal distribution with mean μ and variance 1 under $\{p_\Delta(x)\}$, where μ is given by*

$$\sqrt{\alpha(1-\alpha)}\,\Delta\,\mathrm{Cov}\,(\phi(U),\ g(U))/\sqrt{\mathrm{Var}\,(\phi(U))} \tag{3.3}$$

for random variable U having the uniform distribution on $(0, 1)$.

PROOF. Taylor's series expansion of the logarithm of the likelihood ratio obtained by (3.2) yields

$$\begin{aligned} L_\Delta &=\log\{p_\Delta(X)/p_0(X)\} \\ &=\log\left[\prod_{j=1}^{n_2}\{(1-\Delta/\sqrt{N})+(\Delta/\sqrt{N})g(F(X_{2j}))\}\right] \\ &=\sum_{j=1}^{n_2}(\Delta/\sqrt{N})\{g(U_{2j})-1\}-\sum_{j=1}^{n_2}\{\Delta^2/(2N)\}\{g(U_{2j})-1\}^2 \\ &\quad+\sum_{j=1}^{n_2}\{\Delta^3/(3N\sqrt{N})\}[\{g(U_{2j})-1\}^3/\{1+\delta_j(\Delta/\sqrt{N})(g(U_{2j})-1)\}^3]\ , \end{aligned} \tag{3.4}$$

where $U_{2j}=F(X_{2j})$ and δ_j satisfies $0<\delta_j<1$ for $j=1,\cdots,n_2$. Under H, the first term of the last expression in (3.4), namely, $\sum_{j=1}^{n_2}(\Delta/\sqrt{N})\{g(U_{2j})-1\}$ has asymptotically a normal distribution with mean 0 and variance $\alpha\Delta^2\,\mathrm{Var}\,(g(U))$ from the central limit theorem, the second term converges to $\alpha\Delta^2\,\mathrm{Var}\,(g(U))/2$ in probability from the law of large numbers and the third term converges to 0. Thus we get

$$L_\Delta \xrightarrow[\mathcal{L}]{} N(\mu_1, \sigma_1^2) \tag{3.5}$$

where $\xrightarrow[\mathcal{L}]{}$ denotes convergence in law and $\mu_1 = -\alpha\Delta^2 \operatorname{Var}(g(U))/2$ and $\sigma_1^2 = -2\mu_1$. Hence from VI.1.2 corollary of Hájek and Šidák [2], the family of densities $\{p_\Delta(x)\}$ is contiguous to $\{p_0(x)\}$. On the other hand, let us put $T = \sum_{j=1}^{n_2} \{\phi(U_{2j}) - \bar{\phi}(U)\}/\sqrt{N}$ where $\bar{\phi}(U) = \sum_{i=1}^{2} \sum_{j=1}^{n_i} \phi(U_{ij})/N$ and $U_{ij} = F(X_{ij})$, then we find that $(S - n_2\bar{a}_N)/\sqrt{N} - T$ converges to zero in probability under H in the proof of V.1.5 theorem a of Hájek and Šidák [2]. Hence $(L_\Delta, (S - n_2\bar{a}_N)/\sqrt{N})$ and (L_Δ, T) have asymptotically the same normal distribution under H. Also under H, (L_Δ, T) has asymptotically a bivariate normal distribution with mean $(\mu_1, 0)$ and covariance matrix $\begin{pmatrix} -2\mu_1 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$ where $\sigma_{12} = \alpha(1-\alpha)\Delta \operatorname{Cov}(g(U), \phi(U))$ and $\sigma_2^2 = \alpha(1-\alpha) \operatorname{Var}(\phi(U))$. So from LeCam's third lemma stated in VI.1.4 of Hájek and Šidák [2], under $\{p_\Delta(x)\}$, $(S - n_2\bar{a}_N)/\sqrt{N}$ has asymptotically a normal distribution with mean σ_{12} and variance σ_2^2. Since $\lim_{N\to\infty} \left[\sum_{k=1}^{N} \{a_N(k) - \bar{a}_N\}^2/(N-1)\right] = \int_0^1 \left\{\phi(u) - \int_0^1 \phi(v)dv\right\}^2 du$ from Assumption (1), normalizing the statistic $(S - n_2\bar{a}_N)/\sqrt{N}$ by σ_2 gives the desired result.

If we modify the assumptions of VI. problem 7 of Hájek and Šidák [2], we find that the result of Theorem 1 is equivalent to that of the problem. But in order to prove next Corollaries 1 and 2, the arguments on the proof of Theorem 1 are used.

The asymptotic power of the test based on S in Theorem 1 is maximized when $\phi(u) = g(u)$ in (3.3), which is induced by $a_N(k) = \mathrm{E}\{g(U_N^{(k)})\}$ giving the locally most powerful rank test. We shall compare this locally most powerful rank test with the most powerful test against the alternative of density $p_\Delta(x)$ having critical region $\{L_\Delta \geq t_{N_r}\}$, where $p_\Delta(x)$ and L_Δ are defined by (3.2) and (3.4) respectively. Then we get

COROLLARY 1. *Under the assumptions of Theorem* 1, *the asymptotic relative efficiency of the locally most powerful rank test with respect to the most powerful test based on* L_Δ *under* $\{p_\Delta(x)\}$ *is given by* $\operatorname{ARE}(S, L_\Delta) = 1 - \alpha$.

PROOF. From expression (3.5) in the proof of Theorem 1 and LeCam's third lemma, under $\{p_\Delta(x)\}$,

$$L_\Delta \xrightarrow[\mathcal{L}]{} N(\mu_2, \sigma_2^2) \tag{3.6}$$

where U is defined by a random variable having the uniform distribution on $(0, 1)$ and $\mu_2 = \alpha\Delta^2 \operatorname{Var}(g(U))/2$ and $\sigma_2^2 = 2\mu_2$. Expressions (3.5) and (3.6) show that the asymptotic power of the test based on L_Δ is

$$(3.7) \qquad 1-\Phi(t_\gamma-\Delta\sqrt{\alpha\,\mathrm{Var}\,(g(U))})\ ,$$

where t_γ is the upper 100γ percentage point of the standard normal distribution. Comparing asymptotic power of S in Theorem 1 with (3.7), we get the desired result.

When $\lim_{N\to\infty}(n_2/N)=\alpha$ is small, we find that the asymptotic power of the locally most powerful rank test is nearly equal to the power of the most powerful test irrespective of the value of Δ from Corollary 1. Further though we may doubt that the asymptotic relative efficiency depends on α, the reason is that the rank test is not compared with the t-test but with the most powerful test. Also the asymptotic relative efficiency of the locally most powerful rank test with respect to the most powerful test for H versus K'_N: $F_1(t)=F(t)$ and $F_2(t)=F(t-\Delta/\sqrt{N})$ is equal to $1-\alpha$. So we assent to the result of Corollary 1.

The likelihood ratio test for the equality of means with a common variance under two normal populations is to reject H when the following t-statistic is too large.

$$(3.8) \qquad \sqrt{n_1n_2(N-2)/N}\,(\bar{X}_{2\cdot}-\bar{X}_{1\cdot})\Big/\sqrt{\sum_{j=1}^{n_1}(X_{1j}-\bar{X}_{1\cdot})^2+\sum_{j=1}^{n_2}(X_{2j}-\bar{X}_{2\cdot})^2}$$

where $\bar{X}_{i\cdot}=\sum_{j=1}^{n_i}X_{ij}/n_i$ for $i=1, 2$.

COROLLARY 2. *Under the assumptions of Theorem* 1, *the asymptotic relative efficiency of the test based on S given by* (1.1) *with respect to the t-test given by* (3.8) *under* $\{p_\Delta(x)\}$ *is*

$$(3.9) \qquad \sigma^2\{\mathrm{Cov}\,(\phi(U), g(U))\}^2/[\{\mathrm{Var}\,(\phi(U))\}\{\mathrm{Cov}\,(g(F(X)), X)\}^2]\ ,$$

where X has a distribution function F(t) and $U=F(X)$ *and* $\sigma^2=\mathrm{Var}\,(X)$.

PROOF. The similar argument as in the proof of Theorem 1 shows that the t-test has asymptotically a normal distribution with mean $\sqrt{\alpha(1-\alpha)}\,\Delta\,\mathrm{Cov}\,(g(F(X)), X)/\sigma$ and variance 1 under $\{p_\Delta(x)\}$. The ratio of squares of asymptotic means for T and the t-statistic gives the result.

4. Asymptotic property of multi-sample rank test T

The likelihood ratio test for the equality of means with a common variance in normal populations more than two is to reject H when the following F-statistic, except for constant factor, is too large.

$$(4.1) \qquad (N-1)\sum_{i=1}^{I}n_i(\bar{X}_{i\cdot}-\bar{X}_{\cdot\cdot})^2\Big/\sum_{i=1}^{I}\sum_{j=1}^{n_i}(X_{ij}-\bar{X}_{i\cdot})^2\ ,$$

where $\bar{X}_{i\cdot}=\sum_{j=1}^{n_i} X_{ij}/n_i$ for $i=1,\cdots,I$ and $\bar{X}_{\cdot\cdot}=\sum_{i=1}^{I}\sum_{j=1}^{n_i} X_{ij}/N$. This test is asymptotically distribution-free under H. We consider the following sequence of density functions corresponding to (3.2)

$$q_\Delta(x)=\prod_{i=1}^{I}\prod_{j=1}^{n_i}\{(1-\Delta_i/\sqrt{N})f(x_{ij})+(\Delta_i/\sqrt{N})g(F(x_{ij}))f(x_{ij})\}\,, \tag{4.2}$$

where $\Delta_i \geqq 0$ for all i and the null hypothesis H is specified by $\Delta_1=\cdots=\Delta_I$. By the similar argument as in the proofs of Theorem 1 and Corollary 1, we get

THEOREM 2. *Suppose that $g(u)$ is bounded and $\lim_{N\to\infty}(n_i/N)=\alpha_i$ with $\alpha_i>0$ for all i. If Assumption* (1) *is satisfied, under $\{q_\Delta(x)\}$, the rank statistic T defined by* (1.2) *has asymptotically a noncentral chi-square distribution with $(I-1)$ degrees of freedom and noncentrality parameter*

$$[\{\mathrm{Cov}\,(g(U),\phi(U))\}^2/\mathrm{Var}\,(\phi(U))]\sum_{i=1}^{I}\alpha_i\Big(\Delta_i-\sum_{j=1}^{I}\alpha_j\Delta_j\Big)^2\,. \tag{4.3}$$

Further the asymptotic relative efficiency of the test based on T with respect to the F-test under the alternative densities $\{q_\Delta(x)\}$ is given by the same formula (3.9) *in Corollary* 2.

5. Numerical results of ARE's

Using the normal scores function in $a_N(\cdot)$, Chernoff and Savage [1] showed that the asymptotic relative efficiency of the rank test based on S (T) with respect to the t-test (F-test) against the sequence of location alternatives is equal to 1 under the normal distribution and is always larger than 1 under the other distributions.

On the other hand, under the alternatives of contaminated distributions discussed in this paper, Corollary 2 (Theorem 2) implies that the asymptotic relative efficiency of the rank test based on S (T) with respect to the t-test (F-test) is equal to 1 irrespective of $G(u)$; (1) if normal score is used for $a_N(k)$ and $F(x)$ is normal or (2) if Wilcoxon score $2k/(N+1)-1$ is used for $a_N(k)$ and $F(x)$ is the distribution function from the uniform random variable on a finite interval.

In Table 1, we show the ARE of the rank test S (T) with respect to t-test (F-test) for $G(u)=u^k$ or $1-(1-u)^k$ with $k=1.1$, 1.3, 1.6, 2, 3, 5, 10; $F(x)=$uniform, normal, logistic, double exponential, exponential distributions; and $a_N(\cdot)=$Wilcoxon, normal, sign scores. From Table 1, we can see that except for exponential distribution (1) for Wilcoxon scores, ARE's are always nearly equal to 1 irrespective of the form of $G(u)$, $F(x)$ and k chosen; (2) for normal scores, ARE's are slightly larger

Table 1. Values of the asymptotic relative efficiencies of the test based on S (T) with respect to the t-test (F-test) under the alternative distributions $\prod_{i=1}^{I}\prod_{j=1}^{n_i}[(1-\Delta_i/\sqrt{N})F(x_{ij})+(\Delta_i/\sqrt{N})G(F(x_{ij}))]$ with $G(u)=u^k$ or $1-(1-u)^k$.

k	Distribution of $F(t)$				
	Uniform	Normal	Logistic	Double-exponential	Exponential
		Wilcoxon scores $a_N(k)=2k/(N+1)-1$			
1.1	1	.946	.950	.974	1.739
1.3	1	.986	1.009	1.067	1.624
1.6	1	1.024	1.063	1.134	1.483
2	1	1.047	1.097	1.185	1.333
3	1	1.047	1.097	1.185	1.080
5	1	.986	1.011	1.057	.810
10	1	.847	.826	.804	.487
		Normal scores $a_N(k)=\mathrm{E}\{Z_N^{(k)}\}$			
1.1	1.057	1	1.004	1.030	1.838
1.3	1.014	1	1.023	1.070	1.647
1.6	.977	1	1.038	1.108	1.448
2	.955	1	1.047	1.132	1.273
3	.955	1	1.047	1.132	1.031
5	1.014	1	1.025	1.072	.821
10	1.179	1	.973	.947	.575
		Sign scores $a_N(k)=\mathrm{sign}(2k/(N+1)-1)$			
1.1	.659	.624	.626	.642	1.146
1.3	.691	.681	.697	.729	1.122
1.6	.725	.742	.770	.822	1.074
2	.750	.785	.822	.889	1.000
3	.750	.785	.822	.889	.810
5	.659	.650	.666	.697	.534
10	.496	.421	.409	.399	.242

than 1 for small k; and (3) for sign scores, ARE's are always smaller than 1 irrespective of $G(u)$, $F(x)$ and k. If $F(x)$ is an exponential distribution, ARE's are always larger than 1 for all of the above scores with small k. In Table 2, we show the asymptotic relative efficiencies

Table 2. Values of the asymptotic relative efficiency of the Wilcoxon test with respect to the normal scores test under the alternative distributions $\prod_{i=1}^{I}\prod_{j=1}^{n_i}[(1-\Delta_i/\sqrt{N})F(x_{ij})+(\Delta_i/\sqrt{N})G(F(x_{ij}))]$ with $G(u)=u^k$ or $1-(1-u)^k$.

k	1.1	1.3	1.6	2	3	5	10
ARE	.946	.986	1.024	1.027	1.027	.986	.848

of the rank test based on Wilcoxon scores with respect to the rank test based on normal scores for various $G(u)$ and k used in Table 1 since the efficiencies do not depend on $F(x)$. From Table 2, we find that ARE's always about 1.

As a conclusion, the Wilcoxon rank test has generally no loss of the relative efficiency even against the alternative hypothesis of contaminated distributions discussed here. The ARE of the Wilcoxon rank test with respect to the normal scores rank test is nearly equal to 1. However Wilcoxon test has a simple form and is distribution-free.

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UNIVERSITY OF TSUKUBA

REFERENCES

[1] Chernoff, H. and Savage, I. R. (1958). Asymptotic normality and efficiency of certain nonparametric test statistics, *Ann. Math. Statist.*, **29**, 972-994.

[2] Hájek, J. and Šidák, Z. (1967). *Theory of Rank Tests*, Academic Press.

[3] Hoeffding, W. (1950). "Optimum" non-parametric tests, *Proc. 2nd Berkeley Symposium*, 83-92.

[4] Lehmann, E. L. (1953). The power of rank tests, *Ann. Math. Statist.*, **24**, 23-43.

[5] Puri, M. L. (1964). Asymptotic efficiency of a class of *c*-sample tests, *Ann. Math. Statist.*, **35**, 102-121.