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MEAN INTEGRATED SQUARED ERROR OF KERNEL ESTIMATORS WHEN THE DENSITY AND ITS DERIVATIVE ARE NOT NECESSARILY CONTINUOUS*

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Summary

Asymptotic properties of the mean integrated squared error (MISE) of kernel estimators of a density function, based on a sample $X_1, \cdots, X_n$, were obtained by Rosenblatt [4] and Epanechnikov [1] for the case when the density f and its derivative f' are continuous. They found, under certain additional regularity conditions, that the optimal choice h_{n0} for the scale factor $h_n = Kn^{-\alpha}$ is given by $h_{n0} = K_0 n^{-1/5}$ with K_0 depending on f and the kernel; they also showed that MISE $(h_{n0}) = O(n^{-4/5})$ and Epanechnikov [1] found the optimal kernel.

In this paper we investigate the robustness of these results to departures from the assumptions concerning the smoothness of the density function. In particular it is shown, under certain regularity conditions, that when f is continuous but its derivative f' is not, the optimal value of α in the scale factor becomes 1/4 and MISE $(h_{n0}) = O(n^{-3/4})$; for the case when f is not continuous the optimal value of α becomes 1/2 and MISE $(h_{n0}) = O(n^{-1/2})$. For this last case the optimal kernel is shown to be the double exponential density.

1. Introduction

Let $X_1, \cdots, X_n$ be independent, identically distributed random variables with density function f and let

$$(1.1) \qquad f_n(x) = \frac{1}{nh_n} \sum_{i=1}^{n} w\left(\frac{x - X_i}{h_n}\right), \qquad -\infty < x < \infty$$

be a kernel estimator of f. Rosenblatt [4] and Epanechnikov [1] (see

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also Rosenblatt [5]) studied the asymptotic behaviour of the mean integrated squared error (MISE) of f_n; they found the optimal value of h_n, the asymptotically minimum value of MISE and the optimal kernel, assuming, among other regularity conditions, that w is a density symmetric around zero and that f has two continuous derivatives. Nadaraya [3] extended the results of Rosenblatt and Epanechnikov to the case where f has s ($s \geqq 2$, s even) derivatives with $f^{(s)}$ bounded.

In this paper the results of Rosenblatt and Epanechnikov are extended to the case where the density f and its derivative f' are not necessarily continuous; a bounded density, symmetric around zero, is used for the kernel and it will be shown, under certain additional regularity conditions, that, with h_{n0} the asymptotically optimum value of h_n, $\text{MISE}(h_{n0}) = O(n^{-3/4})$ if f is continuous and f' is not continuous, whereas $\text{MISE}(h_{n0}) = O(n^{-1/2})$ if f is not continuous. Expressions for $\lim_{n\to\infty} n^{3/4}$ $\text{MISE}(h_{n0})$, respectively $\lim_{n\to\infty} n^{1/2}\,\text{MISE}(h_{n0})$, in terms of w and f will be given; further it will be shown that the double exponential density is the optimal kernel for the case when f is not continuous.

Section 2 contains the conditions on w and f, as well as some properties of, and some examples of, densities f satisfying these conditions. The main results are given in Section 3; a sketch of the proofs of these results is given in Section 4. Full details of these proofs can be found in the technical report by van Eeden [6]; this report is available from the author on request.

2. The conditions on w and f

The kernel w will be assumed to satisfy

Condition A.

w is a bounded density, symmetric around zero with $0 < \int_{-\infty}^{+\infty} t^2 w(t)dt < \infty$.

The density f will be assumed to satisfy

Condition B.

1. f has k ($k \geqq 0$) points of discontinuity $a_1 < \cdots < a_k$; at each of these points f has a left-hand and a right-hand limit and $f(a_i^-) \neq f(a_i^+)$, $i=1, \cdots, k$.
2. For each $i=1, \cdots, k+1$, the function g_i defined on $[a_{i-1}, a_i]$ by

$$g_i(x) = \begin{cases} f(a_{i-1}^+) & \text{if } x = a_{i-1} \\ f(x) & \text{if } a_{i-1} < x < a_i \\ f(a_i^-) & \text{if } x = a_i , \end{cases}$$

where $a_0=-\infty$, $a_{k+1}=\infty$, has, except at the points $b_{i1}<\cdots<b_{il_i}$ ($l_i\geqq 0$, $a_{i-1}<b_{i1}$, $b_{il_i}<a_i$), a derivative g_i'. At each of the points $b_{i1},\cdots,b_{il_i}$, g_i has a left-hand and a right-hand derivative; $g_i'(b_{ij}^-)\neq g_i'(b_{ij}^+)$, $j=1,\cdots,l_i$ and

$$g_i(y)-g_i(x)=\int_x^y g_i'(u)du\ , \qquad a_{i-1}\leqq x<y\leqq a_i\ .$$

Further f' is integrable.

3. For each pair (i,j), $i=1,\cdots,k+1$, $j=1,\cdots,l_i+1$, the function g_{ij} defined on $[b_{ij-1},b_{ij}]$ by

$$g_{ij}(x)=\begin{cases} g_i'(b_{ij-1}^+) & \text{if } x=b_{ij-1} \\ g_i'(x) & \text{if } b_{ij-1}<x<b_{ij} \\ g_i'(b_{ij}^-) & \text{if } x=b_{ij}\ , \end{cases}$$

where $b_{i0}=a_{i-1}$ and $b_{il_i+1}=a_i$, is absolutely continuous and g_{ij}' is continuous almost everywhere. Further $\int_{-\infty}^{+\infty}|f''(x)|dx<\infty$ and g_{ij}' is bounded.

The following Lemmas 2.2 and 2.3, needed for the proofs of the main results, give some properties of densities satisfying Condition B. Lemma 2.1 is needed for the proofs of the Lemmas 2.2 and 2.3.

LEMMA 2.1. *If $G(x)$, $-\infty<x<\infty$, is absolutely continuous and G and G' are integrable, then*

$$\text{(2.1)} \qquad \begin{aligned} &\text{(a)}\quad G(x)=\int_{-\infty}^x G'(y)dy \qquad -\infty<x<\infty \\ &\textit{and} \\ &\text{(b)}\quad G(x) \textit{ is bounded}\ . \end{aligned}$$

PROOF. (a) The absolute continuity of $G(x)$ implies that

$$\text{(2.2)} \qquad G(x)-G(y)=\int_y^x G'(u)du \qquad -\infty<y<x<\infty\ .$$

Further, the integrability of G and G' and the absolute continuity of G imply (see the proof of Lemma I.2.4.a of Hájek and Sidák [2])

$$\text{(2.3)} \qquad G(x)\to 0 \qquad \text{as } x\to\pm\infty\ .$$

The result then follows from (2.2) and (2.3).

(b) The boundedness of G follows from (2.3) and the fact that G is absolutely continuous, and hence of bounded variation, on finite intervals.

LEMMA 2.2. i) *For each $i=1,\cdots,k+1$ the function g_i is absolutely continuous on $[a_{i-1},a_i]$ and bounded.* ii) *For each pair (i,j), $i=1,\cdots,k+1$, $j=1,\cdots,l_i+1$, the function g_{ij} is bounded.*

PROOF. For the absolute continuity of g_i see Hájek and Sidák ([2], Theorem I.2.1). For $i=2,\cdots,k$ the fact that g_i is bounded follows from its definition. For $i=1$ and for $i=k+1$ the boundedness of g_i follows from Condition B.2 and Lemma 2.1. For all but the pairs ($i=1$, $j=1$) and ($i=k+1$, $j=l_{k+1}+1$) the fact that g_{ij} is bounded follows from its definition. That g_{11} and $g_{k+1,l_{k+1}+1}$ are bounded follows from Lemma 2.1 and Condition B.3.

Note that Lemma 2.2 i) implies that f is bounded, which implies that f is square integrable. Further, Lemma 2.2 ii) implies that f' is bounded and Condition B.3 implies that f'' is bounded and square integrable. Also, each of the functions f, f' and f'' is continuous almost everywhere. If $k=0$ then $f(x)$, $-\infty<x<\infty$, is, by Lemma 2.2 i), absolutely continuous and if $k=0$, $l_1=0$ then $f'(x)$, $-\infty<x<\infty$, is, by Condition B.3, absolutely continuous.

Now let

$$\begin{aligned} &\delta_i=f(a_i^-)-f(a_i^+) \qquad i=1,\cdots,k \\ &\Delta_{ij}=f'(b_{ij}^-)-f'(b_{ij}^+) \qquad i=1\cdots,k+1,\ j=1,\cdots,l_i+1 \end{aligned} \tag{2.4}$$

and

$$\delta=\sum_{i=1}^{k}\delta_i^2\,, \qquad \Delta=\sum_{i=1}^{k+1}\sum_{j=1}^{l_i+1}\Delta_{ij}^2 \tag{2.5}$$

then

LEMMA 2.3. *If the density f satisfies Condition B then*

$$\delta>0 \quad or \quad \Delta>0 \quad or \quad \int_{-\infty}^{+\infty}\{f''(x)\}^2dx>0\,. \tag{2.6}$$

PROOF. It is sufficient to prove that

$$\{\delta=0,\ \Delta=0\}\Longrightarrow\int_{-\infty}^{+\infty}\{f''(x)\}^2dx>0\,. \tag{2.7}$$

To prove (2.7) first note that

$$\begin{aligned} \{\delta=0,\ \Delta=0\}\Longrightarrow\{k=0,\ l_1=0\}\Longrightarrow\{&f(x),\ -\infty<x<\infty, \\ &\text{and } f'(x),\ -\infty<x<\infty, \text{ are absolutely continuous}\}\,. \end{aligned} \tag{2.8}$$

Then, by Lemma 2.1 and the fact that f, f' and f'' are integrable, one obtains that $\{\delta=0,\ \Delta=0\}$ implies

$$f(x)=\int_{-\infty}^{x}\left(\int_{-\infty}^{y}f''(z)dz\right)dy\,, \qquad -\infty<x<\infty\,. \tag{2.9}$$

Finally, $\int_{-\infty}^{+\infty}\{f''(x)\}^2dx=0$ contradicts (2.9) because f is a density.

The following are some examples of densities satisfying Condition B.

1. The normal, logistic and Cauchy densities satisfy Condition B with $k=l_1=0$, $\delta=\Delta=0$ and $\int_{-\infty}^{+\infty}\{f''(x)\}^2dx>0$,

2. the double exponential density $f(x)=(1/2)e^{-|x|}$, $-\infty<x<\infty$, satisfies Condition B with $k=0$, $l_1=1$, $\delta=0$, $\Delta>0$, $\int_{-\infty}^{+\infty}\{f''(x)\}^2dx>0$,

3. the density

$$f(x)=\begin{cases} 1+x & -1\leqq x\leqq 0 \\ 1-x & 0\leqq x\leqq 1 \end{cases}$$

satisfies Condition B with $k=0$, $l_1=3$, $\delta=\int_{-\infty}^{+\infty}\{f''(x)\}^2dx=0$ and $\Delta>0$,

4. the uniform density $f(x)=1/2$, $-1\leqq x\leqq 1$, satisfies Condition B with $k=2$, $l_1=l_2=l_3=0$, $\delta>0$ and $\Delta=\int_{-\infty}^{+\infty}\{f''(x)\}^2dx=0$,

5. the density

$$f(x)=\begin{cases} \dfrac{a+x}{2a-1} & -1\leqq x\leqq 0 \\[2ex] \dfrac{a-x}{2a-1} & 0\leqq x\leqq 1 \end{cases}$$

where $a>1$, satisfies Condition B with $k=2$, $l_1=l_3=0$, $l_2=1$, $\delta>0$, $\Delta>0$ and $\int_{-\infty}^{+\infty}\{f''(x)\}^2dx=0$,

6. an example of a density for which $\delta>0$, $\Delta=0$ and $\int_{-\infty}^{+\infty}\{f''(x)\}^2dx>0$ can be constructed as follows. Let f be a density satisfying Condition B with $k=l_1=0$. Then $\int_{-\infty}^{+\infty}\{f''(x)\}^2dx>0$. Let $c>0$ and let f^* be defined by

$$f^*(x)=\begin{cases} \dfrac{f(x)}{1+2c} & \text{for } |x|>1 \\[2ex] \dfrac{f(x)+c}{1+2c} & \text{for } |x|\leqq 1\,. \end{cases}$$

Then $f^*(x)$, $-\infty<x<\infty$, is a density; further

$$\frac{d}{dx}f^*(x)=\frac{f'(x)}{1+2c}$$

and f^* satisfies Condition B with $k=2$, $l_1=l_2=l_3=0$, $\delta>0$, $\Delta=0$ and

$$\int_{-\infty}^{+\infty}\{f^{*\prime\prime}(x)\}^2dx=\frac{1}{(1+2c)^2}\int_{-\infty}^{+\infty}\{f''(x)\}^2dx>0\ ,$$

7. an example where $\delta>0$, $\Delta>0$ and $\int_{-\infty}^{+\infty}\{f''(x)\}^2dx>0$ is the density

$$f(x)=\begin{cases}\frac{3}{8}[1+(x+1)^2] & -1\leqq x\leqq 0\\ \frac{3}{8}[1+(x-1)^2] & 0\leqq x\leqq 1\end{cases}$$

for which $k=2$, $l_1=l_3=0$, $l_2=1$.

3. The main results

The main results of this paper are given in the following Theorems 3.1, 3.2 and 3.3; the proofs of these theorems are sketched in Section 4.

THEOREM 3.1. *If the Conditions A and B are satisfied then, for* $n\to\infty$ *and* $h_n\to 0$,

$$\text{MISE}\,(h_n)=\begin{cases}\frac{1}{nh_n}\int_{-\infty}^{+\infty}w^2(t)dt+\frac{1}{4}h_n^4\left\{\int_{-\infty}^{+\infty}t^2w(t)dt\right\}^2\int_{-\infty}^{+\infty}\{f''(x)\}^2dx\\ \qquad+o\left(\frac{1}{nh_n}+h_n^4\right) \qquad \textit{if } \delta=\Delta=0\\ \frac{1}{nh_n}\int_{-\infty}^{+\infty}w^2(t)dt+2h_n^3\Delta\int_{-\infty}^{0}\left\{\int_{-\infty}^{y}(y-t)w(t)dt\right\}^2dy\\ \qquad+o\left(\frac{1}{nh_n}+h_n^3\right) \qquad \textit{if } \delta=0,\ \Delta>0\\ \frac{1}{nh_n}\int_{-\infty}^{+\infty}w^2(t)dt+2h_n\delta\int_{0}^{\infty}\left\{\int_{y}^{\infty}w(x)dx\right\}^2dy\\ \qquad+o\left(\frac{1}{nh_n}+h_n\right) \qquad \textit{if } \delta>0\,.\end{cases}$$

When f satisfies Condition B with $k=l_1=0$, Theorem 3.1 reduces to the results of Rosenblatt [4], Epanechnikov [1] and to Nadaraya's [3] result for $s=2$.

The following Theorem 3.2 gives, for $h_n=Kn^{-\alpha}$, $\alpha>0$, the asymptotically optimum values α_0 and K_0 of α and K, as well as the value of

$$M_0=\lim_{n\to\infty}n^{1-\alpha_0}\,\text{MISE}\,(K_0n^{-\alpha_0})\,.$$

THEOREM 3.2. *If the Conditions A and B are satisfied and* $h_n=Kn^{-\alpha}$, $\alpha>0$, *then* (a) *the asymptotically optimum value* α_0 *of* α *is given by* $\alpha_0=1/5$ (*if* $\delta=\Delta=0$) *or* 1/4 (*if* $\delta=0$, $\Delta>0$) *or* 1/2 (*if* $\delta>0$). (b) *the asymptotically optimum value* K_0 *of* K *is given by*

$$K_0=\begin{cases}\left\{\int_{-\infty}^{+\infty} w^2(t)dt\Big/\left[\left\{\int_{-\infty}^{+\infty} t^2w(t)dt\right\}^2\int_{-\infty}^{+\infty}\{f''(x)\}^2dx\right]\right\}^{1/5} & \text{if } \delta=\Delta=0\\ \left\{\int_{-\infty}^{+\infty} w^2(t)dt\Big/\left[6\Delta\int_{-\infty}^{0}\left\{\int_{-\infty}^{y}(y-t)w(t)dt\right\}^2dy\right]\right\}^{1/4} & \text{if } \delta=0,\ \Delta>0\\ \left\{\int_{-\infty}^{+\infty} w^2(t)dt\Big/\left[2\delta\int_{0}^{\infty}\left\{\int_{y}^{\infty}w(t)dt\right\}^2dy\right]\right\}^{1/2} & \text{if } \delta>0\,,\end{cases}$$

(c) *the asymptotically minimum value of MISE,* MISE $(K_0n^{-\alpha_0})$, *satisfies*

$$\lim_{n\to\infty} n^{1-\alpha_0}\,\text{MISE}\,(K_0n^{-\alpha_0})$$

$$=\begin{cases}\frac{5}{4}\left[\int_{-\infty}^{+\infty} w^2(t)dt\right]^{4/5}\left[\int_{-\infty}^{+\infty} t^2w(t)dt\right]^{2/5}\left[\int_{-\infty}^{+\infty}\{f''(x)\}^2dx\right]^{1/5} & \text{if } \delta=\Delta=0\\ \frac{4}{3}\left[\int_{-\infty}^{+\infty} w^2(t)dt\right]^{3/4}\left[6\Delta\int_{-\infty}^{0}\left\{\int_{-\infty}^{y}(y-t)w(t)dt\right\}^2dy\right]^{1/4} & \text{if } \delta=0,\ \Delta>0\\ 2\left[\int_{-\infty}^{+\infty} w^2(t)dt\right]^{1/2}\left[2\delta\int_{0}^{\infty}\left\{\int_{y}^{\infty}w(t)dt\right\}^2dy\right]^{1/2} & \text{if } \delta>0\,.\end{cases}$$

As for Theorem 3.1, Rosenblatt's [4] result, Epanechnikov's [1] result and Nadaraya's [3] result for $s=2$ are a special case of Theorem 3.2.

If it is known that the density to be estimated satisfies Condition B, but it is not known whether f, nor whether f' is continuous, then the optimal choice for $h_n=Kn^{-\alpha}$ is unknown; in this case, for any choice of α among the values 1/5, 1/4 and 1/2 that is not the optimal choice, the asymptotic efficiency relative to the optimal choice is zero; that is

$$\lim_{n\to\infty}\frac{\text{MISE}\,(K_0n^{-\alpha_0})}{\text{MISE}\,(Kn^{-\alpha})}=0 \qquad K>0,\ \alpha\in\{1/5, 1/4, 1/2\},\ \alpha\neq\alpha_0\,.$$

On the other hand, if α_0 is known and one uses $h_n=Kn^{-\alpha_0}$ then the asymptotic efficiency relative to the optimal choice is positive for all $K>0$ and is equal to one if a consistent estimator of K_0 is used for K. Nadaraya [3] gives such an estimator for the case when $\delta=\Delta=0$.

Finally, the optimal kernel, that is the kernel w that minimizes

$$M_0(w)=\lim_{n\to\infty} n^{1-\alpha_0}\,\text{MISE}\,(K_0n^{-\alpha_0})\,,$$

depends on the unknown density only through α_0. For the case where $\delta=\Delta=0$, it is well-known (see Epanechnikov [1]) that the optimal kernel is given by $w(t)=(3/4)(1-t^2)$ if $t^2\leqq 1$. The following Theorem 3.3 gives the optimal kernel for the case when $\delta>0$.

THEOREM 3.3. *If the Conditions A and B are satisfied and* $\delta>0$ *then the optimal kernel is given by* $w(t)=(1/2)e^{-|t|}$, $-\infty<t<\infty$.

We have been unable to find an optimal kernel for the case when $\delta=0$, $\Delta>0$.

The following Table 1 gives, for several kernels, the asymptotic efficiency, $e(w)$, of the kernel w as defined by

$$e(w)=\frac{M_0(w_0)}{M_0(w)}\ ,$$

where w_0 is the optimal kernel. The kernels used in Table 1 are the same as those used by Epanechnikov [1]; he gives, in his Table 1, the values of $[e(w)]^{-5/4}$ for the case where $\delta=\Delta=0$.

Table 1. Asymptotic efficiencies $e(w)$ of the kernel w

w		$\delta=\Delta=0$	$\delta>0$
$\frac{3}{4}(1-t^2)$	$\lvert t\rvert\leqq 1$	1	.940
$\frac{1}{2}\cos t$	$\lvert t\rvert\leqq\frac{\pi}{2}$	1.000	.945
$1-\lvert t\rvert$	$\lvert t\rvert\leqq 1$	.989	.968
$\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$	$-\infty<t<\infty$	.961	.974
$\frac{1}{2}$	$\lvert t\rvert\leqq 1$	.943	.866
$\frac{1}{2}e^{-\lvert t\rvert}$	$-\infty<t<\infty$	.802	1

4. Proofs of the theorems in Section 3

In order to simplify the notation the index n on h_n will be omitted.

For the proofs of the theorems in Section 3 the following lemmas are needed. The proofs of these lemmas, which are straightforward but lengthy and tedious, are omitted; they can be found in van Eeden [6].

LEMMA 4.1. *If the Conditions A and B are satisfied then*

$$\begin{aligned}(4.1)\quad &\int_{-\infty}^{+\infty}\mathcal{E}\{f_n(x)-f(x)\}^2dx\\ &=\frac{1}{nh}\int_{-\infty}^{+\infty}w^2(t)dt-\frac{2}{n}\int_{-\infty}^{+\infty}f(x)\int_{-\infty}^{+\infty}w(t)f(x-th)dtdx\\ &\quad+\frac{1}{n}\int_{-\infty}^{+\infty}f^2(x)dx+\frac{n-1}{n}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}w(t)\{f(x-th)-f(x)\}dt\right]^2dx\ .\end{aligned}$$

LEMMA 4.2. *If the Conditions A and B are satisfied then*

$$\lim_{h\to 0}\int_{-\infty}^{+\infty} f(x)\int_{-\infty}^{+\infty} w(t)f(x-th)dtdx=\int_{-\infty}^{+\infty} f^2(x)dx\ . \tag{4.2}$$

The following three lemmas are needed to obtain the behaviour of (see (4.1))

$$\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} w(t)\{f(x-th)-f(x)\}dt\right]^2 dx$$

as $h\to 0$; the first two of these lemmas give Taylor series-like expansions for functions f satisfying Condition B; if $k=l_1=0$ these expansions are Taylor series expansions.

LEMMA 4.3. *If f satisfies Condition B then, for $b_{ij_1-1}<x<b_{ij_1}$, $b_{ij_2-1}<x-th<b_{ij_2}$, $1\leqq j_1\leqq l_i+1$, $1\leqq j_2\leqq l_i+1$, $i=1,\cdots,k+1$*

$$f(x-th)-f(x)+thf'(x)-h^2\int_0^t (t-s)f''(x-sh)ds \tag{4.3}$$
$$=\begin{cases} \text{(i)}\ \ 0 & \text{if } j_1=j_2 \\ \text{(ii)}\ \ \sum_{r=j_1}^{j_2-1}(b_{ir}-x+th)\Delta_{ir} & \text{if } j_1<j_2 \\ \text{(iii)}\ \ -\sum_{r=j_2}^{j_1-1}(b_{ir}-x+th)\Delta_{ir} & \text{if } j_1>j_2\ . \end{cases}$$

LEMMA 4.4. *If f satisfies Condition B then, for $b_{i_1j_1-1}<x<b_{i_1j_1}$, $b_{i_2j_2-1}<x-th<b_{i_2j_2}$, $1\leqq j_1\leqq l_{i_1}+1$, $1\leqq j_2\leqq l_{i_2}+1$, $1\leqq i_1\leqq k+1$, $1\leqq i_2\leqq k+1$, $i_1\neq i_2$,*

$$f(x-th)-f(x)+thf'(x)-h^2\int_0^t (t-s)f''(x-sh)ds \tag{4.4}$$
$$=\begin{cases} \text{(i)}\ \ \sum_{r=j_2}^{l_{i_1}+1}(b_{i_1r}-x+th)\Delta_{i_1r}+\sum_{r=i_1+1}^{i_2-1}\sum_{r'=1}^{l_r+1}(b_{rr'}-x+th)\Delta_{rr'} \\ \qquad +\sum_{r=1}^{j_2-1}(b_{i_2r}-x+th)\Delta_{i_2r}-\sum_{\mu=i_1}^{i_2-1}\delta_\mu \qquad \text{if } i_1<i_2 \\ \text{(ii)}\ \ -\sum_{r=j_2}^{l_{i_2}+1}(b_{i_2r}-x+th)\Delta_{i_2r}-\sum_{r=i_2+1}^{i_1-1}\sum_{r'=1}^{l_r+1}(b_{rr'}-x+th)\Delta_{rr'} \\ \qquad -\sum_{r=1}^{j_1-1}(b_{i_1r}-x+th)\Delta_{i_1r}+\sum_{\mu=i_2}^{i_1-1}\delta_\mu \qquad \text{if } i_1>i_2\ . \end{cases}$$

Now let

$$I(x,h)=\int_{-\infty}^{+\infty} w(t)\{f(x-th)-f(x)\}dt\ ,\qquad -\infty<x<\infty\ , \tag{4.5}$$

$$G(x,h)=h^2\int_{-\infty}^{+\infty} w(t)\int_0^t (t-s)f''(x-sh)dsdt\ ,\qquad -\infty<x<\infty\ . \tag{4.6}$$

(4.7) $$H_{ij1}(x,h)=\sum_{\gamma=j}^{l_i+1}\Delta_{i\gamma}\int_{-\infty}^{(x-b_{i\gamma})/h}(b_{i\gamma}-x+th)w(t)dt + \sum_{\gamma=i+1}^{k+1}\sum_{\gamma'=1}^{l_\gamma+1}\Delta_{\gamma\gamma'}\int_{-\infty}^{(x-b_{\gamma\gamma'})/h}(b_{\gamma\gamma'}-x+th)w(t)dt\,,$$
$$b_{ij-1}<x<b_{ij},\ j=1,\cdots,l_i+1,\ i=1,\cdots,k+1\,,$$

(4.8) $$H_{ij2}(x,h)=-\sum_{\gamma=1}^{j-1}\Delta_{i\gamma}\int_{(x-b_{i\gamma})/h}^{\infty}(b_{i\gamma}-x+th)w(t)dt - \sum_{\gamma=1}^{i-1}\sum_{\gamma'=1}^{l_\gamma+1}\Delta_{\gamma\gamma'}\int_{(x-b_{\gamma\gamma'})/h}^{\infty}(b_{\gamma\gamma'}-x+th)w(t)dt\,,$$
$$b_{ij-1}<x<b_{ij},\ j=1,\cdots,l_i+1,\ i=1,\cdots,k+1\,,$$

(4.9) $$V_{i1}(x,h)=\sum_{\mu=1}^{i-1}\delta_\mu\left\{1-W\left(\frac{x-a_\mu}{h}\right)\right\}\,,\qquad a_{i-1}<x<a_i,\ i=1,\cdots,k+1\,,$$

(4.10) $$V_{i2}(x,h)=-\sum_{\mu=i}^{k}\delta_\mu W\left(\frac{x-a_\mu}{h}\right)\,,\qquad a_{i-1}<x<a_i,\ i=1,\cdots,k\,,$$

where

(4.11) $$W(y)=\int_{-\infty}^{y}w(t)dt\,,\qquad -\infty<y<\infty\,.$$

Further let, $j=1,\cdots,l_i+1,\ i=1,\cdots,k+1,$

(4.12) $$H_{ij}(x,h)=H_{ij1}(x,h)+H_{ij2}(x,h)$$
$$V_i(x,h)=V_{i1}(x,h)+V_{i2}(x,h)$$

then it follows from Lemma 4.3 and Lemma 4.4 that

LEMMA 4.5. *If the Conditions A and B are satisfied, then*

(4.13) $$\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}w(t)\{f(x-th)-f(x)\}dt\right]^2dx = \sum_{i=1}^{k+1}\sum_{j=1}^{l_i+1}\int_{b_{ij-1}}^{b_{ij}}\{G(x,h)+H_{ij}(x,h)+V_i(x,h)\}^2dx\,.$$

The following four lemmas give the behaviour, as $h\to 0$, of each of the six terms obtained by expanding the square in the right-hand side of (4.13).

LEMMA 4.6. *If the Conditions A and B are satisfied then*

(4.14) $$\lim_{h\to 0}\frac{1}{h^4}\int_{-\infty}^{+\infty}G^2(x,h)dx=\frac{1}{4}\left\{\int_{-\infty}^{+\infty}t^2w(t)dt\right\}^2\int_{-\infty}^{+\infty}\{f''(x)\}^2dx\,.$$

LEMMA 4.7. *If the Conditions A and B are satisfied then*

$$\lim_{h\to 0}\frac{1}{h^3}\int_{b_{ij-1}}^{b_{ij}} H_{ij}^2(x, h)dx=(\Delta_{ij}^2+\Delta_{ij-1}^2)\int_{-\infty}^{0}\left\{\int_{-\infty}^{y}(y-t)w(t)dt\right\}^2 dy\ , \qquad 1\leqq j\leqq l_i+1,\ i=1,\cdots,k+1\ . \tag{4.15}$$

LEMMA 4.8. *If the Conditions A and B are satisfied then, for* $i=1,\cdots,k+1$,

$$\lim_{h\to 0}\frac{1}{h}\int_{a_{i-1}}^{a_i} V_i^2(x, h)dx=(\delta_i^2+\delta_{i-1}^2)\int_0^{\infty}\{1-W(y)\}^2 dy\ . \tag{4.16}$$

LEMMA 4.9. *If the Conditions A and B are satisfied then*

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h^3}&\int_{b_{ij-1}}^{b_{ij}} G(x, h)H_{ij}(x, h)dx\\ &=\lim_{h\to 0}\frac{1}{h}\int_{a_{i-1}}^{a_i} G(x, h)V_i(x, h)dx\\ &=\lim_{h\to 0}\frac{1}{h}\int_{b_{ij-1}}^{b_{ij}} H_{ij}(x, h)V_i(x, h)dx=0\\ &\qquad 1\leqq j\leqq l_i+1,\ i=1,\cdots,k+1\ .\end{aligned} \tag{4.17}$$

PROOF OF THEOREM 3.1. From Lemmas 4.1, 4.2 and 4.5 it follows that, for $h\to 0$,

$$\begin{aligned}\text{MISE}\,(h)=&\frac{1}{nh}\int_{-\infty}^{+\infty} w^2(t)dt-\frac{1}{n}\int_{-\infty}^{+\infty} f^2(x)dx+o(h)\\ &+\frac{n-1}{n}\sum_{i=1}^{k+1}\sum_{j=1}^{l_i+1}\int_{b_{ij-1}}^{b_{ij}}\{G(x, h)+H_{ij}(x, h)+V_i(x, h)\}^2dx\ .\end{aligned} \tag{4.18}$$

The theorem then follows from the Lemmas 4.6, 4.7, 4.8 and 4.9 and from the fact that

$$\sum_{i=1}^{k+1}\sum_{j=1}^{l_i+1}\Delta_{ij-1}^2=\sum_{i=1}^{k+1}\sum_{j=1}^{l_i+1}\Delta_{ij}^2=\Delta \quad\text{and}\quad \sum_{i=1}^{k+1}\delta_{i-1}^2=\sum_{i=1}^{k+1}\delta_i^2=\delta\ .$$

PROOF OF THEOREM 3.2. For the case where $\delta=\Delta=0$, the proof can be found in Rosenblatt [4] for the uniform kernel and in Epanechnikov [1] for the more general case of a kernel satisfying Condition A. The proofs for the other two cases are analogous to these proofs of Rosenblatt [4] and Epanechnikov [1].

PROOF OF THEOREM 3.3. The kernel that minimizes

$$M_0(w)=\lim_{n\to\infty} n^{1-\alpha_0}\,\text{MISE}\,(K_0 n^{-\alpha_0})$$

when $\delta>0$, is the kernel that minimizes

$$\int_{-\infty}^{+\infty} w^2(t)dt\int_0^{\infty}\{1-W(y)\}^2dy=2\int_0^{\infty} w^2(t)dt\int_0^{\infty}\{1-W(y)\}^2dy\ . \tag{4.19}$$

By Schwartz's inequality

$$\int_0^\infty w^2(t)dt\int_0^\infty \{1-W(y)\}^2dy \geqq \left[\int_0^\infty w(t)(1-W(t))dt\right]^2 = \frac{1}{64}\,, \tag{4.20}$$

with equality if and only if

$$W(t)=aw(t) \qquad 0<t<\infty \text{ for some } a>0 \tag{4.21}$$

or, equivalently, if and only if

$$w(t)=\frac{a}{2}e^{-a|t|} \qquad -\infty<t<\infty,\ a>0\,. \tag{4.22}$$

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REFERENCES

[1] Epanechnikov, V. A. (1969). Non-parametric estimation of a multivariate probability density, *Theory Prob. Appl.*, **14**, 153-158.

[2] Hájek, J. and Sidák, Z. (1967). *Theory of Rank Tests*, Academic Press, New York.

[3] Nadaraya, E. A. (1974). On the integral mean square error of some nonparametric estimates for the density function, *Theory Prob. Appl.*, **19**, 133-141.

[4] Rosenblatt, M. (1956). Remarks on some nonparametric estimates of a density function, *Ann. Math. Statist.*, **27**, 832-837.

[5] Rosenblatt, M. (1971). Curve estimates, *Ann. Math. Statist.*, **42**, 1815-1842.

[6] van Eeden, C. (1982). Mean integrated squared error of kernel estimators when the density and its derivative are not necessarily continuous, *Rapport de Recherches*, **82-7**, Département de Mathématiques et de Statistique, Université de Montréal.