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UNBIASED SEQUENTIAL ESTIMATION OF $1/p$: SETTLEMENT OF A CONJECTURE

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Summary

We present a complete characterization of the class of (unbounded) sampling plans providing unbiased (sequential) estimation of the reciprocal of the Bernoulli parameter p. This settles a conjecture set forth by Sinha and Sinha (1975, *Ann. Inst. Statist. Math.*, **27**, 245–258) regarding the nature of such plans as sought out by Gupta (1967, *Ann. Inst. Statist. Math.*, **19**, 413–416). Incidentally, a special type of sampling plans (termed 'infinite-step generalizations of the inverse binomial plans'), studied by Sinha and Bhattacharyya (1982, Institute of Statistics Mimeo Series, Raleigh), are seen to play a central role in this study.

1. Introduction

Under the set-up of independent Bernoulli trials, various aspects of inference on p and $1/p$ have been studied in the literature. We are interested in the problem of unbiased (sequential) estimation of the reciprocal of the Bernoulli parameter p. More specifically, we are interested in a characterization of the *class* of sampling plans providing unbiased sequential estimation of $1/p$. This specific problem was studied first by Gupta [4] and subsequently, by Sinha and Sinha [6] and Sinha and Bhattacharyya [7]—hereafter abbreviated as SS and SB respectively. In SS, a conjecture was set forth on the nature of such plans as typified by Gupta [4]. We settle the conjecture here and resolve the characterization problem completely. A very special type of sampling plans, termed 'infinite-step generalizations of the inverse binomial plans', were studied in SB. These plans are seen to play a central role in this study.

Key words and phrases: Sequential estimation, Bernoulli parameter, inverse binomial plans and their generalizations.

2. Nomenclature

The word point will refer to points in the XY-plane with positive integral co-ordinates (X measuring the 'failures', Y the 'successes'). A *region* R is a set of points including the point $(0, 0)$. The point (x', y') is *immediately beyond* the point (x, y) if either $x'=x+1$, $y'=y$ or $x'=x$, $y'=y+1$. A *path* in R from the point a_0 to the point a_n is a sequence of points $a_0, a_1, \cdots, a_n$ such that a_i $(i>0)$ is immediately beyond a_{i-1}. We order the points in the XY-plane lexicographically. A *boundary point* (element on the boundary B of R) is a point not in R which is the last point of a path from the origin. *Accessible* points are the points in R which can be reached by accessible paths from the origin. The *index* of a point is the sum of its co-ordinates. A *finite* (or *bounded*) region is a region for which the indices of the accessible points are less than some number n. The probability of a boundary point $(x, y)=\alpha$ is $P(\alpha)=K(\alpha)p^y q^x$, where $K(\alpha)$ is the number of paths from the origin to α. A region is *closed* if $\sum_{\alpha \in B} P(\alpha)=1$. The corresponding sampling plan is said to be a *closed plan*. Unless otherwise stated all plans occurring are assumed to be closed. An *estimator* f is defined only at the boundary points $\alpha \in B$; it is said to be unbiased for $1/p$ if $\sum_{\alpha \in B} f(\alpha)K(\alpha)p^y q^x=1/p$ identically in p. f is said to be *proper* estimator if $f(\alpha)\geqq 1$ $\forall \alpha \in B$.

3. N.S. conditions for unbiased estimation of $1/p$

The following results are known for arbitrary closed sampling plans.

a) The plan must be unbounded along the X-direction (the X-unboundedness condition). However, not all X-unbounded plans provide unbiased estimation of $1/p$ (Vide SS).

b) If the closed plan P with boundary $B=\{\alpha=(x, y)\}$ be such that by changing its boundary points from α to $\alpha'=(x, y+1)$, we get a closed plan P' with boudary $B'=\{\alpha'=(x, y+1)\}$, then $1/p$ is estimable for the plan P; and an unbiased estimate is given by $f(\alpha) =K'(\alpha')/K(\alpha)$, $\alpha \in B$ where $K'(\alpha')$ is the number of accessible paths from the origin to $\alpha' \in B'$. This is Gupta's sufficient condition. It may be noted that the above estimate is *proper*. In SB, the plan P' has been technically called the '*shifted*' form of P.

c) If no point on the line '$Y=1$' is inaccessible, $1/p$ is estimable. (This provides a simple application of Gupta's sufficient condition. Moreover, in some cases, this leads to other estimates, though improper. Vide SS).

In SS, various other implications of Gupta's sufficient condition were studied so much so that it was *conjectured* that the *same condition* would be *necessary* as well. We establish the *validity* of this conjecture (see Section 5). For this we need some further tools. Below is one of them.

LEMMA 1. *A necessary condition for unbiased estimability of $1/p$ is that an infinite number of boundary points of the plan lie below the line $Y=-b+\beta X$, for every $b\geqq 0$, $\beta>0$.*

PROOF. It is enough to consider the case $b=0$. Let $f(\alpha)$ be an unbiased estimator for $1/p$, so that

$$(3.1)\qquad 1/p=\sum_{\alpha\in B} f(\alpha)K(\alpha)p^y q^x\,,\qquad 0<p<1 \text{ holds}\,.$$

Suppose now that for some $\beta>0$, there are only finitely many boundary points of the plan below the line $Y=\beta X$.

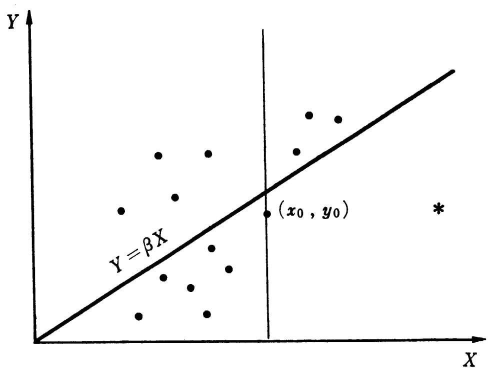

Fig. 1. * No boundary point in this region.

Then there exists an x_0 such that

$$K(\alpha)=0\quad \forall\,\alpha=(x, y)\qquad \text{with } x>x_0,\ y\leqq\beta x\,.$$

Rewrite (3.1) as

$$(3.2)\qquad \frac{1}{p}=\sum_x\sum_y f(x, y)K(x, y)p^y q^x\qquad (\text{taking } K(x, y)=0 \text{ if } (x, y)\notin B)$$

$$=\sum_{x\leqq x_0} q^x\sum_y f(x, y)K(x, y)p^y+\sum_{x>x_0}\sum_y f(x, y)K(x, y)p^y q^x\,.$$

Clearly, in (3.2), the first expression remains bounded as $p\to 0$. The second expression is bounded by (note that $K(x, y)=0$ if $y\leqq\beta x$)

$$\sum_{x>x_0}\sum_{y>\beta x}|f(x, y)|K(x, y)p^{y-\beta x}(p^\beta q)^x\,,$$

which in turn is bounded by

$$\sum_{x>x_0}\sum_{y>\beta x} |f(x, y)| K(x, y) p_0^{y-\beta x}(p_0^\beta q_0)^x \qquad \text{for all } p \leq p_0 ,$$

where p_0 $(0<p_0<1)$ maximizes $p^\beta q$.

Hence, letting $p \to 0$ on both sides of (3.1), we arrive at a contradiction. This settles the lemma.

Incidentally, this technique provides a simpler proof of the statement in (a) above.

In the next section, we study some specific sampling plans useful for our purpose.

4. The family of inverse binomials and their generalizations

A special, simple and interesting class of sampling plans is that of the inverse binomials $P(c)|c=1, 2, \cdots$ where $B(c)=\{\alpha=(x, c),\ x=0, 1, 2, \cdots\}$. In other words, the points on the line '$Y=c$' form the boundary points of the plan $P(c)$, $c=1, 2, \cdots$. It is well-known that these plans are closed and, moreover, they provide unbiased estimation of $1/p$. The following generalizations of such plans were introduced in SB.

For a *finite* k, let $1 \leq C_1 < C_2 < \cdots < C_k < \infty$ be a set of k positive integers. Further, let $0 \leq n_1 < n_2 < \cdots < n_{k-1} < \infty$ be another set of positive integers. The plan sketched below is termed *finite-step generalization of the inverse binomial plans.* It is fairly obvious that such plans as also their shifted forms are necessarily closed.

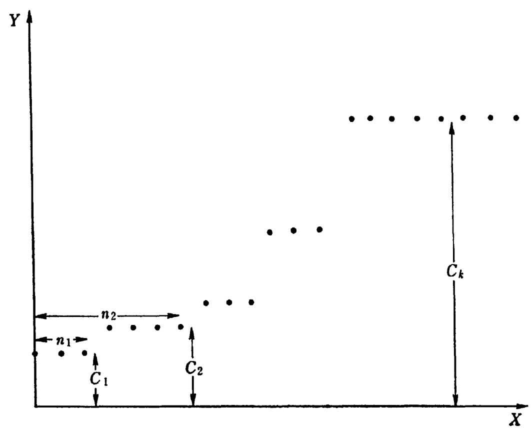

Fig. 2

Next suppose there are two sequences of positive integers $\{C_k \uparrow\}$ $\{n_k \uparrow, n_1 \geq 0\}$. Consider the readily extended form of the above plan. Clearly, this will now be unbounded in both directions. We sketch such a plan below.

These are termed *infinite-step generalizations of the inverse binomials.*

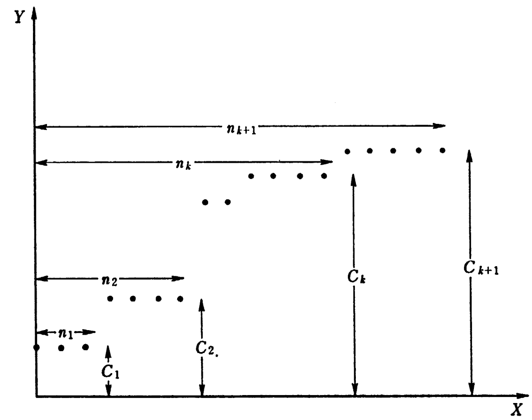

Fig. 3

They are very similar to doubly simple regions (Vide Wolfowitz [9]), with the exception that there are no lower boundary points.

Such plans are not necessarily closed. Regarding their closure we have the following.

LEMMA 2. *An infinite-step generalized inverse binomial plan with parameters* $\{C_k, n_k\}$ *is closed if and only if* $\underline{\lim}\, C_k/n_k=0$.

PROOF. See SB for the 'if' part. As regards the 'only if' part, we argue as follows. Suppose the plan is closed. Then one can immediately see that the shifted form of the plan is also closed; hence, $1/p$ is estimable. But, then, this would mean that below any line $Y=\beta X$ ($\beta>0$, arbitrary), there lie an infinitely many boundary points of the original plan. Hence, for every $\beta>0$, there is a pair $\{C_{k(\beta)}, n_{k(\beta)}\}$ such that $C_{k(\beta)}/n_{k(\beta)}<\beta$. This gives $\underline{\lim}\, C_k/n_k=0$ and proves the lemma.

In the next section we will utilize the above result to establish that Gupta's sufficient condition is also necessary for unbiased estimation of $1/p$.

5. Characterizations of closed plans: settlement of the conjecture

Consider an arbitrary closed sampling plan which provides unbiased estimation of $1/p$. We want to show that the shifted form of the plan will be necessarily closed. Following SS and SB we argue as follows. If the original plan does *not* contain any boundary point along the X-axis, then we are readily done. Otherwise, if again *no point* (all points) on any *parallel* line '$Y=m$' is (are) inaccessible, then also we are readily done. With the exception of such plans, therefore, *every* closed plan contains one (and, hence, an infinite number of) inaccessible point(s) on *each* line '$Y=m$', $m=0, 1, 2, \cdots$. The closure of the shifted form of such a plan will *solely* depend on its behaviour with reference to the *totally* new paths (arising out of the 'shift') which are *now* accessible but were otherwise *inaccessible* in the *original plan*. See SB for details of the argument to feel sure that the problem of closure of the shifted plan is connected with the problem of closure of a *corresponding* infinite-step generalized inverse binomial plan, constructed through a definite set of rules. According to Lemma 1, however, there lie an infinity of boundary points of the original plan below any line $Y=\beta X$, $\beta>0$ and hence, the same is true of the related infinite-step generalized inverse binomial plan (This is evident from its construction). But this would mean that for this latter plan, $\underline{\lim}\, C_k/n_k=0$. Lemma 2 now establishes its closure and hence, that of the shifted form of the original plan.

Remark 1. Lemma 1 can now be modified to make the stated condition sufficient as well.

Remark 2. The same characterization applies also to the problems of estimation of p^{-2}, p^{-3} etc.

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