Ann. Inst. Statist. Math.
37 (1985), Part A, 361-364

SOME COMMENTS ON A PAPER ON k-HNBUE LIFE DISTRIBUTIONS*

BENGT KLEFSJÖ

(Received July 23, 1984; revised Jan. 7, 1985)

Summary

In a recent paper by A. P. Basu and N. Ebrahimi (1984, Ann. Inst. Statist. Math., A, 36, 87–100) a new class of life distributions called k-HNBUE (with dual k-HNWUE) is introduced. Closure properties and bounds on the moments and on the survival function to a k-HNBUE (k-HNWUE) life distribution are presented. However, some of the results presented are incorrect.

1. Introduction

In a recent paper Basu and Ebrahimi [1] present a new class of life distributions called *k-HNBUE* (*HNBUE=harmonic new better than used in expectation*). This class consists of those life distributions F with survival function $\bar{F}=1-F$ and finite mean μ for which

$$\frac{1}{\frac{1}{t}\int_0^t e_F^{-k}(x)dx} \leqq \mu^k \qquad \text{for } t \geqq 0, \tag{1}$$

where

$$e_F(x)=\begin{cases} \left\{\int_x^\infty \bar{F}(s)ds\right\} \Big/ \bar{F}(x) & \text{if } \bar{F}(x)>0 \\ 0 & \text{if } \bar{F}(x)=0 \end{cases}$$

denotes the mean residual life at age x. If the reversed inequality in (1) holds $\bar{F}$ is said to be *k-HNWUE*. When $k=1$ we get the HNBUE (HNWUE) class introduced by Rolski [6] and further studied by Klefsjö [2]–[4].

* This research was supported by Swedish Natural Science Research Council Post Doctoral Fellowship F-PD 1564-100.
Key words: Survival function, HNBUE, HNWUE, k-HNBUE, k-HNWUE.

In their paper Basu and Ebrahimi [1] discuss closure properties and present bounds on the moments and on the survival function of a k-HNBUE (k-HNWUE) life distribution. However, the bounds presented are based on the incorrect facts that $\int_t^\infty \bar{F}(x)dx \leqq \mu \exp(-t/\mu^k)$, $t \geqq 0$, for a k-HNBUE life distribution and $t^{1/k}\int_t^\infty \bar{F}(x)dx \geqq \mu\bar{F}(t)$, $t \geqq 0$, for a k-HNWUE life distribution. Detailed comments on the paper by Basu and Ebrahimi can be found in Klefsjö [5].

The aim of this note is to present a new upper bound on a k-HNBUE survival function.

2. Bounds on a k-HNBUE survival function

We have not succeeded to get sharp upper bounds on the moments and on the survival function of a k-HNWUE life distribution. However, it should be mentioned that the only lower bound we can get on the survival function is the trivial one that $\bar{F}(t) \geqq 0$. This is easily seen by studying $\bar{F}(t) = \varepsilon \exp(-\varepsilon t/\mu)$, $t \geqq 0$, which is k-HNWUE.

We now turn to the situation when $\bar{F}$ is k-HNBUE.

THEOREM 2.1. *If $\bar{F}$ is survival function of a k-HNBUE life distribution, $k>1$, then*

$$\int_t^\infty \bar{F}(x)dx \leqq \left(\frac{\mu^k}{\mu+(k-1)t}\right)^{1/(k-1)} \qquad \text{for } t \geqq 0 . \tag{2}$$

PROOF. Set

$$F^*(x) = \int_x^\infty \bar{F}(s)ds \qquad \text{for } x \geqq 0 .$$

Then

$$\int_0^t \frac{(\bar{F}(x))^k}{(F^*(x))^k}dx \leqq \int_0^t \frac{(\bar{F}(0))^{(k-1)}\bar{F}(x)}{(F^*(x))^k}dx = \int_0^t \frac{\bar{F}(x)}{(F^*(x))^k}dx , \tag{3}$$

here the second step follows from the fact that if F is k-HNBUE then $\bar{F}(0)=1$ (see Remark 1 below). The integral on the right hand side is equal to

$$\frac{1}{1-k}(\mu^{1-k} - (F^*(x))^{1-k}) .$$

Since the k-HNBUE property means that

$$\int_0^t \frac{(\bar{F}(x))^k}{(F^*(x))^k}dx \geqq \frac{t}{\mu^k} \qquad \text{for } t\geqq 0$$

the inequality follows.

Remark 1. If $\bar{F}$ is k-HNBUE then $\bar{F}(0)=1$. This can be seen in the following way: Suppose that $\bar{F}(0)=a<1$ and choose t_0 so that $\int_{t_0}^\infty \bar{F}(x)dx>\mu a$ (this is possible since $\int_0^\infty \bar{F}(x)dx=\mu$). Then $\left\{\bar{F}(x)\Big/\int_x^\infty \bar{F}(s)ds\right\}$ $<a/a\mu=1/\mu$ for $x<t$. This means that $\int_0^{t_0} e_{\bar{F}}^{-k}(x)dx<t_0/\mu^k$ which contradicts the definition in (1).

By using this theorem we can present an upper bound on a k-HNBUE survival function.

THEOREM 2.2. *If $\bar{F}$ is survival function of a k-HNBUE life distribution, $k>1$, then*

$$\bar{F}(t)\leqq\begin{cases} 1 & \text{for } t<\mu \\ \left(\dfrac{k\mu}{\mu+(k-1)t}\right)^{k/(k-1)} & \text{for } t\geqq\mu\,. \end{cases} \tag{4}$$

PROOF. By using (2) and the fact that $\int_s^t \bar{F}(x)dx\leqq\int_s^\infty \bar{F}(x)dx$ for $s<t$ we get that

$$\bar{F}(t)\leqq \inf_{0<s<t}\left\{\left(\frac{\mu^k}{\mu+(k-1)s}\right)^{1/(k-1)}\cdot\frac{1}{t-s}\right\}\,.$$

Then the result follows by straightforward calculations.

Remark 2. If $k\to 1$ in the bound (4) above we get the upper bound on a HNBUE survival function presented in Klefsjö [4],

$$\bar{F}(t)\leqq\begin{cases} 1 & \text{for } t<\mu \\ \exp\left(1-\dfrac{t}{\mu}\right) & \text{for } t\geqq\mu\,. \end{cases}$$

Remark 3. It is also possible to get a lower bound on the survival function by using Theorem 2.1. We get

$$\bar{F}(t)\geqq\sup_{s>t}\frac{\mu-t-(\mu^k/(\mu+(k-1)s))^{1/(k-1)}}{s-t}\,.$$

By calculating the expression at the right hand side we get a rather complicated bound of a form similar to that in Klefsjö [4] in the HNBUE case.

LULEÅ UNIVERSITY, SWEDEN

REFERENCES

[1] Basu, A. P. and Ebrahimi, N. (1984). On k-order harmonic new better than used in expectation distributions, *Ann. Inst. Statist. Math.*, A, **36**, 87-100.

[2] Klefsjö, B. (1980). Some properties of the HNBUE and HNWUE classes of life distributions, *Research Report 1980-8*, Department of Mathematical Statistics, University of Umeå, Sweden.

[3] Klefsjö, B. (1981). HNBUE survival under some shock models, *Scand. J. Statist.*, 8, 39-47.

[4] Klefsjö, B. (1982). The HNBUE and HNWUE classes of life distributions, *Nav. Res. Logist. Quart.*, **29**, 331-344.

[5] Klefsjö, B. (1984). Comments and corrections to a paper on k-order harmonic new better than used in expectation distributions by A. P. Basu and N. Ebrahimi, *Research Report 1984-3*, Department of Mathematics, Luleå University, Sweden.

[6] Rolski, T. (1975). Mean residual life, *Bull. Int. Statist. Inst.*, **46**, 266-270.