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LATTICE SQUARE APPROACH TO CONSTRUCTION OF MUTUALLY ORTHOGONAL F-SQUARES

B. L. RAKTOE AND W. T. FEDERER

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Summary

Factorial design theory has been effectively used in solving problems associated with many combinatorial structures. Hedayat, Raghavarao and Seiden [7] clearly demonstrated this in obtaining various results on $F(n, \lambda)$-squares. Indeed, both theorems in their paper are cute observations if knowledge of factorial design theory is assumed. The objective of this paper is to present the lattice square method for obtaining sets of mutually orthogonal $F(n, \lambda)$-squares of given order n. As a by-product a generalization of a theorem, due to the above authors, is obtained and a lower bound is presented for the number of mutually orthogonal $F(n, \lambda)$-squares on sets of various cardinalities associated with the canonical decomposition of n as a product of powers of distinct primes.

1. Introduction

It is well known that factorial design theory can be used as a tool in solving problems in many mathematical and statistical areas. Conversely, the theory associated with many mathematical and statistical structures lead to solutions in factorial design problems. This intimate relationship is typically climaxed in the construction of factorial designs. A discussion of this aspect can be found in Chapter 12 of the recent book by Raktoe, Hedayat and Federer [13].

The connection between Latin squares and factorial designs has been known for a long time and it is not surprising that F-squares, which are generalizations of Latin squares, also exhibit this property. F-squares were introduced by Finney [3]-[5] and Freeman [6], and subsequently various aspects have been developed by Hedayat and Seiden [8], Hedayat, Raghavarao and Seiden [7], Federer [2], and Mandeli, Lee and Federer [10] In their paper Hedayat, Raghavarao and Seiden [7]

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have effectively used concepts and results from factorial design theory to establish their two theorems concerning F-squares. Their first theorem follows immediately from some basic results in fractional factorial design theory and their second theorem relies on confounding theory for prime powered symmetrical factorials.

The objectives of this paper are as follows: (i) In Section 2 the prime-powered lattice square approach is used to construct sets of mutually orthogonal $F(n, \lambda)$-squares. The development not only results in exhibiting to the reader the various algebraic-geometric-combinatorial aspects associated with this approach, but also leads to re-emergence of the second theorem of Hedayat, Raghavarao and Seiden [7] in the lattice square framework. (ii) In Section 3 we generalize the "easy" case of Section 2 to the mixed prime powered lattice and it is shown, via the canonical decomposition of n, how to obtain various sets of mutually orthogonal $F(n, \lambda)$-squares for given n. This generalization leads to a lower bound on the number of mutually orthogonal $F(n, \lambda)$-squares of the various types associated with n. (iii) Finally, in Section 4 we provide a discussion of the results and indicate directions of further research in this area.

All the results obtained in this paper are illustrated with a detailed example and if deemed necessary some preliminary notions are briefly recalled.

2. Symmetrical prime powered lattice squares and corresponding sets of mutually orthogonal F-squares

For basic terminology on F-squares we refer the reader to the paper by Hedayat and Seiden [8] and that on lattice squares to standard texts, such as Cochran and Cox [1] and Kempthorne [9]. In their paper Hedayat, Raghavarao and Seiden [7] have shown how to obtain a complete set of mutually orthogonal $F(n, \lambda)$-squares in the prime powered case. Since we will present a unified and generalized lattice square method it is both illuminating and essential to develop the prime powered case fully from the lattice square viewpoint.

Let n be a given prime power, which is written as $n=p^u$, where p is a prime. Further, let d be a divisor of u so that $u=hd$ for some h, and n is rewritten as $p^{hd}=(p^h)^d=s^d$. Consider the $n\times n=s^d\times s^d$ lattice square with row-column confounding schemes given by the two-tuple of flats ($(d-1)$-flat, $(d-1)$-flat) of the finite projective geometry $PG(2d-1, s)$ over the Galois field $GF(s)$ such that these flats generate $PG(2d-1, s)$. This implies that the two confounding schemes have no common elements and that $(s^d-1)/(s-1)$ effects of $PG(2d-1, s)$ are confounded with rows and the same number with columns of the lattice square. The allocation of the s^{2d} treatment combinations to the n^2 cells of the lattice square proceeds by taking simultaneously the parallel pencils of the finite Euclidean geometry $EG(2d, s)$ corresponding to the generators of the two-tuple of flats. Now, note that there are precisely $t=(s^{2d}-1)/(s-1)-2(s^d-1)/(s-1)=(s^d-1)^2/(s-1)$ effects which are unconfounded with rows and columns. These effects are orthogonal to each other and each will produce an $F(n, \lambda=s^{d-1})$-square on s symbols in $GF(s)$ by taking linear combinations of levels in the treatment combinations of the n^2 cells corresponding to the point in $PG(2d-1, s)$ representing the unconfounded effect. Hence this will lead to a complete set of mutually orthogonal $F(n, s^{d-1})$-squares on s symbols. This same result was obtained by Hedayat, Raghavarao and Seiden [7] without resorting to lattice square terminology.

In the next section we generalize the above symmetrical prime powered lattice square procedure to the asymmetrical or mixed prime powered lattice square procedure for obtaining sets of mutually orthogonal F-squares.

3. Asymmetrical prime powered lattice squares and corresponding sets of mutually orthogonal F-squares

Let $n=p_1^{u_1}p_2^{u_2}\cdots p_v^{u_v}$ be the canonical representation of the integer n as a product of powers of distinct primes, and let d_i be a divisor of the exponent u_i, $i=1, 2, \cdots, v$. This implies that $u_i=h_id_i$ for some h_i. Further, let n be rewritten as $n=(p_1^{h_1})^{d_1}(p_2^{h_2})^{d_2}\cdots(p_v^{h_v})^{d_v}=(s_1)^{d_1}(s_2^{d_2})\cdots(s_v^{d_v})$, where h_i corresponds to d_i and $s_i=p_i^{h_i}$.

In the ensuring development we assume that the reader is familiar with the calculus of mixed prime powered factorials as developed by White and Hultquist [14] and Raktoe [11], [12]. In this calculus the Galois fields $GF(s_i)$ are imbedded in a finite commutative ring R of order $\prod_{i=1}^{v} s_i$ such that they are isomorphic to mutually annihilating subrings $R(s_i)$ of R. As a consequence the ring R is the direct sum of the $GF(s_i)$'s, i.e., $R=\sum_{i=1}^{v}\oplus(R(s_i)\cong GF(s_i))$ and, both $EG(2d_i, s_i)$ and $PG(2d_i-1, s_i)$ have their isomorphic representations over the subrings $R(s_i)$ of R. Denoting these representations by $EG^*(2d_i, s_i)$ and $PG^*(2d_i-1, s_i)$ it follows that the set of treatment combinations T and the set of effects E of the $s_1^{2d_1}\times s_2^{2d_2}\times\cdots\times s_v^{2d_v}$ factorial are the direct sums of the $EG^*(2d_i, s_i)$'s and the $PG^*(2d_i-1, s_i)$'s, i.e., $T\cong T^*=\sum_{i=1}^{v}\oplus(EG^*(2d_i, s_i)\cong EG(2d_i, s_i))$ and $E\cong E^*=\sum_{i=1}^{v}\oplus(PG^*(2d_i-1, s_i)\cong PG(2d_i-1, s_i))$. This means that all arithmetic with treatment combinations and effects (including generalized interactions) associated with the $s_1^{2d_1} \times s_2^{2d_2} \times \cdots \times s_v^{2d_v}$ factorial will take place in T^* and E^*, respectively. Detailed examples of this unified approach for mixed prime powered factorials can be found in the above-mentioned papers. However, to put the reader at ease we provide an exhaustive example.

Example 3.1. Let $n=6$ with the canonical decomposition $6=2^1 \times 3^1$. The underlying Galois fields are $GF(2)=\{0, 1\}$ and $GF(3)=\{0, 1, 2\}$ with mod. 2 and mod. 3 arithmetics. Let R be the ring of residue classes modulo 6, i.e. $R=\{0, 1, 2, 3, 4, 5\}$ with mod. 6 arithmetic. The reader may verify that $R(2)=\{0, 3\}$ and $R(3)=\{0, 4, 2\}$ are mutually annihilating subrings of R. Via the isomorphisms ϕ_1 and ϕ_2

$$\begin{array}{ccc} \phi_1 & & \phi_2 \\ 0 \longrightarrow 0 & \text{and} & 0 \longrightarrow 0 \\ 1 \longrightarrow 3 & & 1 \longrightarrow 4 \\ & & 2 \longrightarrow 2 \end{array}$$

we observe that $GF(2) \cong R(2)$ and $GF(3) \cong R(3)$, so that $R=GF(2) \oplus GF(3)$. The isomorphic representations of $EG(2, 2)$ and $EG(2, 3)$ are, respectively, equal to $EG^*(2, 2)=\{(0, 0), (3, 0), (0, 3), (3, 3)\}$ and $EG^*(2, 3)=\{(0, 0), (4, 0), (2, 0), (0, 4), (4, 4), (2, 4), (0, 2), (4, 2), (2, 2)\}$. Hence the set of 36 treatment combinations in the $n \times n=2^2 \times 3^2$ mixed factorial is given by the set $T^*=\{(x_{11}, x_{12}, x_{21}, x_{22})\}$, where x_{11}, x_{12} are elements of $R(2)$ and x_{21}, x_{22} are elements of $R(3)$ or equivalently $(x_{11}, x_{12}) \in EG^*(2, 2)$ and $(x_{21}, x_{22}) \in EG^*(2, 3)$. Similarly, the isomorphic representations of $PG(1, 2)$ and $PG(1, 3)$ are, respectively, $PG^*(1, 2)=\{(3, 0), (0, 3), (3, 3)\}$ and $PG^*(1, 3)=\{(4, 0), (0, 4), (4, 4), (4, 2)\}$. These sets represent the sets of effects $\{A_{11}^3, A_{12}^3, A_{11}^3, A_{12}^3\}$ of the 2^2 factorial and $\{A_{21}^4, A_{22}^4, A_{21}^4 A_{22}^4, A_{21}^4 A_{22}^2\}$ of the 3^2 factorial. The set of $3+4+(3)(4)=19$ effects in the $2^2 \times 3^2$ mixed factorial are represented by the set $E^*=\{A_{11}^{x_{11}} A_{12}^{x_{12}} B_{21}^{x_{21}} B_{22}^{x_{22}}\}$, where $\{x_{11}, x_{12}\} \subset R(2)$ and $\{x_{21}, x_{22}\} \subset R(3)$ with the understanding that two effects $A_{11}^{x_{11}} A_{12}^{x_{12}} A_{21}^{x_{21}} A_{22}^{x_{22}}$ and $A_{11}^{y_{11}} A_{12}^{y_{12}} A_{21}^{y_{21}} A_{22}^{y_{22}}$ are the same if $(y_{11}, y_{12}, y_{21}, y_{22})=\rho(x_{11}, x_{12}, x_{21}, x_{22})$ is not a zero divisor in R. Equivalently, the set E^* can be described by taking $(x_{11}, x_{12}) \in PG^*(1, 2)$ and $(x_{21}, x_{22}) \in PG^*(1, 3)$ with the understanding that (x_{11}, x_{12}) represents the class $\{\rho_1^*(x_{11}, x_{12}) : \rho_1^* \neq 0 \in R(2)\}$ and (x_{21}, x_{22}) represents the class $\{\rho_2^*(x_{21}, x_{22}) : \rho_2^* \neq 0 \in R(3)\}$. The generalized interaction of the two effects $A_{11}^{x_{11}} A_{12}^{x_{12}} B_{21}^{x_{21}} A_{22}^{x_{22}}$ and $A_{11}^{y_{11}} A_{12}^{y_{12}} A_{21}^{y_{21}} A_{22}^{y_{22}}$ in E is given by the set of effects $\{A_{11}^{\rho_1 x_{11}+\rho_2 y_{11}} A_{12}^{\rho_1 x_{12}+\rho_2 y_{12}} A_{21}^{\rho_1 x_{21}+\rho_2 y_{21}} A_{22}^{\rho_1 x_{22}+\rho_2 y_{22}}\}$, where ρ_1 and ρ_2 are not zero divisors in R. If in a $2^2 \times 3^2$ factorial blocks of 6 units are considered, then it is sufficient to confound an interaction between a two-level factor and a three-level factor in order to confound the effects making up the interaction. For example, if $A_{11}^3 B_{21}^4$ is confounded with blocks then so are A_{11}^3 and B_{21}^4 also. Conversely, if A_{11}^3 and B_{21}^4 are confounded with blocks then so is the interaction $A_{11}^3 B_{21}^4$. To find the allocation of treatment combinations to the blocks when $A_{11}^3 B_{21}^4$ is confounded we have to solve six equations in T^*. The solutions are given in traditional notation by the sets $(A_{11}^3 A_{21}^4)_0$, $(A_{11}^3 A_{21}^4)_1$, $(A_{11}^3 A_{21}^4)_2$, $(A_{11}^3 A_{21}^4)_3$, $(A_{11}^3 A_{21}^4)_4$ and $(A_{11}^3 A_{21}^4)_5$. Here the "second level of the effect $(A_{11}^3 A_{21}^4)$" $=(A_{11}^3 A_{21}^4)_2=\{(x_{11}, x_{12}, x_{21}, x_{22}) \in T^*: 3x_{11}+4x_{21}=2\}$. Since each element in R has a unique decomposition in terms of the elements of $R(2)$ and $R(3)$, the equation $3x_{11}+4x_{21}=2$ can be decomposed in the two equations $3x_{11}+0x_{21}=0$ and $0x_{11}+4x_{21}=2$, i.e., $3x_{11}=0$ and $4x_{21}=2$. Hence $(A_{11}^3 A_{21}^4)_2=\{(0, 0, 2, 0), (0, 3, 2, 0), (0, 0, 2, 4), (0, 3, 2, 4), (0, 0, 2, 2), (0, 3, 2, 2)\}$. Thus finding the block constituents boils down to finding the six "levels" of $(A_{11}^3 A_{21}^4)$.

The above discussion and example have now brought us in the position to tackle the confounding aspect in a mixed prime powered lattice square. Consider the problem of constructing an $n \times n=(s_1^{d_1} \times s_2^{d_2} \times \cdots \times s_v^{d_v}) \times (s_1^{d_1} \times s_2^{d_2} \times \cdots \times s_v^{d_v})$ lattice square by providing suitable row and column confounding schemes. As has been indicated above, the Galois fields $GF(s_i)$ have as their isomorphic counterparts the subrings $R(s_i)$ of R, and the isomorphic counterparts of $EG(2d_i, s_i)$ and $PG(2d_i-1, s_i)$ are $EG^*(2d_i, s_i)$ and $PG^*(2d_i-1, s_i)$. The set of treatment combinations T and the set of effects E of the $s_1^{2d_1} \times s_2^{2d_2} \times \cdots \times s_v^{2d_v}$ factorial are represented by T^* and E^* respectively. To obtain an $n \times n$ lattice square we must confound a v-tuple of flats ((d_1-1)-flat, (d_2-1)-flat, $\cdots$, (d_v-1)-flat) with rows and another v-tuple of flats ((d_1-1)-flat, (d_2-1)-flat, $\cdots$, (d_v-1)-flat) with columns such that each pair of (d_i-1)-flats in the two-tuples generates $PG^*(2d_i-1, s_i)$, $i=1, 2, \cdots, v$. It follows that all interactions between effects, represented by the points in the flats of the first v-tuple, will also be confounded with rows and similarly in the second v-tuple with columns. For $v=2$ the number of effects confounded with rows (columns) is equal to $(s_1^{d_1}-1)/(s_1-1)+(s_2^{d_2}-1)/(s_2-1)+(s_1^{d_1}-1)(s_2^{d_2}-1)/(s_1-1)(s_2-1)$. For general v this number is equal to the sum of v such terms plus all products of such terms taken two-at-a-time, three-at-a-time, etc., and finally v at-a-time. The treatment combinations for the n^2 cells of the lattice square are obtained by developing simultaneously the "levels" of the generators of the flats in the v-tuples for rows and columns. In this process the interactions of effects from different flats in the row and column v-tuples can also be utilized to obtain the cell constituents. It follows that in $PG^*(2d_i-1, s_i)$ precisely $\gamma_i=(s_i^{d_i}-1)^2/(s_i-1)$ points (or effects) will be unconfounded with rows and columns. Hence from these points we obtain γ_i mutually orthogonal $F(n, \lambda_i=n/s_i)$-squares on s_i symbols in $GF(s_i) \cong R(s_i)$ by taking linear functions of the components of the treatment combinations in the n^2 cells of the lattice square corresponding to the unconfounded points (or effects). Briefly, this process amounts to finding at which "levels" the unconfounded effects are in the n^2 treatment combinations of the lattice square. We thus have established the following theorem:

THEOREM 3.1. *The construction of an* $n \times n = (s_1^{d_1} \times s_2^{d_2} \times \cdots \times s_v^{d_v}) \times (s_1^{d_1} \times s_2^{d_2} \times \cdots \times s_v^{d_v})$ *lattice square implies the construction of* v *sets of mutually orthogonal* $F(n, \lambda_i = n/s_i)$*-squares on* s_i *symbols, the ith set having cardinality* $(s_i^{d_i}-1)^2/(s_i-1)$, $i=1, 2, \cdots, v$.

Before giving a detailed and non-trivial example to illustrate Theorem 3.1 we state two corollaries, which are immediate consequences of the theorem.

COROLLARY 3.1. *Theorem* 3.1 *of Hedayat, Raghavarao and Seiden* [7] *follows by setting* $v=1$.

COROLLARY 3.2. *The number* $\gamma_i = (s_i^{d_i}-1)^2/(s_i-1)$ *in Theorem* 3.1 *is a lower bound on the number of mutually orthogonal* $F(n, \lambda_i = n/s_i)$*-squares on* s_i *symbols.*

Example 3.2. Let $n=6=2^1 \times 3^1$ and consider the problem of constructing a $6 \times 6 = (2^1 \times 3^1) \times (2^1 \times 3^1)$ lattice square. The details of the calculus for the $2^2 \times 3^2$ mixed factorial have been described fully in Example 3.1. To obtain a lattice square select A_{11}^3 and A_{21}^4 (represented by the points (3, 0, 0, 0) and (0, 0, 4, 0) of E^*) for the row confounding and select A_{12}^3 and A_{22}^4 (represented by (0, 3, 0, 0) and (0, 0, 0, 4) of E^*) for the column confounding. Equivalently, we may use the interactions $A_{11}^3 A_{21}^4$ and $A_{12}^3 A_{22}^4$ for the row and column confounding. So the two-tuples of flats representing the confounding schemes are({(3, 0, 0, 0), (0, 0, 4, 0)}) and ({(0, 3, 0, 0), (0, 0, 0, 4)}), and in both cases these are of the type (*o*-flat, *o*-flat). Notice that the points (3, 0, 0, 0) and (0, 3, 0, 0) generate $PG^*(1, 2)$. Similarly, the points (0, 0, 4, 0) and (0, 0, 0, 4) generate $PG^*(1, 3)$. Utilizing the interactions $A_{11}^3 A_{21}^4$ and $A_{12}^3 A_{22}^4$ we obtain the following lattice square in conventional "level" notation:

$L=$	$(A_{12}^3A_{22}^4)_0$	$(A_{12}^3A_{22}^4)_1$	$(A_{12}^3A_{22}^4)_2$	$(A_{12}^3A_{22}^4)_3$	$(A_{12}^3A_{22}^4)_4$	$(A_{12}^3A_{22}^4)_5$
$(A_{11}^3A_{21}^4)_0$	(0, 0, 0, 0)	(0, 3, 0, 4)	(0, 0, 0, 2)	(0, 3, 0, 0)	(0, 0, 0, 4)	(0, 3, 0, 2)
$(A_{11}^3A_{21}^4)_1$	(3, 0, 4, 0)	(3, 3, 4, 4)	(3, 0, 4, 2)	(3, 3, 4, 0)	(3, 0, 4, 4)	(3, 3, 4, 2)
$(A_{11}^3A_{21}^4)_2$	(0, 0, 2, 0)	(0, 3, 2, 4)	(0, 0, 2, 2)	(0, 3, 2, 0)	(0, 0, 2, 4)	(0, 3, 2, 2)
$(A_{11}^3A_{21}^4)_3$	(3, 0, 0, 0)	(3, 3, 0, 4)	(3, 0, 0, 2)	(3, 3, 0, 0)	(3, 0, 0, 4)	(3, 3, 0, 2)
$(A_{11}^3A_{21}^4)_4$	(0, 0, 4, 0)	(0, 3, 4, 4)	(0, 0, 4, 2)	(0, 3, 4, 0)	(0, 0, 4, 4)	(0, 3, 4, 2)
$(A_{11}^3A_{21}^4)_5$	(3, 0, 2, 0)	(3, 3, 2, 4)	(3, 0, 2, 2)	(3, 3, 2, 0)	(3, 0, 2, 4)	(3, 3, 2, 2)

The first cell in L is developed by solving the equations $3x_{11}+4x_{21}=0$ and $3x_{12}+4x_{22}=0$ in T^*. This reduces to $3x_{11}=0$, $3x_{12}=0$, $4x_{21}=0$ and

$4x_{22}=0$, which gives the unique solution (0, 0, 0, 0). The other cells in the first row of L are obtained in the same way. The treatment combinations in the first row form the "intrablock" subgroup and the other rows can be developed by co-setting with the appropriate treatment combinations in the first column.

Using L we obtain $\gamma_1=(2-1)^2/(2-1)=1$ $F(6, \lambda_1=3)$-square on $GF(2) \cong R(2)=\{0, 3\}$ from the unconfounded effect $A_{11}^3 A_{12}^3 \in PG^*(1, 2)$. This is a 6×6 Latin square whose entries are obtained by taking the linear combination $3x_{11}+3x_{12}$ of the components x_{11} and x_{12} in the 36 cells of L. Thus:

$$F_{A_{11}^3 A_{12}^3}=[(3x_{11}+3x_{12})]=\begin{bmatrix} 0 & 3 & 0 & 3 & 0 & 3 \\ 3 & 0 & 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 3 & 0 & 3 \\ 3 & 0 & 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 3 & 0 & 3 \\ 3 & 0 & 3 & 0 & 3 & 0 \end{bmatrix}.$$

By utilizing the unconfounded effects $A_{21}^4 A_{22}^4$ and $A_{21}^4 A_{22}^2$ of $PG^*(1, 3)$ we obtain $\gamma_2=(3-1)^2/(3-1)=2$ mutually orthogonal $F(6, \lambda_2=2)$-squares on $GF(3)\cong R(3)=\{0, 4, 2\}$. The appropriate linear combinations of components of the treatment combinations in L which produce the two squares are $4x_{21}+4x_{22}$ and $4x_{21}+2x_{22}$. Therefore:

$$F_{A_{21}^4 A_{22}^4}=[(4x_{21}+4x_{22})]=\begin{bmatrix} 0 & 4 & 2 & 0 & 4 & 2 \\ 4 & 2 & 0 & 4 & 2 & 0 \\ 2 & 0 & 4 & 2 & 0 & 4 \\ 0 & 4 & 2 & 0 & 4 & 2 \\ 4 & 2 & 0 & 4 & 2 & 0 \\ 2 & 0 & 4 & 2 & 0 & 4 \end{bmatrix}, \quad \text{and}$$

$$F_{A_{21}^4 A_{22}^2}=[(4x_{21}+2x_{22})]=\begin{bmatrix} 0 & 2 & 4 & 0 & 2 & 4 \\ 4 & 0 & 2 & 4 & 0 & 2 \\ 2 & 4 & 0 & 2 & 4 & 0 \\ 0 & 2 & 4 & 0 & 2 & 4 \\ 4 & 0 & 2 & 4 & 0 & 2 \\ 2 & 4 & 0 & 2 & 4 & 0 \end{bmatrix}.$$

These sets of F-squares illustrate Theorem 3.1 and from Corollary 3.2 we know that there are at least 1 mutually orthogonal $F(6, 2)$-square on $\{0, 3\}$ and at least 2 mutually orthogonal $F(6, 3)$-squares on $\{0, 4, 2\}$.

4. Discussion of the results

A generalization of our results could be obtained by abandoning the canonical decomposition and using arbitrary decompositions, e.g. $n=12=6^1\times 2^1$. The penalty carried by this approach would be the loss of the finite geometrical aspects and the subsequent combinatorial complexities. However, it would provide F-squares which are not necessarily based on Galois fields or their isomorphic counterparts.

Another problem is to sharpen the lower bound obtained in Corollary 3.2. A formidable problem is to provide complete sets of mutually orthogonal $F(n, \lambda_i=n/s_i)$-squares.

NATIONAL UNIVERSITY OF SINGAPORE*
CORNELL UNIVERSITY

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* Now at Chaing Mai University.