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ASYMPTOTICALLY MINIMUM VARIANCE UNBIASED ESTIMATION FOR A CLASS OF POWER SERIES DISTRIBUTIONS

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Summary

The problem of finding an asymptotically minimum variance unbiased estimator (A.M.V.U.E.) for the parameter of certain truncated power series distributions, is discussed, when the generating function of their coefficients are i) polynomials of binomial type ii) generalized ascending factorials iii) polynomials with coefficients the well known Eulerian numbers.

1. Introduction and notation

The zero truncated binomial distribution has probability function

$$p_m(n)=\binom{m}{n}\left(\frac{p}{1-p}\right)^n\Big/\sum_{j=1}^{m}\binom{m}{j}\left(\frac{p}{1-p}\right)^j, \qquad n=1,2,\cdots,m,\ 0<p<1$$

and can be seen as a special case of a generalized power series distribution with probability function

$$(1.1)\qquad p_m(n)=A(m,n)\lambda^n/[A_m(\lambda)-A_m(0)] \qquad n=1,2,\cdots,m,\ \lambda>0$$

where

$$A_m(x)=\sum_{j=0}^{m}A(m,j)x^j\,, \qquad A(m,j)\geqq 0\,.$$

Patil ([5], Corollary 1, p. 1052) proved for the distribution (1.1), that there is no M.V.U.E. for λ since the range of values $\{1,2,\cdots,m\}$ is finite. Cacoullos and Charalambides [2] constructed an A.M.V.U.E. ($m\to\infty$) for the parameter $p/(1-p)$ of the zero truncated binomial distribution.

In this paper we consider the problem of finding an A.M.V.U.E.

Key words: Power series distribution, minimum variance unbiased estimator, polynomials of binomial type, generalized ascending factorial, Eulerian numbers.

for the parameter λ of (1.1) when the sequence of polynomials $A_m(x) = \sum_{j=0}^{m} A(m, j)x^j$, $m=0, 1, 2, \cdots$, is defined in the following ways.

(i) By it's exponential generating function

$$\sum_{m=0}^{\infty} A_m(x)z^m/m! = e^{x\{g(z)-g(0)\}}, \qquad x>0,\ g(z)=\sum_{i=0}^{\infty} k_i z^i/i! \tag{1.2}$$

that is, $A_m(x)$ is an exponential binomial type polynomial. From (1.2) we have

$$\sum_{m=n}^{\infty} A(m, n)z^m/m! = [g(z)-g(0)]^n/n! . \tag{1.3}$$

For the numbers $A(m, n)$ the relation (1.3) reveals that

$$A(0, 0)=1, \quad A(m, 0)=A(0, n)=0 \qquad \text{when } n, m\neq 0 \text{ and}$$

$$A(m, n)=\frac{1}{n!}\sum \binom{m}{v_1, \cdots, v_n} k_{v_1}, \cdots, k_{v_n} \tag{1.4}$$

where the summation is over all n-tuples $(v_1, \cdots, v_n)$ with $v_i \geqq 1$ and $\sum_{i=1}^{n} v_i = m$.

(ii) By the following formula

$$A_m(x)=(x+\xi_{m-1})(x+\xi_{m-2})\cdots(x+\xi_0), \qquad m=1, 2, \cdots, \ A_0(x)=1 . \tag{1.5}$$

From (1.5) we have that

$$A(m+1, n)=\xi_m A(m, n)+A(m, n-1) \tag{1.6}$$

with

$$A(0, 0)=1, \quad A(m, 0)=A(0, n)=0 \qquad \text{when } m, n\neq 0$$

that is $A(m, n)$ are the generalized signless Stirling numbers of the first kind which has been defined by Comtet [4].

(iii) By it's exponential generating function which has the form

$$\sum_{m=0}^{\infty} A_m(x)z^m/m! = \frac{x(1-x)}{e^{(x-1)z}-x} \tag{1.7}$$

that is, $A(m, n)$ are the Eulerian numbers (the number of permutations of $1, 2, \cdots, m$ having exactly n rises) for which we have the recurrence relation

$$A(m+1, n)=nA(m, n)+(m-n+2)A(m, n-1) \tag{1.8}$$

with boundary conditions

$$A(0,0)=1\,,\quad A(m,0)=A(0,n)=0 \qquad \text{when } m,n\neq 0\,.$$

The above categories cover in particular most standard cases. For instance $A(m, n)$ may be the signless Stirling numbers of the first kind or the Stirling numbers of the second kind or even the number of permutations of m elements with n cycles or n ordered cycles with given minimal and maximal size, etc.

2. Preparatory results

Let $X_1, X_2, \cdots, X_N$ be a random sample from (1.1). Then the sum $Z=\sum_{i=1}^{N} X_i$ is a complete sufficient statistic and has probability function

$$\text{(2.1)} \qquad \mathrm{P}(Z=z)=\frac{C(N,m,z)\lambda^z}{\left[\sum_{j=1}^{m} A(m,j)\lambda^j\right]^N}\,, \qquad z=N, N+1,\cdots, Nm$$

with

$$C(N,m,z)=\sum A(m,n_1)A(m,n_2)\cdots A(m,n_N)$$

where the summation is extended over all that ordered N-tuples $(n_1, \cdots, n_N)$ of positive integers $n_i\leqq m$, $i=1,2,\cdots,N$ such that $n_1+n_2+\cdots+n_N=Z$. Note that

$$\text{(2.2)} \qquad \sum_{z=N}^{Nm} C(N,m,z)\lambda^z=\left[\sum_{j=1}^{m} A(m,j)\lambda^j\right]^N=[A_m(\lambda)]^N$$

which on differentiating with respect to λ and equating the coefficients of λ^z in both sides of the resulting equation gives the recurrence relation

$$C(N,m,Z+1)=\frac{N}{z+1}\sum_{k=0}^{m-1}(k+1)A(m,k+1)C(N-1,m,z-k)\,, \qquad m=1,2,\cdots$$

with $C(0,m,0)=1$ and $C(N,m,Nm)=[A(m,m)]^N$.

Ignoring the fact that there is no A.M.V.U. estimator for λ, let us denote by $\hat{\lambda}(N,m,z)$ an estimate of λ based on the complete sufficient statistic Z and let us proceed to see where unbiasedness fails. Using (2.1) the condition of unbiasedness is equivalent to

$$\sum_{z=N}^{Nm}\hat{\lambda}(N,m,z)C(N,m,z)\lambda^z\equiv\sum_{z=N}^{Nm}C(N,m,z)\lambda^{z+1}\,.$$

Equating the coefficients of λ^z on both sides shows that the above condition can be satisfied if and only if,

$$(2.3)\qquad \hat{\lambda}(N, m, z)=C(N, m, z-1)/C(N, m, z) \quad z=N, N+1, \cdots, Nm-1 \text{ and } C(N, m, Nm)=0 .$$

Obviously, however $C(N, m, Nm)=0$ can not be satisfied and $\hat{\lambda}(N, m, z)$ as defined in (2.3) has relative bias, caused by the last term z^{Nm+1}, equal to

$$(2.4)\qquad \mathrm{E}\left(\frac{\hat{\lambda}(N, m, z)}{\lambda}\right)-1=-[A_m(\lambda)]^{-N}C(N, m, mN)\lambda^{Nm} = -[A_m(\lambda)]^{-N}[A(m, m)\lambda^m]^N .$$

So, $\hat{\lambda}(N, m, z)$ given by (2.3) has been constructed in a unique way and it is an A.M.V.U.E. of λ in the case that the relative bias given by (2.4) converges to zero.

Remark 1. It can be easily seen from (2.4) that, if the number of samples N tends to infinity the estimator given by (2.3) is an A.M.V.U.E. and if $\hat{\lambda}(N, m, z)$ is an A.M.V.U.E. ($m\to\infty$ or $N\to\infty$), then $\hat{\lambda}(N, m, z)+\hat{\lambda}_0(N, m)$, where $\hat{\lambda}_0(N, m)\to 0$ as $m\to\infty$ or $N\to\infty$, is also an A.M.V.U.E. of λ.

3. A.M.V.U.E. for λ

THEOREM 3.1. *Let $A_m(x)$ be an exponential polynomial in x of degree m defined by* (1.2) with $k_i\geqq 0$, $i=0, 1, 2, \cdots$, $0<a<x<b$. *Assume the existence of a real root r of the equation $xrg'(r)=m$,*

$$(3.1)\qquad D^k[g(r)]\leqq kc_m D^{k-1}[g(r)] , \qquad k=1, 2, 3,\ D^k\equiv d^k/dr^k$$

with $(rc_m)^\gamma/m\to 0$ as $m\to\infty$, γ positive and

$$(3.2)\qquad g(r)-Re[g(re^{i\theta})]\geqq c[g(r)]^{1/(1+l)} , \qquad m^{-3/8}<|\theta|<\pi,\ m\geqq m_0,\ c, l$$

positive constants and Re is real part.

Then, $A_m(x)$ has the following asymptotic expansion

$$A_m(x)=[xg'(r)]^m \exp[x\{g(r)-g(0)\}-m][1+O(1)]/ [1+rg''(r)/g'(r)]^{1/2} \qquad \text{as } m\to\infty .$$

Moreover assuming that $rg'(0)/m\to 0$ the estimator $\hat{\lambda}(N, m, z)$ of (2.3) *is an A.M.V.U.E. of λ with $0<a<\lambda<b$.*

PROOF. Using Caushy's integral formula, $A_m(x)$ may be written in the form

$$A_m(x)=\frac{m!}{2\pi i}\int_C z^{-m-1}\exp[x\{g(z)-g(0)\}]dz$$

where C is the circle $z=re^{i\theta}$, $i=\sqrt{-1}$. Therefore

$$(3.3)\quad A_m(x)=\frac{m!\exp[x\{g(r)-g(0)\}]}{r^m 2\pi}\int_{-\pi}^{\pi}\exp[x\{g(re^{i\theta})-g(r)\}-i\theta m]d\theta$$
$$=\frac{m!\exp[x\{g(r)-g(0)\}]}{r^m\sqrt{2\pi}}\{I_1+I_2\}$$

where

$$I_1=\frac{1}{\sqrt{2\pi}}\int_{-\delta}^{\delta}\exp[G(\theta)]d\theta\ ,$$

$$I_2=\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{-\delta}\exp[G(\theta)]d\theta+\frac{1}{\sqrt{2\pi}}\int_{\delta}^{\pi}\exp[G(\theta)]d\theta$$

with

$$G(\theta)=x\{g(re^{i\theta})-g(r)\}-i\theta m\ .$$

From the assumption of the existence of a real root r of the equation

$$(3.4)\qquad xrg'(r)=m$$

along with the fact that $\psi(r)=xrg'(r)-m$ is an increasing function it follows that this root is unique. Choosing the radius r of the circle of integration to be the root of (3.4) and

$$(3.5)\qquad \delta=m^{-3/8}$$

we can approximate I_1 and show that I_2 is negligible for large m. For some ζ between 0 and θ we have by Taylor and by (3.4) that

$$G(\theta)\equiv x\{g(re^{i\theta})-g(r)\}-i\theta m$$
$$=-\frac{x}{2}\theta^2\{rg'(r)+r^2g''(r)\}-xire^{i\zeta}(\theta^3/6)$$
$$\cdot\ \{g'(re^{i\zeta})+3re^{i\zeta}g''(re^{i\zeta})+r^2e^{2i\zeta}g'''(re^{i\zeta})\}\ .$$

Using the relations (3.1), (3.4) and (3.5) we have for $|\theta|<m^{-3/8}$ $|-xire^{i\zeta}\cdot(\theta^3/6)\{g'(re^{i\zeta})+3re^{i\zeta}g''(re^{i\zeta})+r^2e^{2i\zeta}g'''(re^{i\zeta})\}|\leqq m^{-1/8}(1+6rc_m+6r^2c_m^2)\to 0$ so

$$I_1=\frac{1}{\sqrt{2\pi}}\int_{-\delta}^{\delta}\exp\{G(\theta)\}d\theta=\frac{1}{(\pi x)^{1/2}[rg'(r)+r^2g''(r)]^{1/2}}\int_{-\varepsilon}^{\varepsilon}e^{-\varphi^2}d\varphi[1+O(1)]$$

where

$$\varphi=(x/2)^{1/2}[rg'(r)+r^2g''(r)]^{1/2}\cdot\theta$$

and

$$\varepsilon=[(x/2)\{rg'(r)+r^2g''(r)\}]^{1/2}\cdot\delta\ .$$

Note that $\varepsilon\to\infty$ as $m\to\infty$, since on using (3.4) and (3.5)

$$\begin{aligned}\varepsilon&=m^{-3/8}[(x/2)\{rg'(r)+r^2g''(r)\}]^{1/2}\\&=m^{1/8}\left[\frac{1}{2}\{1+rg''(r)/g'(r)\}\right]^{1/2}\geqq\frac{1}{\sqrt{2}}m^{1/8}.\end{aligned}$$

The asymptotic expansions of the form $\int_{-\varepsilon}^{\varepsilon}e^{-\varphi^2}$ (Polynomial in φ) $d\varphi$ along with these remarks justify the replacement of ε by ∞. Thus

$$I_1=\frac{1+O(1)}{(x)^{1/2}\{rg'(r)+r^2g''(r)\}^{1/2}}.$$

By (3.2) we have $|I_2|\leqq\frac{2(\pi-\delta)}{\sqrt{2\pi}}\exp(-xc(g(r))^{1/(1+l)})$, $\delta=m^{-3/8}$. So $I_2=O\left(\frac{rc_m}{m}\right)$ since on using (3.1) $g(r)\geqq g'(r)/c_m=xrg'(r)/xrc_m=m/xrc_m$, $m/rc_m\to\infty$ as $m\to\infty$ and consequently $\frac{|I_2|}{rc_m/m}\leqq\frac{2(\pi-\delta)}{\sqrt{2\pi}}\exp(-xc(m/xrc_m)^{1/(1+l)})\cdot(m/rc_m)\to0$ as $m\to\infty$. We also have

$$I_2x^{1/2}\{rg'(r)+r^2g''(r)\}^{1/2}=I_2m^{1/2}\{1+rg''(r)/g'(r)\}^{1/2}=O(1).$$

By (3.3) we have

$$A_m(x)=\frac{m!\exp[x\{g(r)-g(0)\}]}{r^m(2\pi x)^{1/2}\{rg'(r)+r^2g''(r)\}^{1/2}}[1+O(1)]$$

or

$$\begin{aligned}A_m(x)=&[xg'(r)]^m\exp[x\{g(r)-g(0)\}-m]\\&\cdot[1+O(1)]/[1+rg''(r)/g'(r)]^{1/2}\end{aligned}$$

by expanding $m!=(2\pi)^{1/2}m^{m+1/2}e^{-m}[1+O(1)]$.

We shall now prove that the relative bias of the estimator $\hat{\lambda}(N, m, z)$ of (2.3) tends to zero as $m\to\infty$. On using the asymptotic expansion of $A_m(x)$, (1.4), (3.1) and the condition $rg'(0)/m\to0$ we have

$$\begin{aligned}\left|\frac{A(m,m)\cdot\lambda^m}{A_m(\lambda)}\right|&=\left|\frac{[eg'(0)\cdot\lambda r/m]^m[1+rg''(r)/g'(r)]^{1/2}}{e^{\lambda\{g(r)-g(0)\}}[1+O(1)]}\right|\\&\leqq K\frac{(m/rc_m)}{e^{m/rc_m}}\to0\quad\text{as}\quad m\to\infty\end{aligned}$$

where K is a positive constant.

Remark 2. The first term in the expansion of $G(\theta)$ which may converge to zero under our assumptions, is the one involving up to third order derivatives of $g(r)$. So in (3.1) we need $k=1, 2, 3$.

Special cases

1. *$g(z)$ is a polynomial of finite degree $\rho>1$, with positive coefficients*

The equation $\lambda rg'(r)=m$ has always a real solution r, since the function $\varphi(r)=\lambda rg'(r)-m$ is an monotone increasing function in r with $\varphi(0)=-m<0$ and there exists a real number a, such that $\varphi(a)>0$. Note that $r\to\infty$ as $m\to\infty$. We can easily see that, the condition (3.1) hold with $c_m=\rho/r$ and $rg'(0)/m=k_1/\lambda g'(r)\to 0$ $(\rho>1)$. By Canfield ([1], Lemma 1) we have that

$$Re\{g(re^{i\theta})-g(r)\geqq c_1r^{1/4} \qquad \text{when } \pi\geqq|\theta|\geqq r^{-\rho/2+1/8}\,.$$

But $r^{-\rho/2+1/8}>m^{-3/8}$ and there are positive constants c, l such that

$$c_1r^{1/4}\geqq c[g(r)]^{1/(1+l)}\,.$$

So the condition (3.2) also hold and Theorem 3.1 is applied.

2. $g(z)=(1+z)^s$, $s>1$, *s integer*

By [3] we have that $A(m, n)$ are the numbers $C(m, n, s)$ appearing in the n-fold convolution of a truncated binomial distribution and satisfy the recurrence relation $A(m+1, n)=(sn-m)A(m, n)+sA(m, n-1)$ with boundary conditions $A(0, 0)=1$, $A(0, n)=0$ if $n\neq 0$ and $sn-m>0$ for $n=1, 2,\cdots, m+1$, $m=0, 1, 2,\cdots, s$. Note that $g(z)$ has positive coefficients when s is an integer and $s\to\infty$ when $m\to\infty$ $(s>m)$. So the case 2 is not a special case of the case 1.

We can easily see that there exists a real solution r, of the equation $\lambda sr(1+r)^{s-1}=m$, where $0<r<1$ and $sr\to\infty$ when $m\to\infty$. We can also see that (3.1) hold with $c_m=s/(1+r)$. Moreover for $\delta=m^{-3/8}<|\theta|<\pi$,

$$\begin{aligned}(1+r)^s-Re(1+re^{i\theta})^s&\geqq(1+r)^s-|1+re^{i\theta}|^s\\&=(1+r)^s-\{(1+r)^2-2r+2r\cos\theta\}^{s/2}\\&\geqq(1+r)^s\left[1-\left\{1+\frac{2r(\cos\delta-1)}{(1+r)^2}\right\}^{s/2}\right]\\&=(1+r)^s\left[1-\sum_{k=0}^{\infty}\binom{s/2}{k}\left\{\frac{2r(\cos\delta-1)}{(1+r)^2}\right\}^{k}\right]\\&=(1+r)^{s-2}sr(1-\cos\delta)\\&\quad\cdot\left[1-\sum_{k=2}^{\infty}\frac{\binom{s/2}{k}}{s/2}\left\{\frac{2r(\cos\delta-1)}{(1+r)^2}\right\}^{k-1}\right].\end{aligned}$$

Since $\binom{s/2}{k}\leqq(s/2)^k$, $k=1, 2,\cdots$, and

$$\left|\frac{sr(1-\cos\delta)}{(1+r)^2}\right|\leqq\frac{sr\delta^2/2}{(1+r)^2}\leqq\frac{sr/(1+r)}{m^{3/4}}\to 0 \qquad \text{as } m\to\infty$$

we have that

$$\left|\sum_{k=2}^{\infty}\frac{\binom{s/2}{k}}{s/2}\left\{\frac{2r(\cos\delta-1)}{(1+r)^2}\right\}^{k-1}\right|$$
$$\leqq\sum_{k=1}^{\infty}\left\{\frac{sr(1-\cos\delta)}{(1+r)^2}\right\}^{k}=\frac{sr(1-\cos\delta)/(1+r)^2}{1-sr(1-\cos\delta)/(1+r)^2}\to 0 \qquad \text{as } m\to\infty$$

and consequently $(1+r)^s-Re(1+re^{i\theta})^s\geqq c(1+r)^{s/4}$ for every $m\geqq m_0$ where we have used the fact that $1-\cos\delta>\delta^2/8$ for small δ. So (3.2) hold and Theorem 3.1 is applied.

3. $g(z)=(1-z)^{-s}$, $s>1$, *s integer*, $0<z<1$

By [3] we have that $A(m,n)$ are the numbers $|C(m,n,-s)|$, appearing in the n-fold convolution of a truncated negative binomial distribution and satisfy the recurrence relation $A(m+1,n)=(sn+m)A(m,n)+sA(m,n-1)$ with boundary conditions $A(0,0)=1$ and $A(0,n)=0$ if $n\neq 0$. We can also see that (3.1) hold with $c_m=s/(1-r)$. Moreover for $\delta=m^{-3/8}<|\theta|<\pi$, $(1-r)^{-s}-Re(1-re^{i\theta})^{-s}\geqq(1-r)^{-s}-|1-re^{i\theta}|^{-s}\geqq(1-r)^{-s}-[(1-r)^2+2r(1-\cos\theta)]^{-s/2}\geqq(1-r)^{-s}-[(1-r)^2+2r(1-\cos\delta)]^{-s/2}=(1-r)^{-s-2}\cdot sr(1-\cos\delta)\left[1-\sum_{k=2}^{\infty}\frac{\binom{s/2+k-1}{k}}{s/2}\left\{\frac{2r(\cos\delta-1)}{(1-r)^2}\right\}^{k-1}\right]$.

Since $\binom{s/2+k-1}{k}\leqq(s/2)^k$, $k=1,2,\cdots$, $s>1$ and

$$\left|\frac{sr(1-\cos\delta)}{(1-r)^2}\right|\leqq\frac{sr\delta^2/2}{(1-r)^2}\leqq\frac{srm^{-3/4}}{(1-r)^2}=\lambda^{-3/4}(sr)^{1/4}(1-r)^{3s/4-5/4}\to 0$$
$$\text{as } m\to\infty\ (r\to 1)$$

we have that

$$\left|\sum_{k=2}^{\infty}\frac{\binom{s/2+k-1}{k}}{s/2}\left\{\frac{2r(\cos\delta-1)}{(1-r)^2}\right\}^{k-1}\right|$$
$$\leqq\sum_{k=1}^{\infty}\left\{\frac{sr(1-\cos\delta)}{(1-r)^2}\right\}^{k}=\frac{sr(1-\cos\delta)/(1-r)^2}{1-sr(1-\cos\delta)/(1-r)^2}\to 0 \qquad \text{as } m\to\infty$$

and consequently $(1-r)^{-s}-Re(1-re^{i\theta})^{-s}\geqq c(1-r)^{-s/4}$ for every $m\geqq m_0$ where we have used the fact that $1-\cos\delta\geqq\delta^2/8$ for small δ. So (3.2) hold and Theorem 3.1 is applied.

4. $g(z)=e^z$

Now $A(m,n)$ are the Stirling numbers of the second kind which satisfy the recurrence relation $A(m+1,n)=nA(m,n)+A(m,n-1)$ with boundary conditions $A(0, 0)=1$, $A(0, n)=0$ if $n\neq 0$. The equation $\lambda re^r=m$ has always a real solution r and $r\to\infty$ as $m\to\infty$. We can easily see that (3.1) hold with $c_m=1$. Moreover for $\delta=m^{-3/8}<|\theta|<\pi$, $e^r-Re\{e^{re^{i\theta}}\}\geqq e^r-|e^{re^{i\theta}}|=e^r-e^{r\cos\theta}=e^r\{1-e^{r(\cos\theta-1)}\}\geqq e^r\{1-e^{r(\cos\delta-1)}\}=e^r r(1-\cos\delta)\cdot\left[1+\sum_{k=2}^{\infty}\frac{\{r(\cos\delta-1)\}^{k-1}}{k!}\right]$.

Since $k!\geqq 2^{k-1}$, $k=1, 2,\cdots$, and $\dfrac{r(\cos\delta-1)}{2}\to 0$ as $m\to\infty$ $(r\to\infty)$ we have that $\left|\sum_{k=2}^{\infty}\dfrac{[r(\cos\delta-1)]^{k-1}}{k!}\right|\leqq\sum_{k=1}^{\infty}\left[\dfrac{r(1-\cos\delta)}{2}\right]^k=\dfrac{r(1-\cos\delta)/2}{1-r(1-\cos\delta/2)}\to 0$ as $m\to\infty$ and consequently $e^r-Re\{e^{re^{i\theta}}\}\geqq ce^{r/4}$ for every $m\geqq m_0$. So (3.2) hold and Theorem 3.1 is applied.

THEOREM 3.2. *Let $A_m(\lambda)$ be a polynomial given by* (1.5) *with* $0<a<\lambda<b$. *If* $\xi_m\geqq 1$, *then* $\hat{\lambda}(N, m, z)$ *given by* (2.3) *is an A.M.V.U.E. of* λ *of* (1.1).

PROOF. Since $A(m, m)=1$ and $\xi_m\geqq 1$ we have that $|A(m, m)\lambda^m/A_m(\lambda)|^N\leqq(\lambda/(1+\lambda))^{mN}\to 0$ as $m\to\infty$. So, the relative bias of $\hat{\lambda}(N, m, z)$ given by (2.3) tends to zero as $m\to\infty$.

Special cases

1. with $\xi_m=1$ we have that $A(m, n)=\binom{m}{n}$ and (1.1) is the zero truncated binomial distribution with $p=\lambda/(1+\lambda)$.
2. with $\xi_m=m$ we have that $A(m, n)$ are the signless Stirling numbers of the first kind.

THEOREM 3.3. *Let $A_m(\lambda)$ be a polynomial given by* (1.7) *with* $0<a<\lambda<b$. *Then* $\hat{\lambda}(N, m, z)$ *given by* (2.3) *is an A.M.V.U.E. of* λ, *of* (1.1).

PROOF. Multiplying both members of (1.8) by e^{sn} and summing for $n=1, 2,\cdots, m+1$ we have

$$F_{m+1}(s)=(1-e^s)F_m'(s)+(m+1)e^sA_m(s) \tag{3.6}$$

where

$$F_m(s)=\sum_{n=0}^{m}A(m, n)e^{sn}\,,\qquad F_0(s)=1\,.$$

The general solution of the difference-differential equation (3.6) is the following

$$F_m(s)=m!\left(\frac{e^s-1}{s}\right)^{m+1}.$$

So

$$A_m(\lambda)=F_m(\ln \lambda)=m!\left(\frac{\lambda-1}{\ln \lambda}\right)^{m+1}. \tag{3.7}$$

Since $A(m, m)=1$ the relative bias of $\hat{\lambda}(N, m, z)$ given by (2.3) tends to zero when $m\to\infty$. In fact

$$|A(m, m)\lambda^m/A_m(\lambda)|^N=\left|\lambda^m\left(\frac{\ln \lambda}{\lambda-1}\right)^{m+1}\Big/m!\right|^N\to 0 \qquad \text{as } m\to\infty .$$

Remark 3. The family of estimators provided by (2.3) does not always yield asymptotically minimum variance unbiased estimators. A specific example is the following.

Consider the set of arrangements of m elements of order j, $(m)_j=m(m-1)\cdots(m-j+1)$, weighted by λ^j, $j=1, 2, \cdots, m$, $\lambda>0$. If an arrangement $(m)_n$ is randomly choosen, let X be the random variable that assigns to $(m)_n$ its order n. Then

$$\mathrm{P}\,(X=n)=p_m(n)=(m)_n\lambda^n\Big/\sum_{j=1}^{m}(m)_j\lambda^j \qquad n=1, 2, \cdots, m .$$

Notice now that the relative bias of the estimator $\hat{\lambda}(N, m, z)$ does not go to zero. In fact

$$\begin{aligned} m!\lambda^m\Big/\sum_{j=1}^{m}(m)_j\lambda^j &= \left[\sum_{j=1}^{m}(1/\lambda)^{m-j}/(m-j)!\right]^{-1} \\ &= \left[\sum_{j=0}^{m}(1/\lambda)^j/j!-(1/\lambda)^m/m!\right]^{-1}\to e^{-1/\lambda} \qquad \text{as } m\to\infty . \end{aligned}$$

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