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DISCRETE DISTRIBUTIONS OF ORDER k ON A BINARY SEQUENCE

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Summary

This paper considers discrete distributions of order k based on a binary sequence which is defined as an extension of independent trials with a constant success probability and is more practical than the independent trials. Some results on calculation of probabilities and characteristics of the distributions are obtained as well as their formal expressions. Examples and an application are also given.

1. Introduction

The exact distribution theory of the discrete distributions of order k was initiated by Philippou, Georghiou and Philippou [10]. They introduced the generalized geometric distribution of order k, which is the distribution of the number of trials until the occurrence of the kth consecutive success in independent trials with success probability p. Before their work, Feller [5] had noted that the distribution can be regarded as an example from renewal theory. They, however, gave exact probability of the distribution explicitly and investigated mutual relationships among a class of distributions of order k. Some distributions of order k were indeed defined such as the negative binomial, the Poisson and the logarithmic series distributions of order k, etc. Various properties of the distributions and mutual relationships among themselves have been studied (cf. e.g. Philippou [9], Philippou and Muwafi [11], Philippou, Georghiou and Philippou [10] and Aki, Kuboki and Hirano [2]). Recently, Hirano [7] defined the binomial distribution of order k and gave the exact probability of it, only normal approximation of which had been given by Feller [5].

The distributions stated above are all based essentially on independent trials with success probability p. The distributions of order k are

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closely related to occurrences of k consecutive successes in the independent trials. It is certainly interesting that the classical fundamental distributions are generalized to the corresponding ones of order k while still retaining the mutual relationships among themselves. However, what is the statistical meaning of considering occurrences of k consecutive successes? This question, to which we can not answer clearly based on the independent trials, leads us to further extension of the distributions to the ones on the binary sequence of order k, which will be defined in Section 2. Examples in Section 2 will give the answer to the question. Moreover, the extension will turn out to be a very practical one.

In Section 3 the geometric distribution of order k will be extended to the corresponding one on the binary sequence of order k. Some of its properties will be also given. We will discuss in Section 4 mutual relationships among the extended geometric, negative binomial, Poisson and logarithmic series distributions of order k, which are all extended on the binary sequence of order k in a natural manner. In Section 5 we will define the extended binomial distribution of order k, which will be found to have useful application to the reliability theory.

2. The binary sequence of order k

Suppose we are given an infinite sequence of $\{0, 1\}$-valued random variables X_i, $i=0, 1, 2, \cdots$, which are defined on some probability space $(\Omega, \mathscr{F}, P)$.

DEFINITION 2.1. A binary sequence $\{X_i\}$ is said to be a *binary sequence of order* k on $(\Omega, \mathscr{F}, P)$ if there exist a positive integer k and k real numbers $0<p_1, p_2, \cdots, p_k<1$ such that

(1) $X_0=0$ almost surely and

(2) $P(X_n=1 \mid X_0=x_0, X_1=x_1, \cdots, X_{n-1}=x_{n-1})=p_j$

is satisfied for any positive integer n, where $j=r-[(r-1)/k]\cdot k$, r is the smallest positive integer which satisfies $x_{n-r}=0$, and we denote, by $[a]$, the largest integer not exceeding a.

Remark 2.1. A binary sequence of order k is usually a dependent sequence. However, it is easily seen that the binary sequence of order k for $p_1=p_2=\cdots=p_k=p$ becomes the independent trials with success probability p. Therefore, we can say that the independent trials with success probability p is a binary sequence of order k for any positive integer k.

Remark 2.2. From the definition, the conditional distribution of $X_n, X_{n+1}, \cdots$ given $X_n=0$ is equal to the distribution of $X_0, X_1, \cdots$ for all positive integer n.

Let $\{X_i\}$ be a binary sequence of order k. Then the finite dimensional distributions of the sequence $\{X_i\}$ are given in the following result.

PROPOSITION 2.1. *Let $\{X_i\}$ be a binary sequence of order k with $p_1, p_2, \cdots, p_k$.*
(1) *For any positive integer n, it holds*

$$P(X_n=1)=\sum_{r=1}^{n} p_1p_2\cdots p_{r-k[r/k]}(p_1\cdots p_k)^{[r/k]} P(X_{n-r}=0) .$$

(2) *Let n be any positive integer. Then, for positive integers $i_1<i_2<\cdots<i_n$, the following formula holds:*

$$\text{(2.1)}\quad P(X_{i_1}=1,\cdots, X_{i_n}=1)$$
$$=P(X_{i_n}=1)+\sum_{r=1}^{n-1}\sum_{\substack{j_1,\cdots,j_r\in\{i_1,\cdots,i_{n-1}\}\\ j_1<j_2<\cdots<j_r}}(-1)^r P(X_{j_1}=0)P(X_{j_2-j_1}=0)$$
$$\times\cdots\times P(X_{j_r-j_{r-1}}=0)P(X_{i_n-j_r}=1) .$$

PROOF. We show (1) first.
The event $\{X_n=1\}$ is written as the disjoint union of the events,

$$\bigcup_{r=1}^{n}\{X_n=1,\ X_{n-1}=1,\cdots, X_{n-r+1}=1,\ X_{n-r}=0\} .$$

By Remark 2.2, we have

$$P(X_n=1)=\sum_{r=1}^{n} P(X_n=1,\ X_{n-1}=1,\cdots, X_{n-r+1}=1,\ X_{n-r}=0)$$
$$=\sum_{r=1}^{n} P(X_n=1,\ X_{n-1}=1,\cdots, X_{n-r+1}=1\,|\,X_{n-r}=0)P(X_{n-r}=0)$$
$$=\sum_{r=1}^{n} p_1p_2\cdots p_{r-k[r/k]}(p_1\cdots p_k)^{[r/k]} P(X_{n-r}=0) .$$

Next, we prove (2) by induction with respect to n. When $n=1$, (2.1) holds obviously. Assume that (2.1) holds for $n-1$, $(n>1)$.

Note that

$$P(X_{i_1}=1,\cdots, X_{i_n}=1)$$
$$=P(X_{i_2}=1,\cdots, X_{i_n}=1)-P(X_{i_1}=0,\ X_{i_2}=1,\cdots, X_{i_n}=1) .$$

From Remark 2.2, it holds

$$P(X_{i_1}=0,\ X_{i_2}=1,\cdots, X_{i_n}=1)=P(X_{i_1}=0)P(X_{i_2-i_1}=1,\cdots, X_{i_n-i_1}=1) .$$

Then, by induction hypothesis, we have

$$
\begin{aligned}
&\mathrm{P}(X_{i_1}=1,\cdots,X_{i_n}=1)\\
&=\mathrm{P}(X_{i_n}=1)+\sum_{r=1}^{n-2}\sum_{\substack{j_1,\cdots,j_r\in\{i_2,\cdots,i_{n-1}\}\\ j_1<j_2<\cdots<j_r}}(-1)^r\,\mathrm{P}(X_{j_1}=0)\\
&\quad\times\cdots\times\mathrm{P}(X_{j_r-j_{r-1}}=0)\,\mathrm{P}(X_{i_n-j_r}=1)-\mathrm{P}(X_{i_1}=0)\,\mathrm{P}(X_{i_n}=1)\\
&\quad+\sum_{r=1}^{n-2}\sum_{\substack{j_1,\cdots,j_r\in\{i_2-i_1,\cdots,i_{n-1}-i_1\}\\ j_1<j_2<\cdots<j_r}}(-1)^{r+1}\,\mathrm{P}(X_{i_1}=0)\,\mathrm{P}(X_{j_1}=0)\\
&\quad\times\cdots\times\mathrm{P}(X_{j_r-j_{r-1}}=0)\,\mathrm{P}(X_{i_n-j_r}=1)\\
&=\mathrm{P}(X_{i_n}=1)+\sum_{r=1}^{n-1}\sum_{\substack{j_1,\cdots,j_r\in\{i_1,\cdots,i_{n-1}\}\\ j_1<j_2<\cdots<j_r}}(-1)^r\,\mathrm{P}(X_{j_1}=0)\\
&\quad\cdots\mathrm{P}(X_{j_r-j_{r-1}}=0)\,\mathrm{P}(X_{i_n-j_r}=1)\,.
\end{aligned}
$$

Consequently, (2.1) holds for every positive integer n. This completes the proof.

Let $\{X_i\}$ be a binary sequence of order k. According to the case of independent trials, X_n is sometimes called nth trial. And the outcomes "1" and "0" are called "success" and "failure", respectively.

Now, we give some examples.

Example 2.1 (Urn model). An urn contains w white and r red balls. Let k be a fixed positive integer such that $k\leqq r$. A ball is drawn at random. If it is a white ball, it is replaced. And if it is a red ball, it is not replaced but laid beside the urn. A new random drawing is made from the urn. If it is a red ball, it is not replaced but laid beside the urn. But if it is white, then it is replaced and, moreover, all red balls outside the urn are replaced if they exist. This procedure is repeated while the number of red balls outside the urn is less than k. If the number of red balls outside the urn becomes k, then all red balls outside the urn are replaced and the above procedure is continued again. A binary sequence of order k is defined by recording 0 or 1 for each random drawing according to whether it is a white ball or a red ball. It is easy to see that $p_i=(r-i+1)/(w+r-i+1)$, $i=1,2,\cdots,k$. In this example, occurrence of consecutive k successes means that the number of red balls outside the urn becomes k.

Example 2.2. An electric bulb is lighted at some spot. It is checked once a day at a certain given time whether it has failed or not. If it is found to be burnt out, then a new one is lighted immediately. And when an electric bulb has been lighted for k days consecutively, it is replaced with a new one even if it has not failed. Let us assume that the distributions of the lifetimes of the electric bulbs are identical. We define a binary sequence of order k by recording 0 or 1 every day,

according to whether the electric bulb has failed or not. Then, $p_1, \cdots, p_k$ are given as follows:
Let X be the lifetime in days of an electric bulb. And let F be the cumulative distribution function of X. Then, we have

$$p_1 = \mathrm{P}(X>1) = 1-F(1) ,$$

$$p_2 = \mathrm{P}(X>2 \mid X>1) = (1-F(2))/(1-F(1)) ,$$

$$\cdots\cdots\cdots\cdots$$

$$p_l = \mathrm{P}(X>l \mid X>l-1) = (1-F(l))/(1-F(l-1)) ,$$

$$\cdots\cdots\cdots\cdots$$

and

$$p_k = \mathrm{P}(X>k \mid X>k-1) = (1-F(k))/(1-F(k-1)) .$$

In this example, occurrence of consecutive k successes means that an electric bulb which is not failed is changed with a new one.

We have the following results relating to Example 2.2.

PROPOSITION 2.2. *If the lifetime X has nondecreasing (nonincreasing) hazard rate, then $p_1 \geqq p_2 \geqq \cdots \geqq p_k$ (resp. $p_1 \leqq p_2 \leqq \cdots \leqq p_k$) holds, where p's are the conditional probabilities given above.*

PROOF. Let h be the hazard rate of X. Since h is nondecreasing (nonincreasing), we have for any integers i and j $(1 \leqq i < j \leqq k)$,

$$\int_i^{i+1} h(x)dx \leqq \int_j^{j+1} h(x)dx \qquad (\text{resp. } \geqq) .$$

Note that

$$\int_i^{i+1} h(x)dx = \int_i^{i+1} \left(\frac{d}{dx}(-\log(1-F(x)))\right)dx$$
$$= -\log((1-F(i+1))/(1-F(i))) .$$

Hence, we have

$$p_i \geqq p_j \text{ for any integers } (1 \leqq i < j \leqq k) \qquad (\text{resp. } \leqq) .$$

This completes the proof.

The following is an immediate consequence of Proposition 2.2 and Remark 2.1.

COROLLARY 2.1. *If the lifetime X has an exponential distribution, the resulting binary sequence of order k is an independent trials with a constant success probability.*

Example 2.3. A manufactured product (e.g. an IC chip) is produced by a sequence of k manufacturing processes. As soon as each process is finished, the material is to be examined whether it is defective or not. If it is not defective, the material immediately goes on to the next process. But if it is found to be defective, the material is given up and the first process is started for a new one. We define a binary sequence by recording 0 or 1 for each examination according to the material is found to be defective or not. If all materials pass the examination for the ith process with identical success probability p_i ($i=1,2,\cdots,k$), the sequence is the binary sequence of order k with $p_1,\cdots,p_k$. In this example occurrence of consecutive k successes means that a product is perfectly produced.

3. The extended geometric distribution of order k

We introduce a geometric distribution on a binary sequence of order k and give some of its properties.

DEFINITION 3.1. A distribution is said to be the extended geometric distribution of order k, to be denoted by $EG_k(p_1, p_2,\cdots, p_k)$, if it is the distribution of the number of trials until the first occurrence of the kth consecutive success in the binary sequence of order k with $p_1, p_2,\cdots, p_k$.

Remark 3.1. By Remark 2.1, if $p_1=p_2=\cdots=p_k=p$ holds, then the extended geometric distribution of order k is reduced to be the geometric distribution of order k which was defined by Philippou, Georghiou and Philippou [10] and is denoted by $G_k(p)$. If $k=1$, then the corresponding distribution is the usual geometric distribution.

PROPOSITION 3.1. *Let X be a random variable distributed as $EG_k(p_1, p_2,\cdots, p_k)$. Then the following recurrence relation holds for any nonnegative integer x:*

$$(3.1)\qquad \mathrm{P}(X=x)=\begin{cases} 0 & \text{for } 0\leqq x<k\,, \\ p_1p_2\cdots p_k & \text{for } x=k\,, \\ \sum_{i=1}^{k} p_1p_2\cdots p_{i-1}q_i\,\mathrm{P}(X=x-i)^{\dagger} & \text{otherwise}\,, \end{cases}$$

where $q_i=1-p_i$ ($i=1,2,\cdots,k$).

PROOF. Let $\{X_n\}$ be the binary sequence of order k. When $x\leqq k$, (3.1) is obvious. If $x>k$, the event $\{X=x\}$ is written as the disjoint union of the events,

† The summand for $i=1$ means $q_1\,\mathrm{P}(X=x-1)$. Such a convention is frequently used in the following.

$$\bigcup_{i=1}^{k} \{X=x \text{ and } X_i \text{ is the first failure}\} .$$

By Remark 2.2, we have

$$\begin{aligned} P(X=x) &= \sum_{i=1}^{k} P(X_i \text{ is the first failure}) P(X=x \mid X_i \text{ is the first failure}) \\ &= \sum_{i=1}^{k} p_1 p_2 \cdots p_{i-1} q_i P(X=x-i) . \end{aligned}$$

This completes the proof.

The recurrence formula (3.1) is useful for computation of the values of probabilities of the extended geometric distribution of order k. Furthermore, by Remark 3.1, the formula (3.1) is available for calculation of the geometric distribution of order k. However, the recurrence relation which is given by setting $p_1=p_2=\cdots=p_k=p$ in (3.1) is different from the recurrence relation derived by Aki, Kuboki and Hirano ([2], Proposition 2.1). Though the latter is proved using the independence of the trials, the former is proved without assuming it.

PROPOSITION 3.2. *Let X be a random variable distributed as $EG_k(p_1, p_2, \cdots, p_k)$. Then the probability generating function (p.g.f.) of X is given by*

$$\frac{p_1 p_2 \cdots p_k t^k}{1-\sum_{i=1}^{k} p_1 p_2 \cdots p_{i-1} q_i t^i} \qquad (|t| \leqq 1) .$$

PROOF. Let $\phi_{EG}(t)$ be the p.g.f. of X; that is,

$$\phi_{EG}(t) = \sum_{x=k}^{\infty} P(X=x) t^x . \tag{3.2}$$

Since $EG_k(p_1, p_2, \cdots, p_k)$ is a probability distribution, the series (3.2) converges for all t such that $|t| \leqq 1$. Let t be a fixed number such that $|t| \leqq 1$. For all $x \geqq k$, we multiply both sides of (3.1) by t^x. By summing each side of such a system of equations, we get

$$\phi_{EG}(t) = p_1 p_2 \cdots p_k t^k + \sum_{i=1}^{k} p_1 p_2 \cdots p_{i-1} q_i t^i \phi_{EG}(t) .$$

This equation implies the desired result.

Putting $p_1=p_2=\cdots=p_k=p$, we obtain the next result.

COROLLARY 3.1. *The p.g.f. of the geometric distribution of order k is given as*

$$\frac{p^k t^k}{1-\sum_{i=0}^{k-1} p^i q t^{i+1}} \qquad (|t|\leqq 1)\,.$$

The result in Corollary 3.1 was first proved by Feller ([5], page 323). And another proof was given by Philippou, Georghiou and Philippou [10]. Our proof is very simple and we do not assume the independence of the trials.

Now, we give the probability distribution of $EG_k(p_1, p_2, \cdots, p_k)$ explicitly by expanding $\phi_{EG}(t)$.

PROPOSITION 3.3. *Let X be a random variable distributed as $EG_k(p_1, p_2, \cdots, p_k)$. Then the probability of X at x is written in the form,*

$$\begin{aligned}\mathrm{P}(X=x)=&\sum_{\substack{x_1,\cdots,x_k\\ x_1+2x_2+\cdots+kx_k=x-k}}\binom{x_1+\cdots+x_k}{x_1,\cdots,x_k}p_1p_2\cdots p_k q_1^{x_1}q_2^{x_2}\cdots q_k^{x_k}\\ &\times p_1^{x_2+x_3+\cdots+x_k}p_2^{x_3+x_4+\cdots+x_k}\cdots p_{k-1}^{x_k}\,, \qquad x\geqq k\,,\end{aligned}$$

where $\binom{a}{a_1, a_2, \cdots, a_k}=\Gamma(a+1)\Big/\Big(\prod_{i=1}^{k}\Gamma(a_i+1)\Big)$.

PROOF. By Proposition 3.2, the p.g.f. of $EG_k(p_1, p_2, \cdots, p_k)$ is written as

$$\frac{p_1p_2\cdots p_k t^k}{1-\sum_{i=1}^{k} p_1p_2\cdots p_{i-1}q_i t^i}\,. \tag{3.3}$$

Noting that $\left|\sum_{i=1}^{k} p_1p_2\cdots p_{i-1}q_it^i\right|<1$ for $|t|\leqq 1$, we have

$$\begin{aligned}\phi_{EG}(t)&=p_1p_2\cdots p_k t^k\sum_{n=0}^{\infty}\Big(\sum_{i=1}^{k}p_1p_2\cdots p_{i-1}q_it^i\Big)^n\\ &=p_1p_2\cdots p_k t^k\sum_{n=0}^{\infty}\sum_{n_1+n_2+\cdots+n_k=n}\binom{n}{n_1,\cdots,n_k}q_1^{n_1}q_2^{n_2}\cdots q_k^{n_k}\\ &\quad\times p_1^{n_2+n_3+\cdots+n_k}p_2^{n_3+n_4+\cdots+n_k}\cdots p_{k-1}^{n_k}t^{n_1+2n_2+\cdots+kn_k}\,.\end{aligned}$$

By setting $x_i=n_i$ $(1\leqq i\leqq k)$ and $x=n_1+2n_2+\cdots+kn_k$, we obtain

$$\begin{aligned}\phi_{EG}(t)=&\,p_1p_2\cdots p_k\sum_{x=0}^{\infty}\sum_{x_1+2x_2+\cdots+kx_k=x}\binom{x_1+\cdots+x_k}{x_1,\cdots,x_k}q_1^{x_1}q_2^{x_2}\cdots q_k^{x_k}\\ &\times p_1^{x_2+x_3+\cdots+x_k}p_2^{x_3+x_4+\cdots+x_k}\cdots p_{k-1}^{x_k}t^{x+k}\,.\end{aligned}$$

But from the definition of $\phi_{EG}(t)$, it holds

$$\phi_{EG}(t)=\sum_{x=0}^{\infty}t^{x+k}\,\mathrm{P}(X=x+k)=\sum_{x=k}^{\infty}t^x\,\mathrm{P}(X=x)\,.$$

This completes the proof.

Now we calculate the mean and variance of the extended geometric distribution of order k. It is easy to see the following lemma.

LEMMA 3.1. *Let $p_1, p_2, \cdots, p_k$ are real numbers such that $0<p_1, \cdots, p_k<1$. Then the following hold:*

(1) $\sum_{i=1}^{k} p_1p_2\cdots p_{i-1}q_i=1-p_1p_2\cdots p_k,$

(2) $\sum_{i=1}^{k} ip_1p_2\cdots p_{i-1}q_i+kp_1p_2\cdots p_k=1+\sum_{i=1}^{k-1} p_1p_2\cdots p_i,$

(3) $\sum_{i=2}^{k} i(i-1)p_1p_2\cdots p_{i-1}q_i+k(k-1)p_1p_2\cdots p_k=2\sum_{i=1}^{k-1} ip_1\cdots p_i.$

PROPOSITION 3.4. *The mean and variance of the extended geometric distribution of order k are given respectively by*

$$\left(1+\sum_{i=1}^{k-1} p_1p_2\cdots p_i\right)\Big/(p_1p_2\cdots p_k)$$

and

$$\left[2(p_1p_2\cdots p_k)\sum_{i=1}^{k-1} ip_1p_2\cdots p_i+\left(1+\sum_{i=1}^{k-1} p_1p_2\cdots p_i\right)^2 +(1-2k)(p_1p_2\cdots p_k)\left(1+\sum_{i=1}^{k-1} p_1p_2\cdots p_i\right)\right]\Big/(p_1p_2\cdots p_k)^2 .$$

PROOF. Since we know the p.g.f. $\phi_{EG}(t)$, for getting the mean and variance, it suffices to calculate $\phi'_{EG}(1)$ and $\phi''_{EG}(1)+\phi'_{EG}(1)-(\phi'_{EG}(1))^2$. Indeed, we have

$$\phi'_{EG}(t)=\left[p_1p_2\cdots p_k kt^{k-1}\left(1-\sum_{i=1}^{k} p_1p_2\cdots p_{i-1}q_it^i\right) +p_1p_2\cdots p_kt^k\sum_{i=1}^{k} p_1p_2\cdots p_{i-1}q_iit^{i-1}\right]\Big/\left(1-\sum_{i=1}^{k} p_1\cdots p_{i-1}q_it^i\right)^2$$

and

$$\begin{aligned}\phi''_{EG}(t)=\Bigg[\Bigg[&p_1p_2\cdots p_k k(k-1)t^{k-2}\left(1-\sum_{i=1}^{k} p_1p_2\cdots p_{i-1}q_it^i\right)\\ &-p_1p_2\cdots p_k kt^{k-1}\sum_{i=1}^{k} p_1p_2\cdots p_{i-1}q_iit^{i-1}\\ &+kp_1p_2\cdots p_kt^{k-1}\sum_{i=1}^{k} p_1p_2\cdots p_{i-1}q_iit^{i-1}\\ &+p_1p_2\cdots p_kt^k\sum_{i=2}^{k} p_1p_2\cdots p_{i-1}q_ii(i-1)t^{i-2}\Bigg]\\ &\times\left(1-\sum_{i=1}^{k} p_1p_2\cdots p_{i-1}q_it^i\right)^2\\ &+2\left(\sum_{i=1}^{k} p_1p_2\cdots p_{i-1}q_iit^{i-1}\right)\left(1-\sum_{i=1}^{k} p_1p_2\cdots p_{i-1}q_it^i\right)\end{aligned}$$

$$\times\Big[p_1p_2\cdots p_kkt^{k-1}\Big(1-\sum_{i=1}^{k}p_1p_2\cdots p_{i-1}q_it^i\Big)$$
$$+p_1p_2\cdots p_kt^k\sum_{i=1}^{k}p_1\cdots p_{i-1}q_iit^{i-1}\Big]\Big]\Big/\Big(1-\sum_{i=1}^{k}p_1\cdots p_{i-1}q_it^i\Big)^4 .$$

By using Lemma 3.1, we can easily obtain the desired result. This completes the proof.

We now discuss higher cumulants of the distribution. Let X be a random variable distributed as $EG_k(p_1, p_2, \cdots, p_k)$. We denote by $\overline{EG}_k(p_1, p_2, \cdots, p_k)$ the distribution of $X-k$.

PROPOSITION 3.5. *Let u be any positive integer. Then the shifted distribution $\overline{EG}_k(p_1, p_2, \cdots, p_k)$ has uth cumulant, which is represented as*

$$\sum_{n=1}^{\infty}\sum_{n_1+2n_2+\cdots+kn_k=n}\binom{n_1+\cdots+n_k-1}{n_1,\cdots,n_k}n^uq_1^{n_1}q_2^{n_2}\cdots q_k^{n_k}$$
$$\times p_1^{n_2+n_3+\cdots+n_k}p_2^{n_3+n_4+\cdots+n_k}\cdots p_{k-1}^{n_k} .$$

PROOF. Let $\psi(t)$ be the cumulant generating function of $\overline{EG}_k(p_1, \cdots, p_k)$; that is,

$$\psi(t)=\log\left(\frac{p_1p_2\cdots p_k}{1-\sum_{i=1}^{k}p_1p_2\cdots p_{i-1}q_ie^{it}}\right) . \tag{3.4}$$

By similar argument as the proof of Proposition 3.3, we can expand it in the following form,

$$\psi(t)=\log(p_1p_2\cdots p_k)+\sum_{n=1}^{\infty}\sum_{n_1+2n_2+\cdots+kn_k=n}\binom{n_1+\cdots+n_k-1}{n_1,\cdots,n_k}q_1^{n_1}q_2^{n_2}\cdots q_k^{n_k}$$
$$\times p_1^{n_2+n_3+\cdots+n_k}p_2^{n_3+n_4+\cdots+n_k}\cdots p_{k-1}^{n_k}\exp(nt) .$$

Let u be any positive integer. If we differentiate formally the right hand side of (3.4) u times with respect to t, we get

$$\sum_{n=1}^{\infty}\sum_{n_1+2n_2+\cdots+kn_k=n}\binom{n_1+\cdots+n_k-1}{n_1,\cdots,n_k}q_1^{n_1}q_2^{n_2}\cdots q_k^{n_k} \tag{3.5}$$
$$\times p_1^{n_2+n_3+\cdots+n_k}p_2^{n_3+n_4+\cdots+n_k}\cdots p_{k-1}^{n_k}n^u\exp(nt) .$$

If we can show that the infinite series (3.5) converges uniformly in t on a region which contains zero, then we can see that the uth cumulant of the distribution exists and it is written as

$$\sum_{n=1}^{\infty}\sum_{n_1+2n_2+\cdots+kn_k=n}\binom{n_1+\cdots+n_k-1}{n_1,\cdots,n_k}n^uq_1^{n_1}q_2^{n_2}\cdots q_k^{n_k}$$
$$\times p_1^{n_2+n_3+\cdots+n_k}p_2^{n_3+n_4+\cdots+n_k}\cdots p_{k-1}^{n_k} .$$

Now, we show the uniform convergence of (3.5) on a region which contains zero. For that, it suffices to show the uniform convergence of

$$(3.6)\qquad \sum_{n=1}^{\infty}\frac{1}{n}\sum_{n_1+n_2+\cdots+n_k=n}\binom{n}{n_1,\cdots,n_k}q_1^{n_1}q_2^{n_2}\cdots q_k^{n_k}$$
$$\times p_1^{n_2+n_3+\cdots+n_k}p_2^{n_3+n_4+\cdots+n_k}\cdots p_{k-1}^{n_k}(n_1+2n_2+\cdots+kn_k)^u$$
$$\times \exp((n_1+2n_2+\cdots+kn_k)t)\ ,$$

since (3.6) is a series obtained by a derangement of the terms in the series (3.5). If $n_1+n_2+\cdots+n_k=n$, then $n_1+2n_2+\cdots+kn_k\leqq kn$ holds. Hence, each term of (3.6) is less than

$$\frac{1}{n}(kn)^u\exp(knt)\left(\sum_{i=1}^{k}p_1p_2\cdots p_{i-1}q_i\right)^n ,$$

which, by Lemma 3.1 (1), is equal to

$$\frac{1}{n}[(kn)^{u/n}\exp(kt)(1-p_1p_2\cdots p_k)]^n .$$

Then, it is sufficient to show that the series,

$$(3.7)\qquad \sum_{n=1}^{\infty}\frac{1}{n}[(kn)^{u/n}\exp(kt)(1-p_1p_2\cdots p_k)]^n ,$$

converges uniformly in t on a region which contains zero. Set $\eta=1-p_1p_2\cdots p_k$. Then $0<\eta<1$ holds from the definition of the binary sequence of order k. Put $t_0=-(1/4k)\log\eta$. Then we have $t_0>0$. Note that it holds $\exp(kt)<\exp(kt_0)=\eta^{-1/4}$ for any $t<t_0$. Since $\lim_{n\to\infty}(kn)^{u/n}=1$, there exists a positive integer n_0 such that $(kn)^{u/n}<\eta^{-1/4}$ holds if $n>n_0$. Consequently, if $t<t_0$ and $n>n_0$, then

$$(kn)^{u/n}\exp(kt)(1-p_1p_2\cdots p_k)<\eta^{1/2}<1 .$$

But if $|x|<1$, then the power series $\sum_{n=1}^{\infty}\frac{1}{n}x^n$ $(=-\log(1-x))$ converges absolutely. Therefore, (3.7) converges uniformly for any $t<t_0$. This completes the proof.

4. The extended negative binomial distribution of order k and some related distributions

In the previous section some properties of the extended geometric distribution of order k, which is naturally defined on the binary sequence of order k, were discussed almost analogously to those of the geometric distribution of order k. We recall that there exist other fundamental distributions of order k such as the negative binomial, the

Poisson, the logarithmic series distributions of order k, etc. and that they preserve mutual relationships which are similar to those of corresponding usual distributions (of order 1).

We now define some other distributions of order k based on the binary sequence of order k and discuss mutual relationships among them.

DEFINITION 4.1. Let $X_1, X_2, \cdots, X_r$ be independent random variables identically distributed as $EG_k(p_1, p_2, \cdots, p_k)$. Then the distribution of $X_1+X_2+\cdots+X_r$ is called to be *the extended negative binomial distribution of order* k and denoted by $ENB_k(r, p_1, p_2, \cdots, p_k)$.

Remark 4.1. The class of the extended negative binomial distributions of order k contains as a special case the usual negative binomial distributions of order k, which was defined by Philippou, Georghiou and Philippou [10]. Indeed, from Definition 4.1 and Remark 2.1, it is easy to see that, if $p_1=p_2=\cdots=p_k=p$ holds, then the corresponding extended negative binomial distribution of order k is equal to the negative binomial distribution of order k.

The definition immediately implies that the p.g.f. of $ENB_k(r, p_1, p_2, \cdots, p_k)$ is given by

$$\psi_{ENB}(t)=\left(\frac{p_1p_2\cdots p_kt^k}{1-\sum_{i=1}^{k} p_1p_2\cdots p_{i-1}q_it^i}\right)^r \qquad (|t|\leqq 1)\,.$$

Now we give the probability distribution of $ENB_k(r, p_1, p_2, \cdots, p_k)$ explicitly. By the binomial expansion, we have

$$\psi_{ENB}(t)=(p_1p_2\cdots p_k)^r t^{kr} \sum_{m=0}^{\infty} \binom{-r}{m}\left(-\sum_{i=1}^{k} p_1p_2\cdots p_{i-1}q_it^i\right)^m .$$

Note that, for any nonnegative integers r and s,

$$(-1)^s\binom{-r}{s}=\binom{r+s-1}{r-1}$$

holds. Then, we can easily see that

$$\begin{aligned}\psi_{ENB}(t)&=(p_1p_2\cdots p_k)^r t^{kr} \sum_{m=0}^{\infty} \binom{r+m-1}{m}\\ &\quad\times \sum_{\substack{m_1,\cdots,m_k\\ m_1+\cdots+m_k=m}} \binom{m_1+\cdots+m_k}{m_1,\cdots,m_k} q_1^{m_1}q_2^{m_2}\cdots q_k^{m_k}\\ &\quad\times p_1^{m_2+m_3+\cdots+m_k}p_2^{m_3+m_4+\cdots+m_k}\cdots p_{k-1}^{m_k}t^{m_1+2m_2+\cdots+km_k}\\ &=(p_1p_2\cdots p_k)^r t^{kr} \sum_{m=0}^{\infty} \sum_{\substack{m_1,\cdots,m_k\\ m_1+\cdots+m_k=m}} \binom{m_1+\cdots+m_k+r-1}{m_1,\cdots,m_k,r-1}\\ &\quad\times q_1^{m_1}q_2^{m_2}\cdots q_k^{m_k}p_1^{m_2+m_3+\cdots+m_k}\cdots p_{k-1}^{m_k}t^{m_1+2m_2+\cdots+km_k}\,.\end{aligned}$$

By setting $y_i=m_i$ $(1\leqq i\leqq k)$ and $y=kr+y_1+2y_2+\cdots+ky_k$, we obtain

$$\phi_{ENB}(t)=\sum_{y=kr}^{\infty} t^y \sum_{\substack{y_1,\cdots,y_k \\ y_1+2y_2+\cdots+ky_k=y-kr}} \binom{y_1+\cdots+y_k+r-1}{y_1,\cdots,y_k,r-1}$$
$$\times(p_1p_2\cdots p_k)^r q_1^{y_1}q_2^{y_2}\cdots q_k^{y_k}p_1^{y_2+y_3+\cdots+y_k}p_2^{y_3+y_4+\cdots+y_k}\cdots p_{k-1}^{y_k}\ .$$

Thus we have the following result.

PROPOSITION 4.1. *Let X be a random variable distributed as $ENB_k(r, p_1, p_2,\cdots, p_k)$. Then the probability of X at x is written in the form,*

$$\text{(4.1)}\qquad P(X=x)=\sum_{\substack{x_1,\cdots,x_k \\ x_1+2x_2+\cdots+kx_k=x-kr}} \binom{x_1+\cdots+x_k+r-1}{x_1,\cdots,x_k,r-1}$$
$$\times(p_1p_2\cdots p_k)^r q_1^{x_1}q_2^{x_2}\cdots q_k^{x_k}p_1^{x_2+x_3+\cdots+x_k}$$
$$\times p_2^{x_3+x_4+\cdots+x_k}\cdots p_{k-1}^{x_k}\ ,\qquad x\geqq kr\ .$$

Remark 4.2. When we take into consideration the complexity of the summation, the formula (4.1) may not be suitable for computing the probability. We can indeed calculate the probability easily by the r-time convolution of $EG_k(p_1, p_2,\cdots, p_k)$ which can be computed by the recurrence formula (3.1).

DEFINITION 4.2. Let X be a discrete random variable. Let $\lambda_1,\cdots, \lambda_k$ be nonnegative constants. We say that X has *the extended Poisson distibution of order k* with $\lambda_1,\cdots,\lambda_k$, to be denoted by $EP_k(\lambda_1, \lambda_2,\cdots, \lambda_k)$, if it holds for any nonnegative integer x,

$$P(X=x)=\sum_{\substack{x_1,\cdots,x_k \\ x_1+2x_2+\cdots+kx_k=x}} \frac{\lambda_1^{x_1}\lambda_2^{x_2}\cdots\lambda_k^{x_k}}{x_1!x_2!\cdots x_k!}\exp\left(-\sum_{i=1}^{k}\lambda_i\right)\ .$$

Remark 4.3. This distribution, which had been introduced by Adelson [1], was called the '*stuttering*' Poisson distribution by Johnson and Kotz [8]. In this paper we however call it the extended Poisson distribution of order k, because it is natural to regard it as an extension of the Poisson distribution of order k, which was defined by Philippou, Georghiou and Philippou [10].

It is easy to see that the p.g.f. of $EP_k(\lambda_1, \lambda_2,\cdots, \lambda_k)$ is given by

$$\phi_{EP}(t)=\exp\left(-\sum_{i=1}^{k}\lambda_i+\sum_{i=1}^{k}\lambda_i t^i\right)\ .$$

For further properties of $EP_k(\lambda_1, \lambda_2,\cdots, \lambda_k)$, see, for example, Aki, Kuboki and Hirano [2].

The following is an extension of Theorem 3.2 in Philippou, Georghiou and Philippou [10].

PROPOSITION 4.2. *Let S_r be a random variable distributed as $ENB_k(r, p_1, \cdots, p_k)$. Assume that $q_j \to 0$ and $rq_j \to \lambda_j$ ($\geqq 0$) ($j=1, 2, \cdots, k$) as $r \to \infty$. Then $S_r - kr$ converges weakly to $EP_k(\lambda_1, \lambda_2, \cdots, \lambda_k)$.*

PROOF. It suffices to show the characteristic function of $S_r - kr$ converges to that of $EP_k(\lambda_1, \lambda_2, \cdots, \lambda_k)$ as $r \to \infty$. The characteristic function of $S_r - kr$ is given by

$$\phi_r(t) = \left(\frac{p_1 p_2 \cdots p_k}{1 - \sum_{j=1}^{k} p_1 p_2 \cdots p_{j-1} q_j \exp(ijt)} \right)^r .$$

where $i = \sqrt{-1}$. We set $\varepsilon(r, j) = rq_j - \lambda_j$, $j = 1, 2, \cdots, k$. Then, it holds

$$\phi_r(t) = \frac{\prod_{j=1}^{k} \left(1 - \frac{\lambda_j}{r} - \frac{\varepsilon(r, j)}{r}\right)^r}{\left(1 - \frac{1}{r} \sum_{j=1}^{k} p_1 p_2 \cdots p_{j-1}(\lambda_j + \varepsilon(r, j)) \exp(ijt)\right)^r} .$$

But it is not difficult to see that, if z is a complex number and $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of complex numbers which converges to zero, then

$$\lim_{n \to \infty} \left(1 + \frac{z}{n} + \frac{\varepsilon_n}{n}\right)^n = e^z .$$

This implies that

$$\lim_{r \to \infty} \phi_r(t) = \exp\left(-\sum_{j=1}^{k} \lambda_j + \sum_{j=1}^{k} \lambda_j \exp(ijt)\right) ,$$

which completes the proof.

Since the usual negative binomial distribution is defined for any positive real r, let us define the extended negative binomial distribution of order k for any positive real r. Then we can see more precise correspondence between distributions of order k. We note that, for a positive real r, $(\psi_{EG}(t))^r$ does not necessarily become a p.g.f. of a discrete distribution, because it may not be a power series.

DEFINITION 4.1′. Let r be any positive real number. A random variable X is said to have *the extended negative binomial distribution of order* k with parameters $r, p_1, p_2, \cdots, p_k$, to be denoted by $ENB_k(r, p_1, p_2, \cdots, p_k)$, if, for every positive integer x greater than or equal to $[kr]$, it holds

$$\mathrm{P}(X = x) = \sum_{\substack{x_1, x_2, \cdots, x_k \\ x_1 + 2x_2 + \cdots + kx_k = x - [kr]}} \binom{x_1 + \cdots + x_k + r - 1}{x_1, \cdots, x_k, r-1} q_1^{x_1} q_2^{x_2} \cdots q_k^{x_k}$$
$$\times (p_1 p_2 \cdots p_k)^r p_1^{x_2 + x_3 + \cdots + x_k} p_2^{x_3 + x_4 + \cdots + x_k} \cdots p_{k-1}^{x_k} .$$

Remark 4.4. The p.g.f. of $EBN_k(r, p_1, p_2, \cdots, p_k)$ is given by

$$(\psi_{EG}(t))^r t^{[kr]-kr}$$

which immediately implies that Definition 4.1′ is equivalent to Definition 4.1 for any positive integer r.

Aki, Kuboki and Hirano [2] defined the logarithmic series distribution of order k as a limiting distribution of a conditional negative binomial distribution of order k. They discussed its properties based on independent trials. We now derive some analogous properties of the corresponding distribution on the basis of the binary sequence of order k.

DEFINITION 4.3. A random variable X is said to have an *extended logarithmic series distribution of order* k, if there exist k real numbers $0<p_1, p_2, \cdots, p_k<1$ such that, for every positive integer x, it is satisfied that

$$\text{(4.2)} \qquad P(X=x)=\sum_{\substack{x_1,\cdots,x_k \\ x_1+2x_2+\cdots+kx_k=x}} \frac{(x_1+x_2+\cdots+x_k-1)!}{-\log(p_1p_2\cdots p_k)x_1!x_2!\cdots x_k!} \times q_1^{x_1}q_2^{x_2}\cdots q_k^{x_k}p_1^{x_2+\cdots+x_k}p_2^{x_3+\cdots+x_k}\cdots p_{k-1}^{x_k} .$$

The distribution is denoted by $ELS_k(p_1, p_2, \cdots, p_k)$.

The following is an extension of Proposition 3.2 in Aki, Kuboki and Hirano [2].

PROPOSITION 4.3. *Let X be a random variable distributed as $ENB_k(r, p_1, p_2, \cdots, p_k)$ and assume that $r\to 0$. Then*

$$P(X=x \mid X\geqq [kr]+1)$$

converges to the right hand side of (4.2).

PROOF. Since

$$P(X=[kr])=(p_1p_2\cdots p_k)^r ,$$

we have

$$\begin{aligned} &P(X=x \mid X\geqq [kr]+1) \\ &\quad =\frac{P(X=x, X\geqq [kr]+1)}{1-P(X=[kr])} \\ &\quad =\sum_{\substack{x_1,\cdots,x_k \\ x_1+2x_2+\cdots+kx_k=x-[kr]}} \binom{x_1+\cdots+x_k+r-1}{x_1,\cdots,x_k,r-1} q_1^{x_1}q_2^{x_2}\cdots q_k^{x_k} \\ &\quad\quad \times (p_1p_2\cdots p_k)^r p_1^{x_2+\cdots+x_k}p_2^{x_3+\cdots+x_k}\cdots p_{k-1}^{x_k}/(1-(p_1p_2\cdots p_k)^r) \end{aligned}$$

$$= \sum_{\substack{x_1,\cdots,x_k \\ x_1+2x_2+\cdots+kx_k=x-[kr]}} \frac{\Gamma(x_1+\cdots+x_k+r)}{x_1!x_2!\cdots x_k!\Gamma(r+1)} \frac{r(p_1p_2\cdots p_k)^r}{1-(p_1p_2\cdots p_k)^r}$$
$$\times q_1^{x_1}q_2^{x_2}\cdots q_k^{x_k}p_1^{x_2+\cdots+x_k}p_2^{x_3+\cdots+x_k}\cdots p_{k-1}^{x_k} .$$

Note that

$$\lim_{r\to 0} \frac{r(p_1p_2\cdots p_k)^r}{1-(p_1p_2\cdots p_k)^r} = \frac{-1}{\log(p_1p_2\cdots p_k)} .$$

Then we get the desired result. This completes the proof.

Remark 4.5. When the binary sequence is generated by independent trials (that is, when $p_1=p_2=\cdots=p_k$), the result was proved for any k by Aki, Kuboki and Hirano [2]. When $k=1$, the corresponding relation was shown by Fisher, Corbet and Williams [6].

PROPOSITION 4.4. *The p.g.f. of* $ELS_k(p_1, p_2,\cdots, p_k)$ *is expressed as*

$$\psi_{ELS}(t)=\alpha(p_1, p_2,\cdots, p_k)\log\left(\frac{1}{1-\sum_{i=1}^{k} p_1p_2\cdots p_{i-1}q_it^i}\right) ,$$

where $\alpha(p_1, p_2,\cdots, p_k)=-1/\log(p_1p_2\cdots p_k)$.

PROOF. Noting that

$$\log(1-y)=-\sum_{n=1}^{\infty}\frac{y^n}{n} ,$$

we have

$$\log\left(\frac{1}{1-\sum_{i=1}^{k} p_1p_2\cdots p_{i-1}q_it^i}\right)$$
$$=\sum_{m=1}^{\infty}\frac{1}{m}\left(\sum_{i=1}^{k} p_1p_2\cdots p_{i-1}q_it^i\right)^m$$
$$=\sum_{m=1}^{\infty}\frac{1}{m}\sum_{m_1+\cdots+m_k=m}\binom{m}{m_1,\cdots, m_k}q_1^{m_1}q_2^{m_2}\cdots q_k^{m_k}p_1^{m_2+\cdots+m_k}$$
$$\times p_2^{m_3+\cdots+m_k}\cdots p_{k-1}^{m_k}t^{m_1+2m_2+\cdots+km_k}$$
$$=\sum_{n=1}^{\infty}\sum_{x_1+2x_2+\cdots+kx_k=n}\binom{x_1+\cdots+x_k-1}{x_1,\cdots, x_k}q_1^{x_1}q_2^{x_2}\cdots q_k^{x_k}$$
$$\times p_1^{x_2+\cdots+x_k}p_2^{x_3+\cdots+x_k}\cdots p_{k-1}^{x_k}t^n .$$

This completes the proof.

We have the following result for moments of $ELS_k(p_1, p_2,\cdots, p_k)$.

PROPOSITION 4.5. *Let n be any positive integer. Then the nth mo-*

ment of $ELS_k(p_1, p_2, \cdots, p_k)$ *is equal to*

$$\alpha(p_1, p_2, \cdots, p_k) \times (nth\ cumulant\ of\ \overline{EG}_k(p_1, p_2, \cdots, p_k)) .$$

PROOF. By Proposition 4.4 and (3.4), we can easily see that

$$\psi_{ELS}(e^t) = \alpha(p_1, p_2, \cdots, p_k) \times (\log \psi_{\overline{EG}}(e^t) - \log (p_1 p_2 \cdots p_k)) .$$

This equation implies the result. This completes the proof.

Remark 4.6. By using Proposition 4.5 and Proposition 3.4, mean and variance of $ELS_k(p_1, p_2, \cdots, p_k)$ are immediately given.

Now we give a recursion formula which is useful for calculation of the extended logarithmic series distribution of order k.

As we saw in the proof of Proposition 4.5, it holds that

$$\psi_{ELS}(t) = \alpha(p_1, p_2, \cdots, p_k) \times (\log \psi_{\overline{EG}}(t) - \log (p_1 p_2 \cdots p_k)) .$$

By differentiating both sides, we get

$$\psi_{\overline{EG}}(t) \psi'_{ELS}(t) = \alpha(p_1, p_2, \cdots, p_k) \psi'_{\overline{EG}}(t) .$$

Therefore, the same argument as in the proof of Proposition 3.3 in Aki, Kuboki and Hirano [2] implies the following:

PROPOSITION 4.6. *The values of probabilities of the extended logarithmic series distribution of order k at n ($n=1, 2, \cdots$), which are denoted by* $\mathrm{P}_{ELS}(n)$, *satisfy the following recursion formula,*

$$\begin{aligned}\mathrm{P}_{ELS}(n) = {} & \alpha(p_1, p_2, \cdots, p_k)[\mathrm{P}_{\overline{EG}}(n)/\mathrm{P}_{\overline{EG}}(0)] \\ & - \frac{1}{n} \sum_{j=1}^{n-1} j [\mathrm{P}_{\overline{EG}}(n-j)/\mathrm{P}_{\overline{EG}}(0)]\, \mathrm{P}_{ELS}(j) ,\end{aligned}$$

where $\mathrm{P}_{\overline{EG}}(i)$ *is the value of probability of* $\overline{EG}_k(p_1, p_2, \cdots, p_k)$ *at* i.

5. The extended binomial distribution of order k

In this section we investigate some properties of the extended binomial distribution of order k and give an application to the reliability theory.

DEFINITION 5.1. Let n be a positive integer. Let $\{X_i\}_{i=0}^{\infty}$ be a binary sequence of order k with $p_1, p_2, \cdots, p_k$. A distribution is said to be *the extended binomial distribution of order k*, to be denoted by $EB_k(n, p_1, p_2, \cdots, p_k)$, if it is the distribution of the number of occurrences of consecutive k successes until the nth trial of the binary sequence of order k.

Remark 5.1. When $p_1=p_2=\cdots=p_k=p$, the corresponding distribution obviously coinsides with the binomial distribution of order k, which was defined explicitly by Hirano [7]. The binomial distribution of order k, which is denoted by $B_k(n, p)$, was firstly introduced by Feller [5] and its normal approximation was discussed.

From the definition, $EB_k(n, p_1, p_2, \cdots, p_k)$ is a distribution on $\{0, 1, 2, \cdots, [n/k]\}$. We denote, by $EB_k(n, p_1, p_2 \cdots, p_k; i)$, the value of probability of $EB_k(n, p_1, p_2, \cdots, p_k)$ at i ($i=0, 1, 2, \cdots, [n/k]$). $B_k(n, p; i)$, $ENB_k\ (r, p_1, p_2, \cdots, p_k; i)$, etc. are also defined similarly.

Let $Y_1, Y_2, \cdots, Y_r$ be independent random variables identically distributed as $EG_k(p_1, p_2, \cdots, p_k)$. Then, for any positive integer r not greater than $[n/k]$, the following equation holds,

$$(5.1)\qquad \sum_{x=kr}^{n} ENB_k(r, p_1, p_2, \cdots, p_k; x) = \mathrm{P}\,(Y_1+Y_2+\cdots+Y_r \leqq n) = \sum_{x=r}^{[n/k]} EB_k(n, p_1, p_2, \cdots, p_k; x)\ .$$

Therefore,

$$(5.2)\qquad EB_k(n, p_1, p_2, \cdots, p_k; 0) = 1-\sum_{x=1}^{[n/k]} EB_k(n, p_1, p_2, \cdots, p_k; x) = 1-\sum_{x=k}^{n} EG_k(p_1, p_2, \cdots, p_k; x)\ .$$

From (5.1), we can immediately calculate the value $EB_k(n, p_1, p_2 \cdots, p_k; x)$ for every x, by using Proposition 3.1 and Remark 4.2.

Furthermore, we have an explicit representation of $EB_k(n, p_1, p_2, \cdots, p_k; x)$. The following is an extension of Proposition 2.2 in Hirano [7]. Since it can be similarly proved, we omit the proof.

PROPOSITION 5.1. *For every nonnegative integer x not greater than $[n/k]$, the value of $EB_k(n, p_1, p_2, \cdots, p_k; x)$ is written in the form,*

$$\sum_{m=0}^{k-1} \sum_{\substack{x_1, \cdots, x_k \\ x_1+2x_2+\cdots+kx_k=n-m-kx}} \binom{x_1+\cdots+x_k+x}{x_1, \cdots, x_k, x} q_1^{x_1} q_2^{x_2} \cdots q_k^{x_k} p_1 p_2 \cdots p_m \times p_1^{x_2+x_3+\cdots+x_k} p_2^{x_3+x_4+\cdots+x_k} \cdots p_{k-1}^{x_k} (p_1 p_2 \cdots p_k)^x\ .$$

We can easily derive some characteristics of the distribution. For example, the mean and the second moment of the distribution are given as follows: Set

$$N=[n/k]\ ,$$

$$A_r=\sum_{x=kr}^{n} ENB_k(r, p_1, p_2, \cdots, p_k; x)\ ,$$

$$B_r = A_r - A_{r+1}, \qquad r=1, 2, \cdots, n-1,$$

$$B_N = A_N,$$

and

$$B_0 = 1 - A_1 .$$

Then, the mean is equal to

$$\sum_{r=0}^{N} rB_r = \sum_{r=1}^{N-1} r(A_r - A_{r+1}) + NA_N = \sum_{r=1}^{N} A_r .$$

The second moment is given by

$$\sum_{r=0}^{N} r^2 B_r = \sum_{r=1}^{N} (2r-1)A_r .$$

As an application of the extended binomial distribution of order k, we now consider a problem on the reliability of a system which is called consecutive-k-out-of-n:F system. The system, which was introduced by Chiang and Niu [3], consists of n components in sequence and fails whenever k consecutive components are failed. The reliability, that is, the probability that the system is functioning, was calculated by Chiang and Niu [3] and Derman, Lieberman and Ross [4] on the assumption that all components fail independently with an identical distribution.

It is easily seen that the reliability of the system is equal to $B_k(n, p; 0)$ on the above assumption and it can be written explicitly by Proposition 5.1, where p is the probability that each component fails.

Moreover, we need not assume the independence of each component for getting the reliability of the system. If we assume that the binary sequence generated by the failure of each component becomes a binary sequence of order k, then the reliability of the system is given by $EB_k(n, p_1, p_2, \cdots, p_k; 0)$, which immediately be calculated by (5.2) and Proposition 3.1.

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