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CHARACTERIZATIONS OF LIFE DISTRIBUTIONS FROM PERCENTILE RESIDUAL LIFETIMES

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Summary

An α-percentile residual life function does not uniquely determine a life distribution; however, a continuous life distribution can be uniquely determined by its α-percentile and β-percentile residual life functions if α and β satisfy a certain condition. Two characterizations in terms of percentile residual lifetimes are given for the Beta $(1, \theta, K)$, Exponential (λ) and Pareto (θ, K) family of distributions.

1. Introduction

Let $F(t)$ be a life distribution. The mean and α-percentile residual life functions at time t are respectively the mean and 100α percentile residual life given survival up to time t (mathematical definitions are given in Section 2). It is well-known that if F has finite mean, then it is uniquely determined by its mean residual life function $m_F(t)$—see, for example, Meilijson [6], Swartz [9], Laurent [5], Galambos and Kotz [2], and Hall and Wellner [3]. In contrast, it is shown in Arnold and Brockett [1] and Joe and Proschan [4] that there can be infinitely many life distributions with the same α-percentile residual life function $q_{\alpha,F}(t)$. Hence it is impossible to characterize life distributions in terms of a single percentile residual life function. In this paper, we show that a continuous life distribution F is uniquely determined by $q_{\alpha,F}$ and $q_{\beta,F}$ if α and β satisfy a certain condition. The Beta $(1, \theta, K)$, Exponential (λ) and Pareto (θ, K) (of the second kind) distributions are characterized as the only absolutely continuous distributions with (i) linear α-percentile residual lifetimes for an interval of α's and (ii) the mean residual life function equivalent to an α-percentile residual life function for some $0<\alpha<1$. Characterizations in terms of linear mean residual life are given in Hall and Wellner [3] and Morrison [7].

Key words and phrases: Residual life, percentile, beta, exponential, Pareto.

2. Definitions

Let F be a life distribution, that is, a distribution such that $F(0^-)=0$, and let $\bar{F}$ be the survival function: $\bar{F}(t)=1-F(t)$, $t>0$. Let F^{-1} be the left continuous inverse of F defined by

$$F^{-1}(u)=\inf\{x:\ F(x)\geqq u\}\ , \qquad 0\leqq u\leqq 1\ .$$

Then $\bar{F}^{-1}(u)=F^{-1}(1-u)$ is the right continuous inverse of $\bar{F}$. Let $T_F=\sup\{x:\ F(x)<1\}=F^{-1}(1)$ be the right-hand endpoint of support and for $0\leqq t\leqq T_F$, let

$$r_F(t)=f(t)/\bar{F}(t)\ ,$$

$$\bar{F}_t(x)=1-F_t(x)=\bar{F}(t+x)/\bar{F}(t)\ ,$$

$$m_F(t)=\int_0^\infty \bar{F}_t(x)dx=\int_t^\infty \bar{F}(x)dx/\bar{F}(t)\ , \qquad \text{and}$$

$$q_{\alpha,F}(t)=F_t^{-1}(\alpha)=F^{-1}(1-\bar{\alpha}\bar{F}(t))-t=\bar{F}^{-1}(\bar{\alpha}\bar{F}(t))-t\ ,$$

be the failure rate, the conditional survival function, the mean residual life function and the α-percentile residual life function, respectively. Whenever possible, we will suppress the subscript F and use T, r, m, q_α in place of T_F, r_F, m_F, $q_{\alpha,F}$ respectively.

3. Uniqueness in terms of two percentile residual lifetimes

In Theorem 1 below, a functional relationship is obtained for two continuous life distributions F and G with the same α-percentile residual life function. Using this result, it is shown that a continuous life distribution is uniquely determined by its α- and β-percentile residual life functions if α and β satisfy the condition in Theorem 2.

THEOREM 1. *Let F and G be continuous life distributions and let $0<\alpha<1$. Then $q_{\alpha,F}\equiv q_{\alpha,G}$ if and only if there exists a periodic function $\Delta(x)$ with period $-\log(1-\alpha)$, such that*

$$\bar{F}(x)=\Delta(-\log\bar{G}(x))\cdot\bar{G}(x)\ , \qquad x\geqq 0\ . \tag{1}$$

PROOF. (Necessity). Suppose that $q_{\alpha,F}\equiv q_{\alpha,G}$. Then $\bar{F}^{-1}(\bar{\alpha}\bar{F}(t))=\bar{G}^{-1}(\bar{\alpha}\bar{G}(t))$, $t\geqq 0$, which implies

$$\bar{\alpha}\bar{F}(t)=\bar{F}\bar{G}^{-1}(\bar{\alpha}\bar{G}(t))\ , \qquad t\geqq 0 \tag{2}$$

since F and G are continuous. The hypotheses imply that F is constant on any interval where G is constant and vice-versa so that (2) implies

$$\bar{\alpha}\bar{F}\bar{G}^{-1}(u)=\bar{F}\bar{G}^{-1}(\bar{\alpha}u)\,, \qquad 0<u<1\,. \tag{3}$$

The function

$$\Delta(x)=\bar{F}\bar{G}(e^{-x})e^{x}\,, \qquad x\geqq 0 \tag{4}$$

satisfies (1), and $\Delta(x)=\Delta(x-\log\bar{\alpha})$, $x\geqq 0$, if and only if

$$\bar{F}\bar{G}^{-1}(e^{-x})=\bar{F}\bar{G}^{-1}(\bar{\alpha}e^{-x})/\bar{\alpha}\,, \qquad x\geqq 0$$

which is equivalent to (3).

(Sufficiency). Suppose that (1) holds for a periodic function with period $-\log(1-\alpha)$. Then

$$\begin{aligned}\bar{F}\bar{G}^{-1}(\bar{\alpha}\bar{G}(t))&=\Delta(-\log\bar{\alpha}\bar{G}(t))\cdot\bar{\alpha}\bar{G}(t)\\ &=\Delta(-\log\bar{G}(t))\cdot\bar{\alpha}\bar{G}(t)=\bar{\alpha}\bar{F}(t)\,, \qquad t\geqq 0\,,\end{aligned}$$

and equation (2) holds. For fixed $t\geqq 0$, $\bar{F}^{-1}(\bar{\alpha}\bar{F}(t))=\bar{F}^{-1}(\bar{F}(y))$, where $y=\bar{G}^{-1}(\bar{\alpha}\bar{G}(t))$. Let $I=[t_1, t_2]$ be the interval of u's for which $\bar{F}(u)=\bar{F}(y)$. Then $\bar{F}^{-1}(\bar{F}(y))=F^{-1}(F(y))=t_1$. (2) implies that $\bar{F}(x)$ is constant on an interval $I\Longleftrightarrow\bar{G}^{-1}(\bar{\alpha}\bar{G}(x))$ is constant on $I\Longleftrightarrow\bar{G}(x)$ is constant on I. Therefore, $\bar{G}(t_1)=\bar{G}(y)=\bar{G}(t_2)=\bar{\alpha}\bar{G}(t)$ and $\bar{G}^{-1}(\bar{\alpha}\bar{G}(t))=t_1=y$. Hence $\bar{F}^{-1}(\bar{\alpha}\bar{F}(t))=\bar{G}^{-1}(\bar{\alpha}\bar{G}(t))$ and $q_{\alpha,F}(t)=q_{\alpha,G}(t)$.

THEOREM 2. *Let F and G be continuous life distributions. If $q_{\alpha,F}\equiv q_{\alpha,G}$ and $q_{\beta,F}\equiv q_{\beta,G}$ for α and β in* $(0,1)$ *such that* $\log(1-\alpha)/\log(1-\beta)$ *is irrational, then $F\equiv G$.*

PROOF. Let $\Delta(x)$ be defined as in (4). By Theorem 1, Δ is periodic with periods $-\log(1-\alpha)$ and $-\log(1-\beta)$. Since $\log(1-\alpha)/\log(1-\beta)$ is irrational, Δ is periodic with a dense set of periods and hence is constant (Semadeni [8]). The hypotheses of the theorem imply that $F(0)=G(0)=0$ and therefore, from (1) of Theorem 1, $F\equiv G$.

Remarks. 1. Arnold and Brockett [1] have a proof of Theorem 2 assuming that F and G are strictly incresing.
2. A corollary of Theorem 2 is: If F and G are continuous life distributions such that $q_{\beta,F}\equiv q_{\beta,G}$ for all $\beta\in I$, where I is a subinterval of $(0,1)$, then $F\equiv G$. This generalizes Theorem 13 of Joe and Proschan [4].
3. Theorem 2 generalizes the following result in Joe and Proschan [4]: If $q_{\alpha,F}$ and $q_{\beta,F}$ are constant functions and $\log(1-\alpha)/\log(1-\beta)$ is irrational, then F is an exponential distribution.

4. Characterization for Beta, Exponential and Pareto distributions

In Remark 3 following Theorem 2, a characterization is given in terms of 2 constant percentile residual lifetimes. The next simplest form of a percentile residual lifetime is a linear one, and for a characterization we will need to consider a continuum of α's. The characterization is for the Beta $(1, \theta, K)$, Exponential (λ) and Pareto (θ, K) (of the second kind) distributions. The survival functions for these 3 families together with the corresponding failure rate, mean residual life and α-percentile residual life functions are stated next before the characterization result. Note that a Beta $(1, \theta, K)$ distribution is the distribution of K times a Beta $(1, \theta)$ random variable.

(i) Beta $(1, \theta, K)$, $0 \leqq t \leqq K$, $\theta > 0$, $K > 0$:

$$\bar{F}(t) = (1 - t/K)^{\theta}, \qquad r(t) = \theta/(K - t),$$

$$m(t) = (K - t)/(\theta + 1), \quad \text{and} \quad q_\alpha(t) = (1 - \bar{\alpha}^{1/\theta})(K - t).$$

(ii) Exponential (λ), $t \geqq 0$, $\lambda > 0$:

$$\bar{F}(t) = e^{-\lambda t}, \qquad r(t) = \lambda,$$

$$m(t) = 1/\lambda, \quad \text{and} \quad q_\alpha(t) = (-\log \bar{\alpha})/\lambda.$$

(iii) Pareto (θ, K), $t \geqq 0$, $\theta > 0$, $K > 0$:

$$\bar{F}(t) = (1 + t/K)^{-\theta}, \qquad r(t) = \theta/(K + t),$$

$$m(t) = (K + t)/(\theta - 1), \quad (\theta > 1), \quad \text{and} \quad q_\alpha(t) = (\bar{\alpha}^{-1/\theta} - 1)/(K + t),$$

$m(t)$ does not exist for $0 < \theta \leqq 1$.

THEOREM 3. *Suppose that F is absolutely continuous and that $q_\alpha(t)$ is linear in t for all $\alpha \in I$ for some subinterval I of $(0, 1)$. Then F is either a Beta $(1, \theta, K)$, Exponential (λ) or Pareto (θ, K) distribution.*

PROOF. Suppose that for $\alpha \in I$, $q_\alpha(t) = a_\alpha + b_\alpha t$, $0 \leqq t < T$, where $a_\alpha > 0$ and $b_\alpha > -1$. Note that $q_\alpha(t) + t = \bar{F}^{-1}(\bar{\alpha}\bar{F}(t))$ is increasing in t. Since F is continuous, then

$$\bar{\alpha}\bar{F}(t) = \bar{F}(q_\alpha(t) + t) = \bar{F}(a_\alpha + (b_\alpha + 1)t), \qquad 0 \leqq t < T. \tag{5}$$

It follows that a_α and b_α are differentiable in α. Let the two derivatives be denoted by a'_α and b'_α. Differentiation of (5) with respect to α and with respect to t yield

$$\bar{F}(t) = f(a_\alpha + (b_\alpha + 1)t) \cdot (a'_\alpha + b'_\alpha t) \quad \text{and} \quad \bar{\alpha} f(t) = f(a_\alpha + (b + 1)t) \cdot (b_\alpha + 1).$$

The ratio of these two equations leads to

$$1/r(t)=\bar{\alpha}(a'_\alpha+b'_\alpha t)/(b_\alpha+1)\,,\quad 0\leqq t<T\,,\quad \alpha\in I\,.$$

Thus $\bar{\alpha}a'_\alpha/(b_\alpha+1)$ and $\bar{\alpha}b'_\alpha/(b_\alpha+1)$ are constants, say, c_1 and c_2, with $c_1>0$. Since a failure rate function uniquely determines a survival function, the only possibilities for F are:

(i) F is Beta $(1, \theta, K)$ if $c_1=K/\theta$, $c_2=-1/\theta$,

(ii) F is Exponential (λ) if $c_1=1/\lambda$, $c_2=0$, and

(iii) F is Pareto (θ, K) if $c_1=K/\theta$, $c_2=1/\theta$.

Remark. We conjecture that the conclusion of Theorem 3 is still valid if the hypothesis of absolute continuity is replaced by $F(0)=0$, F is nondegenerate and F has no mass at $T=F^{-1}(1)$.

Finally, we characterize all absolutely continuous distributions for which $m(t)$ is equivalent to $q_\alpha(t)$ for some $0<\alpha<1$. We first note the following.

(i) Within the Beta $(1, \theta, K)$ family, $m(t)\equiv q_\alpha(t)$ if $\alpha=1-(1+1/\theta)^{-\theta}=\alpha_1(\theta)$. $\alpha_1(\theta)$ is strictly increasing in $\theta>0$ with $\lim_{\theta\to 0}\alpha_1(\theta)=0$ and $\lim_{\theta\to\infty}\alpha_1(\theta)=1-e^{-1}$.

(ii) Within the Exponential (λ) family, $m(t)\equiv q_\alpha(t)$ if $\alpha=1-e^{-1}$.

(iii) Within the Pareto (θ, K) family, $m(t)\equiv q_\alpha(t)$ if $\alpha=1-(1-1/\theta)^{\theta}=\alpha_2(\theta)$. $\alpha_2(\theta)$ is strictly decreasing in $\theta>1$ with $\lim_{\theta\to\infty}\alpha_2(\theta)=1-e^{-1}$ and $\lim_{\theta\to 1}\alpha_2(\theta)=1$.

THEOREM 4. *Suppose that $m(t)\equiv q_\alpha(t)$ for some $\alpha\in(0, 1)$. If F is absolutely continuous, then there is a $\mu\geqq 0$ ($\mu=F^{-1}(0)$) such that $G(t)=F(t-\mu)$ is either a Beta $(1, \theta, K)$ or an Exponential (λ) or a Pareto (θ, K) distribution.*

Remark. A Pareto (θ, K) distribution shifted by $\mu=K$ is a Pareto distribution of the first kind with $\bar{F}(t)=(K/t)^\theta$, $t\geqq K$.

PROOF. If F is absolutely continuous, then from the definition of $m(t)$ and the equation

$$\bar{F}(t)=\exp\left\{-\int_0^t r(y)dy\right\}=[m(0)/m(t)]\bar{F}(0)\exp\left\{-\int_0^t [m(y)]^{-1}dy\right\},$$

$m'(t)$ exists and

$$r(t)=(1+m'(t))/m(t)\,,\qquad 0\leqq t<T\,. \tag{6}$$

From the hypotheses of the theorem,

$$\bar{\alpha}\bar{F}(t)=\bar{F}(q_\alpha(t)+t)=\bar{F}(m(t)+t)\,.$$

Differentiation with respect to t leads to

$$(7)\qquad r(t)=r(t+m(t))\cdot(1+m'(t))\,,\qquad 0\leqq t<T\,.$$

Let $t_1=\sup\{t\colon 1+m'(t)=0,\ 0\leqq t<T\}$. This is well-defined since the continuity of $m(t)$ implies that $\bar{F}^{-1}(\bar{\alpha}\bar{F}(t))$ is continuous which in turn implies that $F^{-1}(u)$ is continuous for $u\geqq\alpha$ or $1+m'(t)>0$ for $t>F^{-1}(\alpha)$. Equating (6) and (7) yields

$$(8)\qquad r(t+m(t))=1/m(t)\,,\qquad t_1<t<T\,.$$

Substitution of (6) on the left-hand side of (8) results in

$$(9)\qquad m'(t+m(t))=[m(t+m(t))-m(t)]/m(t)\,,\qquad t_1<t<T\,.$$

Since $m(t)$ is differentiable, then by (8), $r(y)$ is continuous and differentiable for $y>t_2=t_1+m(t_1)$. By (6), $m'(y)$ is continuous and differentiable for $y>t_2$. Hence both sides of (9) are differentiable,

$$(10)\qquad m''(t+m(t))(1+m'(t))=[m'(t+m(t))-m'(t)]/m(t)$$

for $t_1<t<T$, and $m''(y)$ is continuous for $y<t_2$. By Taylor's theorem, for $t<t_2$,

$$(11)\qquad \frac{m(t+m(t))-m(t)}{m(t)}=m'(t+m(t))-\frac{1}{2}m''(s(t))m(t)\,,$$

where $s(t)$ is between t and $t+m(t)$ and can be assumed continuous. Equations (9) and (11) together imply that $m''(s(t))=0$, $t>t_2$, so that $m''(y)=0$ for $y>t_2+m(t_2)$. Hence there is a constant $b>-1$ such that $m'(y)=b$ for $y>t_2+m(t_2)$. By (10), $m'(y)=b$ for $y>t_2$. Substitution into (9) yields

$$(12)\qquad bm(t)=m(t+m(t))-m(t)\,,\qquad t_1<t<T\,.$$

Differentiation of (12) results in $m'(t)=b$, $t_1<t<T$. There is a constant a such that

$$(13)\qquad m(t)=a+bt\,,\qquad t_1<t<T\,.$$

We now prove by contradiction that $F(t_1)=0$. If $F(t_1)>0$, then there are constants $s_1<s_2<t_1$ such that

$$(14)\qquad 1+m'(t)=0\quad\text{for}\quad s_2<t<t_1\,,$$

and $F(s)>0$, $1+m'(s)>0$ and $s+m(s)>t_1$ for $s_1<s<s_2$. By (6) and (7),

$$r(s+m(s))=(1+b)/(a+b[s+m(s)])=1/m(s)\,, \qquad s_1<s<s_2$$

or

$$(15) \qquad m(s)=a+bs\,, \qquad s_1<s<s_2\,.$$

(13), (14), (15) together contradict the fact that $m(t)$ is continuous, and therefore $F(t_1)=0$ and $t_1=F^{-1}(0)$.

COROLLARY. *If F is absolutely continuous and $F_t(x)$ is a symmetric distribution for all $0\leqq t<T$, then F is a Uniform $[\mu, \mu+K]$ distribution for some $\mu>0$, $K>0$.*

PROOF. By Theorem 4, $m(t)=q_{0.5}(t)$ if and only if $F(t-\mu)$ is a Beta $(1, 1, K)$ or Uniform $[0, K]$ distribution. Therefore F is a Uniform $[\mu, \mu+K]$ distribution; the conditional distribution F_t is Uniform $[\max(0, \mu-t), \mu+K-t)$.

Remark. If the assumption of absolute continuity in Theorem 4 is replaced by $F(0)=0$ and F is nondegenerate, then the condition $m(t)\equiv q_\alpha(t)$ for some α is also satisfied by a subclass of the Geometric family of distributions. If F is Geometric with parameter $0<p<1$, that is, $\bar{F}(j)=(1-p)^j$, $j=0, 1, \cdots$, then $m(t)=q_\alpha(t)=p^{-1}-t+[t]$, $t\geqq 0$, provided $p^{-1}=n$ is a positive integer and $1-(1-p)^{n-1}<\alpha\leqq 1-(1-p)^n$. Except for scale and location modifications of these Geometric distributions, we conjecture that there are no other distributions satisfying $m(t)\equiv q_\alpha(t)$ for some α under the weakened hypothesis.

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REFERENCES

[1] Arnold, B. and Brockett, P. L. (1983). When does the βth percentile residual life function determine the distribution. *Opns. Research*, **31**, 391-396.

[2] Galambos, J. and Kotz, S. (1978). *Characterizations of Probability Distributions*, Lecture Notes in Mathematics #675, Springer-Verlag, New York.

[3] Hall, W. J. and Wellner, J. A. (1981). Mean residual life, *Statistics and Related Topics* (edited by M. Csorjo, D. A. Dawson, J. N. K. Rao and A. K. Md. E. Saleh), North-Holland, Amsterdam, 169-184.

[4] Joe, H. and Proschan, F. (1984). Percentile residual life functions, *Opns. Research*, **32**, 668-678.

[5] Laurent, A. G. (1974). On characterization of some distributions by truncation properties, *J. Amer. Statist. Assoc.*, **69**, 823-827.

[6] Meilijson, I. (1972). Limiting properties of the mean residual life function, *Ann. Math. Statist.*, **43**, 354-357.

[7] Morrison, D. G. (1978). On linearly increasing mean residual lifetimes, *J. Appl. Prob.*, **15**, 617-620.

[8] Semadeni, Z. (1964). Periods of measurable functions and the Stone-Cech compactification, *Amer. Math. Monthly*, **71**, 891-893.

[9] Swartz, G. B. (1973). The mean residual life function, *IEEE Transactions on Reliability*, **R-22**, 108-109.