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SOME CONSTRUCTIONS OF PBIB DESIGNS

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Summary

Constructions of three series of regular GD and semi-regular GD designs are given. Furthermore, a series of rectangular PBIB designs is constructed and particular cases of this series which reduce to PBIB designs with two associate classes are also provided.

1. Introduction

We consider partially balanced incomplete block (PBIB) designs with two associate classes of GD type (semi-regular, regular), L_2 type and with three associate classes of rectangular type, having parameters v, b, r, k, λ_i, n_i, p^i_{jl}. For complete description of balanced incomplete block (BIB) designs and PBIB designs, along with association schemes mentioned above, we refer to Raghavarao [2].

In this note, we give some constructions of these designs.

2. Construction of some GD designs

THEOREM 2.1. *For $n\geqq 3$ and $m\geqq 2$, there exists a semi-regular GD design with parameters*

$$v=mn\,,\qquad b=\left\{\binom{n}{2}\right\}^{m},\qquad r=(n-1)\left\{\binom{n}{2}\right\}^{m-1},\qquad k=2m\,,$$

$$(2.1)\qquad \lambda_1=\left\{\binom{n}{2}\right\}^{m-1},\qquad \lambda_2=(n-1)^2\left\{\binom{n}{2}\right\}^{m-2},\qquad n_1=n-1\,,$$

$$n_2=n(m-1)\,.$$

In particular, when groups of this design have three treatments, i.e., $n=3$, the complement of the design with parameter (2.1) *is a semi-regular GD design with parameters*

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$$v'=3m\,,\quad b'=3^m\,,\quad r'=3^{m-1}\,,\quad k'=m\,,\quad \lambda_1'=0\,,$$
$$\lambda_2'=3^{m-2}\,,\quad n_1=2\,,\quad n_2=3(m-1)\,.$$

PROOF. Let m groups $G_1, G_2, \cdots, G_m$ have the following arrangement of treatments:

G_1	G_2	$\cdots$	G_m
θ_{11}	θ_{21}	$\cdots$	θ_{m1}
θ_{12}	θ_{22}	$\cdots$	θ_{m2}
$\vdots$	$\vdots$		$\vdots$
θ_{1n}	θ_{2n}	$\cdots$	θ_{mn} .

Let A_i be the set of all possible $\binom{n}{2}$ distinct pairs of elements of the type $(\theta_{ir}, \theta_{is})$, $r\neq s$, from the same group G_i, $i=1, 2, \cdots, m$. In this case, any element of the cartesian product $A_1\times A_2\times\cdots\times A_m$, like $\{\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}, \cdots, \theta_{m1}, \theta_{m2}\}$, constitutes a block of the required design. Since $|A_1\times A_2\times\cdots\times A_m|=\left\{\binom{n}{2}\right\}^m$, where $|A|$ denotes the cardinality of the set A, $b=\left\{\binom{n}{2}\right\}^m$. Also, this construction yields the block size $k=2m$. As for the replication number, r, note that, since each treatment of G_i is in $(n-1)$ pairs belonging to A_i and the number of blocks having each such pair is $|A_1\times A_2\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_m|$, i.e., $\left\{\binom{n}{2}\right\}^{m-1}$, any treatment is replicated $(n-1)\left\{\binom{n}{2}\right\}^{m-1}$ $(=r)$ times. Since any pair of elements of G_i is precisely an element of A_i and occurs in $\left\{\binom{n}{2}\right\}^{m-1}$ blocks of the design, it gives the value of λ_1 as $\left\{\binom{n}{2}\right\}^{m-1}$. Now if we consider a pair of treatments like $(\theta_{is}, \theta_{jt})$ for $i\neq j$, there are $(n-1)$ pairs in A_i containing θ_{is} and $(n-1)$ pairs in A_j containing θ_{jt}. Thus, $(n-1)^2$ elements in $A_i\times A_j$ contain pairs of this type and each pair occurs in $\left\{\binom{n}{2}\right\}^{m-2}$ blocks, accounting for the value of λ_2 as $(n-1)^2\cdot\left\{\binom{n}{2}\right\}^{m-2}$. Since $r-\lambda_1=(n-2)\left\{\binom{n}{2}\right\}^{m-1}>0$ for $n\geqq 3$, and $rk-v\lambda_2=0$, the design with parameter (2.1) is a GD design of semi-regular type.

THEOREM 2.2. *A regular GD design with parameters*

$$
\begin{aligned}
&v=mn\,,\quad b=mn(m-1)\binom{n}{t-1}\,,\quad r=t(m-1)\binom{n}{t-1}\,,\\
(2.2)\qquad &k=t,\quad \lambda_1=n(m-1)\binom{n-2}{t-3}\,,\quad \lambda_2=2\binom{n-1}{t-2}\,,\\
&n_1=n-1\,,\quad n_2=n(m-1)\,,
\end{aligned}
$$

for positive integers t, n, $m \geqq 2$ can be constructed, unless $t=m=2$.

PROOF. Let G_i, $i=1, 2, \cdots, m$, be m groups each of size n. For $t-1 \leqq n$, take all possible $(t-1)$-tuples from a fixed group, G_i, say. The number of such $(t-1)$-tuples is $\binom{n}{t-1}$. Adjoin each of the $(m-1)n$ treatments from groups other than G_i to these $(t-1)$-tuples. Then we get $n(m-1)\binom{n}{t-1}$ t-tuples forming blocks of size $k=t$, generated by the fixed group G_i. Repeating this process with other groups, we can obtain $b=mn(m-1)\binom{n}{t-1}$ blocks in total. The number of blocks containing a particular treatment θ, i.e., r, is

$$\binom{n-1}{t-2}n(m-1)+\binom{n}{t-1}(m-1)=t(m-1)\binom{n}{t-1} .$$

It is easy to verify that $\lambda_1=n(m-1)\binom{n-2}{t-3}$ and $\lambda_2=2\binom{n-1}{t-2}$. Finally, since $n \geqq t-1$, $r-\lambda_1$ is always positive, and $rk-v\lambda_2=\binom{n}{t-1}[(m-1)t^2-2m(t-1)]$, which is always positive unless $t=m=2$. Hence, the design with parameter (2.2) is a GD design of regular type, unless $t=m=2$.

Remark 2.1. If $t=m=2$ in Theorem 2.2, then, since $rk-v\lambda_2=0$ in this case, we have a semi-regular GD design with parameters $v=2n$, $b=2n^2$, $r=2n$, $k=2$, $\lambda_1=0$, $\lambda_2=2$, $n_1=n-1$, $n_2=n$, which is well known.

THEOREM 2.3. *For $n \geqq 3$, we can always construct a regular GD design with parameters*

$$\begin{aligned} &v=n^2 , \quad b=n^2(n^2-3n+4)/2 , \quad r=(2n-1)(n^2-3n+4)/2 , \\ (2.3) \quad &k=2n-1 , \quad \lambda_1=n^3-5n^2+9n-4 , \quad \lambda_2=n^2-2n+2 , \\ &n_1=n-1 , \quad n_2=n(n-1) . \end{aligned}$$

PROOF. Let n groups $G_1, G_2, \cdots, G_n$ be as follows:

$$\begin{array}{c|l} G_1 & \theta_{11}, \ \ \theta_{12}, \ \cdots, \theta_{1n}, \\ G_2 & \theta_{21}, \ \ \theta_{22}, \ \cdots, \theta_{2n}, \\ \vdots & \ \vdots \qquad \vdots \qquad\quad \vdots \\ G_n & \theta_{n1}, \ \ \theta_{n2}, \ \cdots, \theta_{nn} . \end{array}$$

Take any element θ_{ij} and write all the elements in the ith row and the jth column, together with θ_{ij}, in one block, making a block of size $k=2n-1$, called a block of type 1. It is illustrated for θ_{11} as follows:

$$\{\theta_{11}, \theta_{12}, \cdots, \theta_{1n}, \theta_{21}, \cdots, \theta_{n1}\} .$$

Excluding the ith row and jth column corresponding to θ_{ij} choose any two rows out of the remaining $(n-1)$ rows and write the elements of these two rows, together with θ_{ij}, in a single block forming a block of type 2. Naturally, the number of blocks of type 2 is $\binom{n-1}{2}$ for fixed θ_{ij}. We form blocks of types 1 and 2 according to this procedure for every θ_{ij}. Thus, the resulting design has $n^2\left[1+\binom{n-1}{2}\right]=n^2(n^2-3n+4)/2$ blocks, each of size $k=2n-1$.

Let us fix any treatment, θ_{11}, say. Now, this treatment occurs in $(n-1)+(n-1)+1$, i.e., $(2n-1)$ blocks of type 1 are generated by elements in the first row and first column, and $\binom{n-1}{2}$ blocks of type 2 already have the element θ_{11}. Further blocks of type 1 corresponding to $(n-1)^2$ elements in rows other than the first and columns other than the first do not contain θ_{11}. On the other hand, the number of blocks of type 2 containing θ_{11} is $(n-2)(n-1)^2$. Thus the replication number, r, is given by

$$r=(2n-1)+\binom{n-1}{2}+(n-2)(n-1)^2=(2n-1)(n^2-3n+4)/2 .$$

For getting λ_1, note that if we fix two treatments from the same group, $(\theta_{11}, \theta_{12})$, say, then this pair occurs in n blocks of type 1 and $(n-2)[(n-2)(n-1)]$ blocks of type 2. The value of λ_1 is, therefore, $\lambda_1=n+(n-2)^2(n-1)=n^3-5n^2+9n-4$.

For λ_2, if we fix any two treatments from two different groups, namely, starting from the pair, $(\theta_{11}, \theta_{21})$, say, it follows that this pair occurs in n blocks of type 1 and $(n-2)(n-1)$ blocks of type 2. Hence $\lambda_2=n+(n-2)(n-1)=n^2-2n+2$. Thus, the proof of the theorem is completed.

Remark 2.2. When $n=3$, a design defined by parameter (2.3) is a BIB design with parameters $v=9$, $b=18$, $r=10$, $k=5$, $\lambda=5$ having its complementary design as $v'=9$, $b'=18$, $r'=8$, $k'=4$, $\lambda'=3$ which is a known design (cf. Raghavarao [2]). Note that $\lambda_1=\lambda_2$ in Theorem 2.3 iff $n=3$.

3. Construction of a series of rectangular PBIB designs

THEOREM 3.1. *For $n-1\geqq t\geqq 2$, there exists a rectangular PBIB design with parameters*

$$
\begin{aligned}
&v=n^2\,, \qquad b=n^2\binom{n-1}{t}\,, \qquad r=\binom{n-1}{t}(nt-t+1)\,,\\
&k=nt-t+1\,, \qquad \lambda_1=(n-1)(n-2)\binom{n-2}{t-1}\,,\\
&\lambda_2=(n-1)(n-2)\binom{n-3}{t-2}\,, \qquad \lambda_3=(n-2)^2\binom{n-3}{t-2}+2\binom{n-2}{t-1}\,,\\
&n_1=n_2=n-1\,, \qquad n_3=(n-1)^2\,.
\end{aligned} \tag{3.1}
$$

PROOF. Let a rectangular association scheme for $m=n$ be as follows:

$$
\begin{matrix}
\theta_{11} & \theta_{21} & \cdots & \theta_{n1}\\
\theta_{12} & \theta_{22} & \cdots & \theta_{n2}\\
\vdots & \vdots & & \vdots\\
\theta_{1n} & \theta_{2n} & \cdots & \theta_{nn}
\end{matrix}
$$

where two elements in the same row are first associates, two elements in the same columns are second associates and two elements from different rows and different columns are third associates. Take any element θ_{ij} in the above scheme. Omit the row and the column in which this particular element appears. Choose t rows out of the $n-1$ remaining rows. Clearly we can get $\binom{n-1}{t}$ sets of such t rows forming, after adjoining θ_{ij}, blocks in the required design. We form blocks in this way for every θ_{ij} of the scheme, giving $b=n^2\binom{n-1}{t}$ blocks, each of size $t(n-1)+1$, i.e., $k=nt-t+1$. It is obvious that the replication number, r, is

$$
r=\binom{n-1}{t}+(n-1)^2\binom{n-2}{t-1}=(nt-t+1)\binom{n-1}{t}\,.
$$

For calculation of λ_1, let us fix any row, j, say, and consider a pair $(\theta_{pj}, \theta_{lj})$, say, where $p\neq l$. This pair occurs in $\lambda_1=(n-1)(n-2)\cdot\binom{n-2}{t-1}$ blocks.

For λ_2, let us fix two rows, say q and s, where $q\neq s$, and fix two treatments forming a pair $(\theta_{iq}, \theta_{is})$ from these rows. This pair occurs in $(n-1)(n-2)\binom{n-3}{t-2}$ blocks in our design.

For λ_3, take a pair $(\theta_{ij}, \theta_{lp})$, where $i\neq l$ and $j\neq p$. If we fix θ_{ij}, then the pair $(\theta_{ij}, \theta_{lp})$ occurs in $\binom{n-2}{t-1}$ blocks of the design, and if we fix θ_{lp}, then the pair $(\theta_{ij}, \theta_{lp})$ occurs in $\binom{n-2}{t-1}$ blocks of the design.

Now fixing the two rows, one containing θ_{ij} and the other θ_{lp}, there are $(n-2)^2\binom{n-3}{t-2}$ blocks containing the pair $(\theta_{ij}, \theta_{lp})$. Therefore $\lambda_3=(n-2)^2\binom{n-3}{t-2}+2\binom{n-2}{t-1}$. Thus, the proof of this theorem is completed.

Now we investigate cases when a PBIB design defined by (3.1) reduces to a PBIB design with two associate classes.

Case (1): $\lambda_1=\lambda_2$, i.e., $n=t+1$. In this case, we get an L_2 PBIB design with parameters $v^*=(t+1)^2=b^*$, $r^*=k^*=t^2+1$, $\lambda_1^*=\lambda_2^*=t(t-1)$, $\lambda_3^*=t^2-2t+3$, and this family is believed to be new. But for $t=2$, $v^*=b^*=9$, $r^*=k^*=5$, $\lambda_1^*=2$, $\lambda_2^*=3$ is listed in Clatworthy [1] as LS49, and for $t=3$, we get $v^*=b^*=16$, $r^*=k^*=10$, $\lambda_1^*=\lambda_2^*=\lambda_3^*=6$ which is a known symmetrical BIB design with $(v, k, \lambda)=(16, 10, 6)$ (cf. Raghavarao [2]).

Case (2): $\lambda_2=\lambda_3$, i.e., $t=3$. In this case, for $n>4$, we get a regular GD design with parameters $v'=n^2$, $b'=n^2\binom{n-1}{3}$, $r'=(3n-2)\binom{n-1}{3}$, $k'=3n-2$, $\lambda_1'=(n-1)(n-2)\binom{n-2}{2}$, $\lambda_2'=(n-1)(n-2)(n-3)$ and $m'=n'=n$. But, for $n=4$, we get a symmetrical BIB design with $(v, k, \lambda)=(16, 10, 6)$.

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