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CONNECTEDNESS OF PBIB DESIGNS HAVING ASYMMETRICAL ASSOCIATION SCHEMES*

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Summary

A necessary and sufficient condition for the connectedness of m-associate partially balanced incomplete block (PBIB) designs having an asymmetrical association scheme is given, only in terms of design parameters, without inner structure parameters of designs.

1. Introduction

A block design is originally defined to be connected (Bose [2]) if, given any two treatments i and j, it is possible to construct a chain of treatments $i=i_0, i_1, i_2, \cdots, i_n=j$ such that every consecutive two treatments in the chain occur together in a block. This definition is obviously equivalent to the following statistical statement: A block design is said to be connected if all elementary treatment contrasts are estimable. This fact shows the usefulness of connected designs in statistical analysis.

As a very well-known result, we have (cf. Raghavarao [9]) that a block design with parameters v, b, r, k is connected if and only if the rank of a matrix $rI_v-(1/k)NN'$ is exactly $v-1$, where I_v is the identity matrix of order v and N is the incidence matrix of the design. However, this condition uses inner structure parameters and it is not so easy to use them in a practical sense. Connectedness of m-associate PBIB designs having usual association schemes due to Bose and Mesner [3], who developed the idea of using association matrices, has been investigated, only in terms of design parameters, by Mohan [5], Kageyama [4], Ogawa, Ikeda and Kageyama [7], Saha and Kageyama [10] and others. It is well-known that the relation of association in the usual association scheme is symmetrical or reflexive. By relaxing some axiom of the usual association scheme, Shah [11] and Nair [6] intro-

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duced the concept of association schemes other than the usual one. In this case, Shah [11] generalized usual PBIB designs by relaxing a condition $p^i_{jk}=p^i_{kj}$. He gave an axiom that if two treatments are ith associates, then the number of treatments common to the jth associates of the first treatment and the kth associates of the second treatment, plus the number of treatments common to the kth associates of the first treatment and the jth associates of the second treatment, is equal to $2q^i_{jk}$ and is the same for all pairs of treatments that are ith associates. Even under this axiom, he proved that his condition is equivalent to the usual PBIB design association scheme for a two-associate case. His result shows that a necessary and sufficient condition for the connectedness of 2-associate PBIB designs having the association scheme of Shah [11] is entirely the same as an expression described in Ogawa, Ikeda and Kageyama [7].

Here, from a point of view of deriving a condition easy to use, we shall discuss the connectedness of m-associate PBIB designs having an asymmetrical association scheme due to Nair [6]. A necessary and sufficient condition for the connectedness of such PBIB designs is given by the same approach as in Saha and Kageyama [10].

For usual association schemes and some technical terms in designs, we refer to Raghavarao [9].

2. Discussion

At first, we present the main result of Ogawa, Ikeda and Kageyama [7] to derive a solution of the present problem:

THEOREM. *An m-associate PBIB design with $\lambda_1=\lambda_2=\cdots=\lambda_s=0<\lambda_{s+1}, \lambda_{s+2},\cdots,\lambda_m$ is connected if and only if the following holds:*

(i) *$p^{\alpha}_{iu}>0$ for some $\alpha\in\{1,2,\cdots,s\}$ and some i, $u\in\{s+1,\cdots,m\}$, and for each $\alpha'(\neq\alpha)\in\{1,2,\cdots,s\}$*

(ii) *$p^{\alpha'}_{\beta l}>0$ for some $\beta(\neq\alpha')\in\{1,2,\cdots,s\}$ (including $\beta=\alpha$) and some $l\in\{s+1,\cdots,m\}$.*

Remark. We do not need the condition (ii) if the condition (i) holds for *all* $\alpha\in\{1,2,\cdots,s\}$.

Next, we introduce the Nair [6] design: Given v treatments $1,2,\cdots,v$, a relation satisfying the following axioms is said to be an asymmetrical association scheme with m associate classes.

(a) Given any treatment, the other treatments can be classified as 1st, 2nd, $\cdots$, mth associates of it, the relation of association being not necessarily symmetrical, i.e., if α is the ith associate of β, β need not be the ith associate of α.

(b) Each treatment α has n_i ith associates, the number n_i being independent of α.

(c) If α is the ith associate of β, then the number of treatments which are jth associates of α and uth associates of β is p^i_{ju} and is independent of this pair. Also, p^i_{ju} need not $=p^i_{uj}$.

If we have this association scheme, then we get a PBIB design with r replications and b blocks if we can arrange v treatments into b blocks such that (i) each block contains k treatments (all distinct), (ii) each treatment is contained in r blocks and (iii) if α is the ith associate of β, then the pair (β, α) occurs together in that order in λ_i blocks, λ_i being independent of this pair.

Remark. An example of showing a relation between the usual association scheme and the Nair association scheme is given in Aggarwal and Raghavarao [1].

Under this definition of the Nair design, it is obvious that the Ogawa, Ikeda and Kageyama [7] condition for a PBIB design having the usual association scheme is sufficient for the connectedness of the Nair PBIB design. Now, in the present association scheme, the relation of association is not necessarily symmetrical. So, we have to make some modification of the Ogawa, Ikeda and Kageyama condition to derive a necessary and sufficient condition for the connectedness. However, by considering the meaning of the axiom (a) in the point of constituting chains of treatments for the connectedness, the Saha and Kageyama [10] approach can easily produce an answer for the present problem. Thus, we can state the following result without giving any special proof.

PROPOSITION. *Let D be an m-associate PBIB design, having an asymmetrical association scheme of Nair* [6], *with parameters v, b, r, k, λ_i, n_i, p^i_{jl} $(i, j, l=0, 1, \cdots, m)$ satisfying*

$$\lambda_1=\lambda_2=\cdots=\lambda_s=0<\lambda_{s+1}, \lambda_{s+2}, \cdots, \lambda_m$$

for $0<s<m\geqq 2$. Then a necessary and sufficient condition for D to be connected is that

(i) *for some t (being an integer of $1\leqq t\leqq s$) distinct positive integers $\alpha_1, \alpha_2, \cdots, \alpha_t$ $(1\leqq\alpha_i\leqq s)$,*

$$p^{\alpha_i}_{uu'}>0 \quad or \quad p^{\alpha_i}_{u'u}>0 \qquad for\ some\ u, u'\in\{s+1, \cdots, m\}$$

for each α_i $(i=1, 2, \cdots, t)$; and

(ii) *[for the case $d=s-t>0$] for each β_j of the integers $\{\beta_1, \beta_2, \cdots, \beta_d\}=\{1, 2, \cdots, s\}\backslash\{\alpha_1, \alpha_2, \cdots, \alpha_t\}$, there exist some n $(1\leqq n\leqq d)$ distinct integers, $\beta_{j_0}=\beta_j, \beta_{j_1}, \cdots, \beta_{j_{n-1}}, \beta_{j_n}=\alpha_i$, having an $\alpha_i\in\{\alpha_1, \alpha_2, \cdots, \alpha_t\}$ with $1\leqq j_1, \cdots, j_{n-1}\leqq d$, such that*

$$p^{\beta j}_{\beta j_1 u_1}>0 \quad or \quad p^{\beta j}_{u_1 \beta j_1}>0\,, \qquad p^{\beta j_1}_{\beta j_2 u_2}>0 \quad or \quad p^{\beta j_1}_{u_2 \beta j_2}>0\,,$$
$$\cdots,\ p^{\beta j_{n-1}}_{\alpha_i u_n}>0 \quad or \quad p^{\beta j_{n-1}}_{u_n \alpha_i}>0$$

for some $u_1, u_2, \cdots, u_n \in \{s+1, \cdots, m\}$.

Remark. If all the λ_i's are positive, then the design is obvious to be connected, because of the definition of connectedness. So, a restriction $s>0$ is natural. On the other hand, if all the λ_i's are zero, then it is clear that the design is disconnected. Then a condition $s<m$ is reasonable. Thus, we can put a condition $0<s<m$, for the present problem, without loss of generality.

In a practical sense, the following is immediate from the Proposition when $m=2$ and 3.

COROLLARY. (a) *A 2-associate PBIB design, having an asymmetrical association scheme, with* $\lambda_1=0<\lambda_2$ *is connected if and only if* $p^1_{22}>0$.

(b) *A necessary and sufficient condition for a 3-associate PBIB design having an asymmetrical association scheme to be connected is that*

(I) *when* $\lambda_1=0<\lambda_2, \lambda_3$,

$$p^1_{uu'}>0 \quad or \quad p^1_{u'u}>0 \qquad for\ some\ u, u' \in \{2, 3\}\ ;$$

(II) *when* $\lambda_1=\lambda_2=0<\lambda_3$,

(i) $p^1_{33}>0$ *and* $p^2_{33}>0$; *or*

(ii) $p^1_{33}>0$ *and* $\{p^2_{13}>0$ *or* $p^2_{31}>0\}$; *or*

(iii) $p^2_{33}>0$ *and* $\{p^1_{23}>0$ *or* $p^1_{32}>0\}$.

Remark. The above condition (a) in 2-associate PBIB designs is entirely the same as that for 2-associate PBIB designs having the usual association scheme (cf. Mohan [5]).

By use of the Proposition or Corollary, we can check the connectedness of some PBIB designs having the asymmetrical association schemes. For instance, we consider individual examples of PBIB designs mentioned in Nair [6].

(1) A 3-associate PBIB design with parameters $v=b=9$, $r=k=3$, $\lambda_1=2$, $\lambda_2=1$, $\lambda_3=0$, $n_1=n_2=1$, $n_3=6$ given on page 822 is disconnected.

(2) A 3-associate PBIB design with parameters $v=6$, $b=24$, $r=8$, $k=2$, $\lambda_1=0$, $\lambda_2=\lambda_3=1$, $n_1=1$, $n_2=n_3=2$ given on page 826, and a 4-associate PBIB design with parameters $v=8$, $b=32$, $r=12$, $k=3$, $\lambda_1=0$, $\lambda_2=\lambda_3=\lambda_4=2$, are both connected.

For other examples,

(3) A 6-associate PBIB design with parameters $v=b=t(t+1)(t^2+t+2)/2$, $r=k=t(t+2)$, $\lambda_1=t+2$, $\lambda_2=t$, $\lambda_3=\lambda_4=\lambda_5=2$, $\lambda_6=0$, $n_1=2t^2$, $n_2=2(t-1)$, $n_3=t^2(t-1)^2/2$, $n_4=t^2(t-1)=n_5$, $n_6=(t-1)(t-2)/2$, given on pages 137 and 138 of Patwardhan and Vartak [8], is connected.

Thus, since our conditions are expressed only in terms of design parameters, it would be easy to use them.

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