Ann. Inst. Statist. Math.
37 (1985), Part A, 95-108

# ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTIONS OF SOME TEST STATISTICS

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(Received Dec. 1, 1983; revised Apr. 2, 1984)

## Summary

A modified Wald statistic for testing simple hypothesis against fixed as well as local alternatives is proposed. The asymptotic expansions of the distributions of the proposed statistic as well as the Wald and Rao statistics under both the null and alternative hypotheses are obtained. The powers of these statistics are compared and it is shown that for special structures of parameters some statistics have same power in the sence of order $1/\sqrt{n}$. The results obtained are applied for testing the hypothesis about the covariance matrix of the multivariate normal distribution and it is shown that none of the tests based on the above statistics is uniformly superior.

## 1. Introduction

Let $X=[\underset{\sim}{x}_1, \underset{\sim}{x}_2, \cdots, \underset{\sim}{x}_n]$ be an $m\times n$ observation matrix, where $\underset{\sim}{x}_\alpha$'s are independently and identically distributed with a probability density function $f(\underset{\sim}{x}|\underset{\sim}{\theta})$ depending on an unknown parameter $\underset{\sim}{\theta}=(\theta_1, \theta_2, \cdots, \theta_p)'$, $p\geqq 1$.

Denoting the log-likelihood function of $X$ by $L(\underset{\sim}{\theta})=\sum_{\alpha=1}^{n}\log f(\underset{\sim}{x}_\alpha|\underset{\sim}{\theta})$, the following notations and assumptions will be adopted.

(i) The function $L(\underset{\sim}{\theta})$ is regular with respect to $\underset{\sim}{\theta}$.

(ii) Any function evaluated at $\underset{\sim}{\theta}=\hat{\underset{\sim}{\theta}}$ will be distinguished by the addition of a circumflex ^.

(iii) Any function evaluated at $\underset{\sim}{\theta}=\underset{\sim}{\theta}_0$ will be distinguished by the addition of a tilde ~.

(iv) Let

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* Research supported by the National Science Foundation Grant MCS 830149.
Key words and phrases: Likelihood ratio criterion, Rao statistic, modified Wald statistics, asymptotic expansion, local alternative, power function.

$$y_i = \frac{1}{\sqrt{n}} \frac{\partial L}{\partial \theta_i}, \qquad y_{ij} = \frac{1}{n} \frac{\partial^2 L}{\partial \theta_i \partial \theta_j},$$

$$y_{ijk} = \frac{1}{n\sqrt{n}} \frac{\partial^3 L}{\partial \theta_i \partial \theta_j \partial \theta_k}, \qquad y_{ijkl} = \frac{1}{n^2} \frac{\partial^4 L}{\partial \theta_i \partial \theta_j \partial \theta_k \partial \theta_l},$$

for $i, j, k, l = 1, 2, \cdots, p$ and let

$$\underset{\sim}{y} = (y_1, y_2, \cdots, y_p)', \qquad Y = (y_{ij}), \qquad Y_{...} = (y_{ijk}), \qquad Y_{....} = (y_{ijkl}),$$

$$\kappa_{i,j} = \mathrm{E}(y_i y_j), \qquad \kappa_{ij} = \mathrm{E}(y_{ij}),$$

$$\kappa_{i,j,k} = \sqrt{n}\, \mathrm{E}(y_i y_j y_k), \qquad \kappa_{i,jk} = \sqrt{n}\, \mathrm{E}(y_i y_{jk}), \qquad \kappa_{ijk} = \sqrt{n}\, \mathrm{E}(y_{ijk})$$

$$\kappa_{i,j,k,l} = n\, \mathrm{E}(y_i y_j y_k y_l), \qquad \kappa_{i,j,kl} = n\, \mathrm{E}(y_i y_j y_{kl}),$$

$$\kappa_{ij,kl} = n\, \mathrm{E}(y_{ij} y_{kl}), \qquad \kappa_{ijkl} = n\, \mathrm{E}(y_{ijkl})$$

$$K = (\kappa_{i,j}) = \mathrm{E}(\underset{\sim}{y}\underset{\sim}{y}') = -\mathrm{E}(Y) = -K_{..}$$

$$K_{...} = (\kappa_{ijk}), \qquad K_{.,..} = (\kappa_{i,jk}), \qquad K_{.,.,.} = (\kappa_{i,j,k})$$

$$K_{....} = (\kappa_{ijkl}), \qquad K_{.,...} = (\kappa_{i,jkl}), \qquad K_{.,.,..} = (\kappa_{i,j,kl}), \qquad K_{..,..} = (\kappa_{ij,kl})$$

$$K_{.,.,.,.} = (\kappa_{i,j,k,l}).$$

(v) For three and four suffix quantities, the following summation notations are adopted.

Let $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ be $p \times 1$ column vectors, and $A$, $B$ and $C$ be $p \times p$ matrices,

$$K_{...} \circ \underset{\sim}{a} \circ \underset{\sim}{b} \circ \underset{\sim}{c} = \sum \kappa_{ijk} a_i b_j c_k,$$

$$K_{...} \circ \underset{\sim}{a} \circ \underset{\sim}{b} = \left(\sum \kappa_{ijk} a_j b_k\right), \qquad K_{...} \circ \underset{\sim}{a} = \left(\sum \kappa_{ijk} a_k\right)$$

$$K_{...} \circ A \circ \underset{\sim}{b} = \sum \kappa_{ijk} a_{ij} b_k$$

$$K_{...} * A * B * C * K_{...} = \sum \kappa_{ijk} \kappa_{pqr} a_{ip} b_{jq} c_{kr}$$

$$K_{..,..} \circ A \circ B = \sum \kappa_{ij,kl} a_{ij} b_{kl} \quad \text{or} \quad \sum \kappa_{ij,kl} a_{ji} b_{kl}, \quad \text{etc.}$$

$$K_{..,..} \circledast A \circledast B = \sum \kappa_{ij,kl} a_{ik} b_{jl} \quad \text{or} \quad \sum \kappa_{ij,kl} a_{il} b_{jk}, \quad \text{etc.}$$

$$K_{..,..} \circledcirc D_{..,..} = \sum \kappa_{ij,kl} d_{ij,kl}$$

$$K_{.,..} \boxtimes A \boxtimes D_{..,..} \boxtimes K_{.,..} = \sum \kappa_{i,jk} \kappa_{p,qr} a_{ip} d_{jk,qr}.$$

The problem considered is that of testing a simple hypothesis $H_0$: $\underset{\sim}{\theta} = \underset{\sim}{\theta}_0$ against $H_1: \underset{\sim}{\theta} \neq \underset{\sim}{\theta}_0$, where $\underset{\sim}{\theta}_0$ is a specified vector. For testing this hypothesis the useful well known test statistics are the likelihood ratio statistic $\lambda$ proposed by Neyman and Pearson [10], Wald's statistic $W$ [15] and Rao's statistic $R$ [12]. The test statistics are expressed as follows:

$$\lambda=\prod_{\alpha=1}^{n}\frac{f(\underset{\sim}{x}_\alpha|\underset{\sim}{\theta}_0)}{f(\underset{\sim}{x}_\alpha|\hat{\underset{\sim}{\theta}})},$$

$$W=n(\hat{\underset{\sim}{\theta}}-\underset{\sim}{\theta}_0)'\hat{K}(\hat{\underset{\sim}{\theta}}-\underset{\sim}{\theta}_0),$$

$$R=\tilde{\underset{\sim}{y}}'\tilde{K}^{-1}\tilde{\underset{\sim}{y}},$$

where $\hat{\underset{\sim}{\theta}}$ is the maximum likelihood estimator of $\underset{\sim}{\theta}$.

The limiting distribution of $-2\log\lambda$ under $H_0$ was obtained as chi-squared distribution with $p$ degrees of freedom by Wilks [16] and the asymptotic expansion of the distribution of it was obtained up to order $1/n$ by Hayakawa [5], in which composite hypothesis case was studied. The validity of the asymptotic expansion of the distribution of the likelihood ratio criterion was shown in Chandra and Ghosh [1]. It is of interest to note that for a simple hypothesis we always have a correction factor $\rho$ which makes the term of order $1/n$ in the asymptotic expansion of the distribution of $-2\rho\log\lambda$ vanish.

Wald [15] considered the limiting distribution of $-2\log\lambda$ under Pitman's local alternative converging to the null hypothesis at the rate $1/\sqrt{n}$ and proposed $W$ as an estimator of a non-centrality parameter of the non-central chi-squared distribution. Peers [11] and Hayakawa [4] have considered in great detail asymptotic expansions of the distributions of the test statistics under Pitman's alternative. Peers [11] studied the Wald statistic and the Rao statistic for the simple hypothesis. The Wald's statistic and Rao's statistic for the composite hypothesis were also studied by Hayakawa [4] and Harris and Peers [3], respectively.

The Wald statistic $W$ is not easy to handle in the practical situation, because of the complexity of the information matrix $K(\hat{\underset{\sim}{\theta}})$ whose elements are functions of MLE $\hat{\underset{\sim}{\theta}}$. Here we propose a modified Wald statistic $\tilde{W}$

$$\tilde{W}=n(\hat{\underset{\sim}{\theta}}-\underset{\sim}{\theta}_0)'\tilde{K}(\hat{\underset{\sim}{\theta}}-\underset{\sim}{\theta}_0)$$

as a measure of the departure from the null hypothesis. We show in the later section that $\tilde{W}$ is more powerful than other statistics for parameters of the specific structure and in the region of certain alternative parameters.

## 2. The expansions of the distribution functions of statistics

To have the moment generating function (MGF) of test statistic we use the multivariate Edgeworth expansion for the joint density function of $\underset{\sim}{y}$, $Y$, $Y_{...}$ and $Y_{....}$ up to the order $1/n$, which is expressed as follows:

$$(1)\qquad f_1=f_0[1+A/\sqrt{n}+B/n]+o(1/n)\ ,$$

where

$$
\begin{aligned}
f_0&=(2\pi)^{-p/2}|K|^{-1/2}\exp\left\{-\frac{1}{2}\underset{\sim}{y}'K^{-1}\underset{\sim}{y}\right\}\prod\delta(y_{ij}-\kappa_{ij})\\
&\quad\cdot\prod\delta(y_{ijk}-\kappa_{ijk}/\sqrt{n})\prod\delta(y_{ijkl}-\kappa_{ijkl}/n)\\
A&=\frac{1}{6}\{K_{.,.,.}(\circ\, K^{-1}\underset{\sim}{y})^3-3K_{.,.,.}\circ K^{-1}\circ K^{-1}\underset{\sim}{y}\}-K_{.,..}\circ K^{-1}\underset{\sim}{y}\circ D_{..}\\
B&=\frac{1}{2}K_{..,..}\circledcirc D_{..,..}-\frac{1}{2}D_{..,..}(\circ\, K^{-1})^2\\
&\quad+\frac{1}{2}\{K_{.,.,..}\circ K^{-1}\circ D_{..}-K_{.,.,..}(\circ\, K^{-1}\underset{\sim}{y})^2\circ D_{..}\}\\
&\quad+\frac{1}{2}(p-\underset{\sim}{y}'K^{-1}\underset{\sim}{y})\operatorname{tr}KD_{..}-K_{.,...}\circ K^{-1}\underset{\sim}{y}\circ D_{...}\\
&\quad-\frac{1}{2}K_{.,..}\boxast A\boxast D_{..,..}\boxast K_{.,..}+\frac{1}{2}D_{..,..}(\circ\,(K_{..,.}\circ K^{-1}\underset{\sim}{y}))^2\\
&\quad+\frac{1}{24}\{K_{.,.,.,.}(\circ\, K^{-1}\underset{\sim}{y})^4-6K_{.,.,.,.}\circ K^{-1}(\circ\, K^{-1}\underset{\sim}{y})^2+3K_{.,.,.,.}(\circ\, K^{-1})^2\}\\
&\quad-\frac{1}{8}\{p(p+2)-2(p+2)\underset{\sim}{y}'K^{-1}\underset{\sim}{y}+(\underset{\sim}{y}'K^{-1}\underset{\sim}{y})^2\}\\
&\quad-\frac{1}{2}K_{.,.,.}\circ K^{-1}\circ K^{-1}(K_{.,..}\circ D_{..})\\
&\quad+\frac{1}{2}K_{.,.,.}(\circ\, K^{-1}\underset{\sim}{y})^2\circ K^{-1}(K_{.,..}\circ D_{..})\\
&\quad+\frac{1}{2}K_{.,.,.}\circ K^{-1}\circ K^{-1}\underset{\sim}{y}\cdot K_{.,..}\circ K^{-1}\underset{\sim}{y}\circ D_{..}\\
&\quad-\frac{1}{6}K_{.,.,.}(\circ\, K^{-1}\underset{\sim}{y})^3\cdot K_{.,..}\circ K^{-1}\underset{\sim}{y}\circ D_{..}\\
&\quad+\frac{1}{72}\{-9K_{.,.,.}\circ K^{-1}\circ K^{-1}(K_{.,.,.}\circ K^{-1})-6K_{.,.,.}(*\,K^{-1})^3*K_{.,.,.}\\
&\qquad+18K_{.,.,.}\circ K^{-1}\circ K^{-1}(K_{.,.,.}(\circ\, K^{-1}\underset{\sim}{y})^2)\\
&\qquad+9(K_{.,.,.}\circ K^{-1}\circ K^{-1}\underset{\sim}{y})^2+18K_{.,.,.}(*\,K^{-1})^2*K^{-1}\underset{\sim}{y}\underset{\sim}{y}'K^{-1}*K_{.,.,.}\\
&\qquad+(K_{.,.,.}(\circ\, K^{-1}\underset{\sim}{y})^3)^2-9K_{.,.,.}(\circ\, K^{-1}\underset{\sim}{y})^2\circ K^{-1}(K_{.,.,.}(\circ\, K^{-1}\underset{\sim}{y})^2)\\
&\qquad-6K_{.,.,.}\circ K^{-1}\circ K^{-1}\underset{\sim}{y}\cdot K_{.,.,.}(\circ\, K^{-1}\underset{\sim}{y})^3\}
\end{aligned}
$$

$$D_{..}=(d_{bc})\ ,\qquad d_{bc}=\delta'(y_{bc}-\kappa_{bc})/\delta(y_{bc}-\kappa_{bc})$$

$$D_{...}=(d_{abc})\,, \qquad d_{abc}=\delta'(y_{abc}-\kappa_{abc}/\sqrt{n})/\delta(y_{abc}-\kappa_{abc}/\sqrt{n})$$

$$D_{..,..}=(d_{ab,cd})\,, \qquad d_{ab,cd}=\begin{cases}\delta^{(2)}(y_{ab}-\kappa_{ab})/\delta(y_{ab}-\kappa_{ab}) & \\ \qquad\qquad \text{for } a=c,\ b=d \text{ or } a=d,\ b=c \\ d_{ab}d_{cd}\,, \qquad \text{otherwise}\,,\end{cases}$$

where $\delta$ is the Dirac delta function, and $\delta'$ and $\delta^{(2)}$ are the first and the second derivatives of $\delta$, respectively.

The expression $B$ of (6) in Hayakawa [5] should be read as the form given above. The correction is due to Iqbal [7].

Using an approach similar to that discussed in Hayakawa [5], the modified Wald statistic $\tilde{W}$, the Wald statistic and the Rao statistic $R$ are expressed asymptotically as follows:

$$(2)\qquad \tilde{W}=\underset{\sim}{y}'Y^{-1}KY^{-1}\underset{\sim}{y}+\tilde{r}_{1w}+\tilde{r}_{2w}+o_p(1/n)\,,$$

$$\tilde{r}_{1w}=Y_{...}\circ Y^{-1}KY^{-1}\underset{\sim}{y}(\circ\ Y^{-1}\underset{\sim}{y})^2$$

$$\begin{aligned}\tilde{r}_{2w}=&\,Y_{...}\circ Y^{-1}KY^{-1}\underset{\sim}{y}\circ Y^{-1}\underset{\sim}{y}\circ Y^{-1}[Y_{...}(\circ\ Y^{-1}\underset{\sim}{y})^2]\\&+\frac{1}{4}Y_{...}(\circ\ Y^{-1}\underset{\sim}{y})^2\circ Y^{-1}KY^{-1}[Y_{...}(\circ\ Y^{-1}\underset{\sim}{y})^2]\\&-\frac{1}{3}Y_{....}\circ Y^{-1}KY^{-1}\underset{\sim}{y}(\circ\ Y^{-1}\underset{\sim}{y})^3\end{aligned}$$

$$(3)\qquad W=\underset{\sim}{y}'Y^{-1}KY^{-1}\underset{\sim}{y}+r_{1w}+r_{2w}+o_p(1/n)\,,$$

$$r_{1w}=\tilde{r}_{1w}+\frac{1}{\sqrt{n}}\{K_{...}(\circ\ Y^{-1}\underset{\sim}{y})^3+K_{.,..}(\circ\ Y^{-1}\underset{\sim}{y})^3\}$$

$$\begin{aligned}r_{2w}=&\,\tilde{r}_{2w}+\frac{1}{\sqrt{n}}\Big\{\frac{3}{2}K_{...}(\circ\ Y^{-1}\underset{\sim}{y})^2\circ Y^{-1}[Y_{...}(\circ\ Y^{-1}\underset{\sim}{y})^2]\\&+K_{.,..}(\circ\ Y^{-1}\underset{\sim}{y})^2\circ Y^{-1}[Y_{...}(\circ\ Y^{-1}\underset{\sim}{y})^2]\\&+\frac{1}{2}K_{.,..}\circ Y^{-1}[Y_{...}(\circ\ Y^{-1}\underset{\sim}{y})^2](\circ\ Y^{-1}\underset{\sim}{y})^2\Big\}\\&-\frac{1}{2n}\{(K_{....}+2K_{.,...}+K_{..,..}+K_{.,.,..})(\circ\ Y^{-1}\underset{\sim}{y})^4\}\end{aligned}$$

$$(4)\qquad R=\underset{\sim}{y}'K^{-1}\underset{\sim}{y}\,.$$

In these expressions all $K$'s are the values at $\underset{\sim}{\theta}=\underset{\sim}{\theta}_0$, so that we have to use $\tilde{K}$'s by the notation (iii). However we use $K$'s because there may not be any confusions.

Hereafter we use $T_0$ for $-2\log\lambda$, $T_1$ for $W$, $T_2$ for $\tilde{W}$ and $T_3$ for $R$, respectively.

The use of Edgeworth expansion of the joint density function of

$(\underset{\sim}{y}, Y, Y_{...}, Y_{....})$ for the calculation of MGF's of these statistics and the inversion of MGF give the asymptotic expansions of the distributions of these as following forms under $H_0$.

$$(5) \qquad \mathrm{P}\{T_\alpha \leqq x\} = \mathrm{P}\{\chi_p^2 \leqq x\} + \frac{1}{n}\sum_{k=0}^{3} a_{2k}^{(\alpha)} \mathrm{P}\{\chi_{p+2k}^2 \leqq x\} + o(1/n) ,$$

where

$$(6) \qquad a_0^{(0)} = -a_2^{(0)} ,$$

$$\begin{aligned} a_2^{(0)} = \frac{1}{24}[&3K_{....}(\circ K^{-1})^2 + 12K_{..,..}(\circledast K^{-1})^2 + 12K_{.,.,..}(\circledast K^{-1})^2 \\ &+12K_{.,...}(\circ K^{-1})^2 + 3K_{...} \circ K^{-1} \circ K^{-1}(K_{...} \circ K^{-1}) \\ &+12K_{.,..} \circ K^{-1} \circ K^{-1}(K_{...} \circ K^{-1}) \\ &+12K_{.,..} \circ K^{-1} \circ K^{-1}(K_{..,.} \circ K^{-1}) \\ &+6K_{...}(* K^{-1})^3 * K_{...} + 4K_{.,.,.}(* K^{-1})^3 * K_{...} \\ &+24K_{.,..}(* K^{-1})^3 * K_{...} + 12K_{.,..}(* K^{-1})^3 * K_{..,.}] \end{aligned}$$

$$a_4^{(0)} = a_6^{(0)} = 0 .$$

$$\begin{aligned} (7) \qquad a_0^{(1)} = \frac{1}{24}[&12K_{..,..}(\circ K^{-1})^2 - 12K_{..,..}(\circledast K^{-1})^2 \\ &+12K_{.,.,..}(\circ K^{-1})^2 + 3K_{.,.,.,.}(\circ K^{-1})^2 \\ &-12K_{.,..} \circ K^{-1}(K_{.,..} \circ K^{-1}) \circ K^{-1} + 12K_{.,..}(\circledast K^{-1})^3 * K_{.,..} \\ &-12K_{.,.,.} \circ K^{-1} \circ K^{-1}(K_{.,..} \circ K^{-1}) \\ &-3K_{.,.,.} \circ K^{-1} \circ K^{-1}(K_{.,.,.} \circ K^{-1}) \\ &-2K_{.,.,.}(* K^{-1})^3 * K_{.,.,.} - 3p(p-2)] \end{aligned}$$

$$\begin{aligned} a_2^{(1)} = \frac{1}{8}[&-6K_{..,..}(\circ K^{-1})^2 - 6K_{.,.,..}(\circ K^{-1})^2 - 4K_{.,.,..}(\circledast K^{-1})^2 \\ &+2K_{....}(\circ K^{-1})^2 - 2K_{.,.,.,.}(\circ K^{-1})^2 \\ &+10K_{.,.,.} \circ K^{-1} \circ K^{-1}(K_{.,..} \circ K^{-1}) \\ &+4K_{.,.,.} \circ K^{-1} \circ K^{-1}(K_{..,.} \circ K^{-1}) \\ &+8K_{.,.,.}(* K^{-1})^3 * K_{.,..} + 3K_{.,.,.} \circ K^{-1} \circ K^{-1}(K_{.,.,.} \circ K^{-1}) \\ &+2K_{.,.,.}(* K^{-1})^3 * K_{.,.,.} + 2K_{...} \circ K^{-1} \circ K^{-1}(K_{.,..} \circ K^{-1}) \\ &+4K_{...}(* K^{-1})^3 * K_{.,..} + K_{...} \circ K^{-1} \circ K^{-1}(K_{...} \circ K^{-1}) \\ &+2K_{...}(* K^{-1})^3 * K_{...} + 9K_{.,..} \circ K^{-1}(K_{.,..} \circ K^{-1}) \circ K^{-1} \\ &+2K_{.,..} \circ K^{-1}(K_{..,.} \circ K^{-1}) \circ K^{-1} + 6K_{.,..} \circ K^{-1} \circ K^{-1}(K_{.,..} \circ K^{-1}) \\ &+2K_{.,..}(* K^{-1})^3 * K_{.,..} + 8K_{.,..}(* K^{-1})^3 * K_{..,.} + 2p^2] \end{aligned}$$

$$a_4^{(1)} = \frac{1}{8}[2K_{..,..}(\circ K^{-1})^2 + 4K_{..,..}(\circledast K^{-1})^2 + 2K_{.,.,..}(\circ K^{-1})^2$$

$$+4K_{.,.,..}(\circledast K^{-1})^2-2K_{....}(\circ K^{-1})^2+K_{.,.,.,.}(\circ K^{-1})^2$$
$$-8K_{.,.,.}\circ K^{-1}\circ K^{-1}(K_{.,..}\circ K^{-1})$$
$$-8K_{.,.,.}\circ K^{-1}\circ K^{-1}(K_{..,.}\circ K^{-1})$$
$$-12K_{.,.,.}(* K^{-1})^3 * K_{.,..}-3K_{.,.,.}\circ K^{-1}\circ K^{-1}(K_{.,.,.}\circ K^{-1})$$
$$-2K_{.,.,.}(* K^{-1})^3 * K_{.,.,.}-K_{...}\circ K^{-1}\circ K^{-1}(K_{...}\circ K^{-1})$$
$$-2K_{...}(* K^{-1})^3 * K_{...}-2K_{...}\circ K^{-1}\circ K^{-1}(K_{.,..}\circ K^{-1})$$
$$-4K_{...}(* K^{-1})^3 * K_{.,..}-6K_{.,..}\circ K^{-1}(K_{.,..}\circ K^{-1})\circ K^{-1}$$
$$-4K_{.,..}\circ K^{-1}(K_{..,.}\circ K^{-1})\circ K^{-1}-8K_{.,..}\circ K^{-1}\circ K^{-1}(K_{.,..}\circ K^{-1})$$
$$-4K_{.,..}\circ K^{-1}\circ K^{-1}(K_{..,.}\circ K^{-1})-8K_{.,..}(* K^{-1})^3 * K_{.,..}$$
$$-12K_{.,..}(* K^{-1})^3 * K_{..,.}-p(p+2)]$$

$$a_6^{(1)}=-(a_0^{(1)}+a_2^{(1)}+a_4^{(1)})\ .$$

(8) $$a_0^{(2)}=\frac{1}{24}[12K_{..,..}(\circ K^{-1})^2-12K_{..,..}(* K^{-1})^2+12K_{.,.,..}(\circ K^{-1})^2$$
$$+3K_{.,.,.,.}(\circ K^{-1})^2-12K_{.,...}\circ K^{-1}(K_{.,..}\circ K^{-1})\circ K^{-1}$$
$$-12K_{.,.,.}\circ K^{-1}\circ K^{-1}(K_{.,..}\circ K^{-1})$$
$$-3K_{.,.,.}\circ K^{-1}\circ K^{-1}(K_{.,..}\circ K^{-1})$$
$$+12K_{.,..}(* K^{-1})^3 * K_{.,..}-2K_{.,.,.}(* K^{-1})^3 * K_{.,.,.}-3p(p-2)]$$

$$a_2^{(2)}=\frac{1}{8}[-4K_{..,..}(\circ K^{-1})^2+4K_{..,..}(\circledast K^{-1})^2+8K_{.,.,..}(\circledast K^{-1})^2$$
$$+4K_{.,...}(\circ K^{-1})^2+2K_{.,.,.,.}(\circ K^{-1})^2$$
$$-16K_{...}\circ K^{-1}\circ K^{-1}(K_{.,..}\circ K^{-1})$$
$$+2K_{...}\circ K^{-1}\circ K^{-1}(K_{.,.,.}\circ K^{-1})+4K_{.,..}\circ K^{-1}\circ K^{-1}(K_{...}\circ K^{-1})$$
$$-16K_{.,..}\circ K^{-1}(K_{..,.}\circ K^{-1})\circ K^{-1}$$
$$-12K_{.,..}\circ K^{-1}\circ K^{-1}(K_{..,.}\circ K^{-1})$$
$$+3K_{.,.,.}\circ K^{-1}\circ K^{-1}(K_{.,.,.}\circ K^{-1})-2K_{...}(* K^{-1})^3 * K_{...}$$
$$-2K_{.,..}(* K^{-1})^3 * K_{...}+4K_{.,..}(* K^{-1})^3 * K_{..,.}$$
$$+2K_{.,.,.}(* K^{-1})^3 * K_{.,..}+2p^2]$$

$$a_4^{(2)}=\frac{1}{8}[-4K_{.,.,..}(\circ K^{-1})^2-8K_{.,.,..}(\circledast K^{-1})^2-4K_{.,...}(\circ K^{-1})^2$$
$$-3K_{.,.,.,.}(\circ K^{-1})^2-K_{...}\circ K^{-1}\circ K^{-1}(K_{...}\circ K^{-1})$$
$$+24K_{...}\circ K^{-1}\circ K^{-1}(K_{.,..}\circ K^{-1})$$
$$-16K_{.,..}\circ K^{-1}\circ K^{-1}(K_{...}\circ K^{-1})$$
$$+8K_{.,..}\circ K^{-1}(K_{..,.}\circ K^{-1})\circ K^{-1}$$
$$+12K_{.,..}\circ K^{-1}\circ K^{-1}(K_{..,.}\circ K^{-1})$$
$$-3K_{.,.,.}\circ K^{-1}\circ K^{-1}(K_{.,.,.}\circ K^{-1})$$
$$-2K_{.,..}(* K^{-1})^3 * K_{.,..}-p(p+2)]$$

$$a_6^{(2)} = -(a_0^{(2)} + a_2^{(2)} + a_4^{(2)}) \,.$$

$$\text{(9)} \quad a_0^{(3)} = \frac{1}{24}[3K_{\cdot,\cdot,\cdot,\cdot}(\circ K^{-1})^2 - 3K_{\cdot,\cdot,\cdot} \circ K^{-1} \circ K^{-1}(K_{\cdot,\cdot,\cdot} \circ K^{-1}) \\ -2K_{\cdot,\cdot,\cdot}(* K^{-1})^3 * K_{\cdot,\cdot,\cdot} - 3p(p+2)]$$

$$a_2^{(3)} = \frac{1}{8}[-2K_{\cdot,\cdot,\cdot}(\circ K^{-1})^2 + 3K_{\cdot,\cdot,\cdot} \circ K^{-1} \circ K^{-1}(K_{\cdot,\cdot,\cdot} \circ K^{-1}) \\ +2K_{\cdot,\cdot,\cdot}(* K^{-1})^3 * K_{\cdot,\cdot,\cdot} + 2p(p+2)]$$

$$a_4^{(3)} = \frac{1}{8}[K_{\cdot,\cdot,\cdot,\cdot}(\circ K^{-1})^2 - 3K_{\cdot,\cdot,\cdot} \circ K^{-1} \circ K^{-1}(K_{\cdot,\cdot,\cdot} \circ K^{-1}) \\ -2K_{\cdot,\cdot,\cdot}(* K^{-1})^3 * K_{\cdot,\cdot,\cdot} - p(p+2)]$$

$$a_6^{(3)} = \frac{1}{24}[3K_{\cdot,\cdot,\cdot} \circ K^{-1} \circ K^{-1}(K_{\cdot,\cdot,\cdot} \circ K^{-1}) + 2K_{\cdot,\cdot,\cdot}(* K^{-1})^3 * K_{\cdot,\cdot,\cdot}] \,.$$

The expression for $T_0$ was given in Hayakawa [5]. By the use of the expressions above, $100\alpha$-percentile point $\chi_\alpha^{(i)}$ of $T_i$ can be expanded in terms of the percentile point of a central chi-squared random variable. It is worth to note that the second term of the expansion of $\chi_\alpha^{(i)}$ is $O(1/n)$. (See, Hill and Davis [6]).

The expression of the asymptotic expansions of distributions suggests that there is no correction factor $\rho$ in general which makes the term of order $1/n$ in the expansion vanish.

## 3. Power comparison between the test criteria

Peers [11] compared the likelihood ratio criterion, Wald's statistic and Rao's statistic for a simple hypothesis and showed that only the likelihood ratio test is locally unbiased in the sense of order $1/\sqrt{n}$. Hayakawa [4] showed that it is not unbiased for a composite hypothesis. Harris and Peers [3] also compared these statistics for a composite hypothesis and showed that there was no uniform superiority property.

We here cite four expressions of the asymptotic expansions of the power functions of these statistics under Pitman's alternative

$$K_n : \underset{\sim}{\theta} = \underset{\sim}{\theta}_0 + \underset{\sim}{\varepsilon}/\sqrt{n}$$

$$\text{(10)} \quad P_\alpha = \mathrm{P}\,\{\chi_p^2(\delta^2) \geqq x^{(\alpha)}\} + \frac{1}{\sqrt{n}} \sum_{k=0}^{3} b_{2k}^{(\alpha)}\, \mathrm{P}\,\{\chi_{p+2k}^2(\delta^2) \geqq x^{(\alpha)}\} + o(1/\sqrt{n}) \,,$$

where $\chi_f^2(\delta^2)$ is a non-central chi-squared random variable with $p$ degrees of freedom and a non-centrality parameter $\delta^2 = \underset{\sim}{\varepsilon}' K \underset{\sim}{\varepsilon}/2$, and coefficients $b_{2k}^{(\alpha)}$'s are as follows.

(11) $b_0^{(0)}=\frac{1}{6}K_{...}(\circ\,\underset{\sim}{\varepsilon})^3,\qquad b_2^{(0)}=\frac{1}{2}K_{.,..}(\circ\,\underset{\sim}{\varepsilon})^3,\qquad b_4^{(0)}=\frac{1}{6}K_{.,.,.}(\circ\,\underset{\sim}{\varepsilon})^3$

(12) $b_0^{(1)}=\frac{1}{6}K_{...}(\circ\,\underset{\sim}{\varepsilon})^3,\qquad b_2^{(1)}=\frac{1}{2}\{K_{.,..}(\circ\,\underset{\sim}{\varepsilon})^3+K_{...}\circ K^{-1}\circ\underset{\sim}{\varepsilon}\}$

$b_4^{(1)}=-b_2^{(1)},\qquad b_6^{(1)}=-b_0^{(1)}$

(13) $b_0^{(2)}=\frac{1}{6}K_{...}(\circ\,\underset{\sim}{\varepsilon})^3$

$$b_2^{(2)}=\frac{1}{2}K_{.,..}(\circ\,\underset{\sim}{\varepsilon})^3-\frac{1}{2}(K_{...}-K_{.,.,.})\circ K^{-1}\circ\underset{\sim}{\varepsilon}$$

$$b_4^{(2)}=-\frac{1}{2}(K_{...}+2K_{.,..})(\circ\,\underset{\sim}{\varepsilon})^3+\frac{1}{2}(K_{...}-K_{.,.,.})\circ K^{-1}\circ\underset{\sim}{\varepsilon}$$

$$b_6^{(2)}=\frac{1}{6}(K_{...}-K_{.,.,.})(\circ\,\underset{\sim}{\varepsilon})^3$$

(14) $b_0^{(3)}=\frac{1}{6}K_{...}(\circ\,\underset{\sim}{\varepsilon})^3$

$$b_2^{(3)}=-\frac{1}{2}K_{.,.,.}\circ K^{-1}\circ\underset{\sim}{\varepsilon}+\frac{1}{2}K_{.,..}(\circ\,\underset{\sim}{\varepsilon})^3$$

$$b_4^{(3)}=\frac{1}{2}K_{.,.,.}\circ K^{-1}\circ\underset{\sim}{\varepsilon}$$

$$b_6^{(3)}=\frac{1}{6}K_{.,.,.}(\circ\,\underset{\sim}{\varepsilon})^3\,.$$

The $x^{(\alpha)}$ in (10) is an upper percentile point of $T_\alpha$. However, $x^{(\alpha)}$ can be replaced by the upper percentile point of the chi-squared random variable with $p$ degrees of freedom, because the second term of the asymptotic expansion of $x^{(\alpha)}$ is the order $1/n$.

The case $T_0$ was obtained by Peers [11] and Hayakawa [4], and the cases of $T_1$ and $T_3$ were obtained by Peers [11]. Chandra and Ghosh [2] showed the validity of the asymptotic expansions of the distributions of some of these statistics under the Pitman's alternative. From the expansions, we can see that the local powers of these test statistics are identical for some special structures of parameters in the sense of order $1/\sqrt{n}$.

(i) If $\kappa_{ijk}=0$, $T_0$ and $T_1$ have identical local power properties.
(ii) If $\kappa_{i,jk}=0$, $T_1$ and $T_3$ have identical local power properties.
(iii) If $\kappa_{i,j,k}=0$, $T_0$ and $T_3$ have identical local power properties.
(iv) If $\kappa_{i,jk}=-\kappa_{i,j,k}$, $T_2$ and $T_3$ have identical local power properties.
(v) If $\kappa_{i,jk}=-\kappa_{ijk}$, $T_1$ and $T_2$ have identical local power properties.

(vi) If $\kappa_{i,j,k}=\kappa_{ijk}$, $T_0$ and $T_2$ have identical local power properties.

Finally, we consider the asymptotic expansions of the distributions of the normalized statistics under the simple alternative $K_{\underset{\sim}{\varepsilon}}: \underset{\sim}{\theta}=\underset{\sim}{\theta}_0+\underset{\sim}{\varepsilon}$, where $\underset{\sim}{\varepsilon}$ is a fixed vector. The distribution of $-2\log\lambda$ was dealt with in Hayakawa [5]. The Wald statistic $W$ is expanded under $K_{\underset{\sim}{\varepsilon}}$ as

$$(W-n\underset{\sim}{\varepsilon}'K\underset{\sim}{\varepsilon})/\sqrt{n}=-2\underset{\sim}{\varepsilon}'KY^{-1}\underset{\sim}{y}-\partial K(\circ\, \underset{\sim}{\varepsilon})^2\circ Y^{-1}\underset{\sim}{y}+B+o_p(1/\sqrt{n}),$$

where

$$B=-Y_{\cdots}(\circ\, Y^{-1}K\underset{\sim}{\varepsilon})(\circ\, Y^{-1}\underset{\sim}{y})^2-\frac{1}{2}\partial K(\circ\, \underset{\sim}{\varepsilon})^2\circ Y^{-1}Y_{\cdots}(\circ\, Y^{-1}\underset{\sim}{y})^2$$
$$+\frac{1}{\sqrt{n}}\left\{\underset{\sim}{y}'Y^{-1}KY^{-1}\underset{\sim}{y}+2\partial K\circ\underset{\sim}{\varepsilon}(\circ\, Y^{-1}\underset{\sim}{y})^2+\frac{1}{2}\partial^2K(\circ\, \underset{\sim}{\varepsilon})^2(\circ\, Y^{-1}\underset{\sim}{y})^2\right\}$$

$$\partial K=\left(\frac{\partial\kappa_{i,j}}{\partial\theta_k}\right),\qquad \frac{\partial\kappa_{i,j}}{\partial\theta_k}=-(\kappa_{ijk}+\kappa_{ij,k})$$

$$\partial^2K=\left(\frac{\partial^2\kappa_{i,j}}{\partial\theta_k\partial\theta_l}\right),\qquad \frac{\partial^2\kappa_{i,j}}{\partial\theta_k\partial\theta_l}=-(\kappa_{ijkl}+\kappa_{ij,kl}+\kappa_{ijl,k}+\kappa_{ij,k,l}+\kappa_{ijk,l}).$$

Inversion of a characteristic function gives the following

(15) $$\mathrm{P}\{(W-n\underset{\sim}{\varepsilon}'K\underset{\sim}{\varepsilon})/\sqrt{n}\,\tau\leqq x\}$$
$$=\Phi(x)-\frac{1}{\sqrt{n}}\{w_1\Phi^{(1)}(x)/\tau+w_3\Phi^{(3)}(x)/\tau^3\}+o(1/\sqrt{n}),$$

where $\tau^2=\underset{\sim}{\tilde{a}}'K\underset{\sim}{\tilde{a}}$, and $\underset{\sim}{\tilde{a}}$ is defined as $\underset{\sim}{\tilde{a}}'\underset{\sim}{y}=2\underset{\sim}{\varepsilon}'\underset{\sim}{y}+\partial K(\circ\, \underset{\sim}{\varepsilon})^2\circ K^{-1}\underset{\sim}{y}$,

$$w_1=K_{\cdots}\circ K^{-1}\circ\underset{\sim}{\varepsilon}+\frac{1}{2}\partial K(\circ\, \underset{\sim}{\varepsilon})^2\circ K^{-1}K_{\cdots}\circ K^{-1}+p+2\partial K\circ\underset{\sim}{\varepsilon}\circ K^{-1}$$
$$+\frac{1}{2}\partial^2K(\circ\, \underset{\sim}{\varepsilon})^2\circ K^{-1}+2K_{\cdot,\cdot\cdot}\circ K^{-1}\circ\underset{\sim}{\varepsilon}+\partial K(\circ\, \underset{\sim}{\varepsilon})^2\circ K^{-1}(K_{\cdot\cdot,\cdot}\circ K^{-1})$$

$$w_3=K_{\cdots}\circ\underset{\sim}{\varepsilon}\circ\underset{\sim}{\tilde{a}}\underset{\sim}{\tilde{a}}'+\frac{1}{2}\partial K(\circ\, \underset{\sim}{\varepsilon})^2\circ K^{-1}(K_{\cdots}\circ\underset{\sim}{\tilde{a}}\underset{\sim}{\tilde{a}}')+\underset{\sim}{\tilde{a}}'K\underset{\sim}{\tilde{a}}+2\partial K\circ\underset{\sim}{\varepsilon}\circ\underset{\sim}{\tilde{a}}\underset{\sim}{\tilde{a}}'$$
$$+\frac{1}{2}\partial^2K(\circ\, \underset{\sim}{\varepsilon})^2\circ\underset{\sim}{\tilde{a}}\underset{\sim}{\tilde{a}}'+2K_{\cdot,\cdot\cdot}(\circ\, \underset{\sim}{\tilde{a}}\underset{\sim}{\tilde{a}}')\circ\underset{\sim}{\varepsilon}+\partial K(\circ\, \underset{\sim}{\varepsilon})^2\circ K^{-1}(K_{\cdot\cdot,\cdot}\circ\underset{\sim}{\tilde{a}}\underset{\sim}{\tilde{a}}').$$

The distribution of the modified Wald statistic is also obtained in a similar way as the case of the Wald statistic and this distribution is given by

(16) $$\mathrm{P}\{(\tilde{W}-n\underset{\sim}{\varepsilon}'K_0\underset{\sim}{\varepsilon})/\sqrt{n}\,\tilde{\tau}\leqq x\}$$
$$=\Phi(x)-\frac{1}{\sqrt{n}}\{\tilde{w}_1\Phi^{(1)}(x)/\tilde{\tau}+\tilde{w}_3\Phi^{(3)}(x)/\tilde{\tau}^3\}+o(1/\sqrt{n}),$$

where

$$\tilde{\tau}^2 = 4\underset{\sim}{\varepsilon}' K_0 K^{-1} K_0 \underset{\sim}{\varepsilon}$$

$$\tilde{w}_1 = K_{\cdots} \circ K^{-1} \circ K^{-1} K_0 \underset{\sim}{\varepsilon} + \mathrm{tr}\, K_0 K^{-1} + 2K_{\cdot,\cdot\cdot} \circ K^{-1} \circ K^{-1} K_0 \underset{\sim}{\varepsilon}$$

$$\tilde{w}_3 = 4K_{\cdots}(\circ K^{-1} K_0 \underset{\sim}{\varepsilon})^3 + 4\underset{\sim}{\varepsilon}' K_0 K^{-2} K_0 K^{-2} K_0 \underset{\sim}{\varepsilon}$$
$$+ \frac{4}{3} K_{\cdot,\cdot,\cdot}(\circ K^{-1} K_0 \underset{\sim}{\varepsilon})^3 + 8K_{\cdot,\cdot\cdot}(\circ K^{-1} K_0 \underset{\sim}{\varepsilon})^3$$

and $K_0 = K(\underset{\sim}{\theta}_0)$ and $K = K(\underset{\sim}{\theta})$, respectively.

To have the distribution of Rao's statistic, we have the normalized expression

$$(R - n\underset{\sim}{k}_0' K_0 \underset{\sim}{k}_0)/\sqrt{n} = 2\underset{\sim}{k}_0' K_0^{-1} \underset{\sim}{w} + \frac{1}{\sqrt{n}} \underset{\sim}{w}' K_0^{-1} \underset{\sim}{w} + o_p(1/\sqrt{n}),$$

where

$$\underset{\sim}{k}_0 = \mathrm{E}\left[\frac{\partial \log f(\underset{\sim}{x} | \underset{\sim}{\theta})}{\partial \underset{\sim}{\theta}}\bigg|_{\underset{\sim}{\theta}_0}\right] = \int \frac{\partial \log f(\underset{\sim}{x} | \underset{\sim}{\theta})}{\partial \underset{\sim}{\theta}}\bigg|_{\underset{\sim}{\theta}_0} f(\underset{\sim}{x} | \underset{\sim}{\theta}) d\underset{\sim}{x}$$

$$\underset{\sim}{w} = \frac{1}{\sqrt{n}} \sum_{\alpha=1}^{n} \left\{\frac{\partial \log f(\underset{\sim}{x}_\alpha | \underset{\sim}{\theta})}{\partial \underset{\sim}{\theta}}\bigg|_{\underset{\sim}{\theta}_0} - \underset{\sim}{k}_0\right\}.$$

The Edgeworth expansion of the probability density function of $\underset{\sim}{w}$ is expressed as

$$f_0\left[1 + \frac{1}{\sqrt{n}} \frac{1}{6}\left\{H_{\cdot,\cdot,\cdot}(\circ M^{-1} \underset{\sim}{w})^3 - 3H_{\cdot,\cdot,\cdot} \circ M^{-1} \circ M^{-1} \underset{\sim}{w}\right\} + o(1/\sqrt{n})\right],$$

where

$$f_0 = (2\pi)^{-p/2} |M|^{-1/2} \exp\left\{-\frac{1}{2} \underset{\sim}{w}' M^{-1} \underset{\sim}{w}\right\},$$

$$M = \mathrm{E}\,[\underset{\sim}{w}\underset{\sim}{w}'],$$

$$H_{\cdot,\cdot,\cdot} = (h_{i,j,k}), \qquad h_{i,j,k} = \mathrm{E}\,(w_i w_j w_k)/\sqrt{n}.$$

Using this expansion, we have the asymptotic expansion of the distribution of the normalized Rao statistic as

$$\text{(17)} \qquad \mathrm{P}\,\{(R - n\underset{\sim}{k}_0' K_0^{-1} \underset{\sim}{k}_0)/\sqrt{n}\,\hat{\tau} \leq x\}$$
$$= \Phi(x) - \frac{1}{\sqrt{n}} \{\gamma_1 \Phi^{(1)}(x)/\hat{\tau} + \gamma_3 \Phi^{(3)}(x)/\hat{\tau}^3\} + o(1/\sqrt{n}),$$

where

$$\hat{\tau}^2 = 4\underset{\sim}{k}_0' K_0^{-1} M K_0^{-1} \underset{\sim}{k}_0$$

$$\gamma_1 = \mathrm{tr}\, K_0^{-1} M$$

$$\gamma_3=\frac{4}{3}H_{\cdot,\cdot,\cdot}(\circ\ K_0^{-1}\underset{\sim}{k}_0)^3+4\underset{\sim}{k}_0'K_0^{-1}MK_0^{-1}MK_0^{-1}\underset{\sim}{k}_0\ .$$

*Example.* Let $\underset{\sim}{x}_1,\cdots,\underset{\sim}{x}_n$ be a random sample taken from the multivariate normal distribution with mean $\underset{\sim}{\mu}=\underset{\sim}{0}$ and covariance matrix $\Sigma$. We wish to test the simple hypothesis,

$$H_0:\Sigma=I_p \quad \text{against} \quad H_1:\Sigma\neq I_p\ .$$

Denoting $S=\sum_{\alpha=1}^{n}\underset{\sim}{x}_\alpha\underset{\sim}{x}_\alpha'$, and parametrizing $\underset{\sim}{\theta}'=(\theta_{11},\theta_{12},\cdots,\theta_{p,p-1})=(\sigma_{11},\sigma_{12},\cdots,\sigma_{p-1p})$, each test statistic can be respectively written as

$$T_0=\left(\frac{e}{n}\right)^{np/2}|S|^{n/2}\operatorname{etr}\left(-\frac{1}{2}S\right)$$

$$T_1=\frac{n}{2}\operatorname{tr}(nS^{-1}-I)^2$$

$$T_2=T_3=\frac{n}{2}\operatorname{tr}(S/n-I)^2\ .$$

In this case the modified Wald statistic and Rao statistic are the same and have been considered by Nagao [8].

It is of interest that if we parametrize $\underset{\sim}{\theta}=(\theta_{11},\cdots,\theta_{p-1,p})=(\sigma^{11},\cdots,\sigma^{p-1,p})$, where $\sigma^{ij}$'s are elements of inverse matrix of $\Sigma$, the likelihood ratio criterion is unchanged, Wald's statistic becomes $(n/2)\operatorname{tr}(S/n-I)^2$ and Rao's statistic $(n/2)\operatorname{tr}(nS^{-1}-I)^2$ for testing $\Sigma=I_p$.

Sugiura and Nagao [14] showed the exact unbiasedness of $T_0$, in which they proved the unbiasedness of a modified likelihood ratio criterion for testing a composite hypothesis $\Sigma=I_p$ under $\underset{\sim}{\mu}\neq\underset{\sim}{0}$.

Under the Pitman alternative $K_n:\Sigma=I_p+\Theta/\sqrt{n}$, the power functions of these are expressed as follows:

$$(18)\qquad P_0=\bar{P}_f(\delta^2)+\frac{1}{\sqrt{n}}\frac{1}{6}\operatorname{tr}\Theta^3\{\bar{P}_{f+4}(\delta^2)-3\bar{P}_{f+2}(\delta^2)+2\bar{P}_f(\delta^2)\}+o(1/\sqrt{n})\ ,$$

$$(19)\qquad P_1=\bar{P}_f(\delta^2)-\frac{1}{\sqrt{n}}\left\{\frac{1}{3}\operatorname{tr}\Theta^3\bar{P}_{f+6}(\delta^2)+\left((p+1)\operatorname{tr}\Theta-\frac{1}{2}\operatorname{tr}\Theta^3\right)\bar{P}_{f+4}(\delta^2)\right.$$
$$\left.-\left((p+1)\operatorname{tr}\Theta-\frac{1}{2}\operatorname{tr}\Theta^3\right)\bar{P}_{f+2}(\delta^2)-\frac{1}{3}\operatorname{tr}\Theta^3\bar{P}_f(\delta^2)\right\}$$
$$+o(1/\sqrt{n})\ ,$$

$$(20)\qquad P_2=\bar{P}_f(\delta^2)+\frac{1}{\sqrt{n}}\left\{\frac{1}{6}\operatorname{tr}\Theta^3\bar{P}_{f+6}(\delta^2)+\frac{1}{2}(p+1)\operatorname{tr}\Theta\bar{P}_{f+4}(\delta^2)\right.$$
$$\left.-\left(\frac{1}{2}(p+1)\operatorname{tr}\Theta+\frac{1}{2}\operatorname{tr}\Theta^3\right)\bar{P}_{f+2}(\delta^2)+\frac{1}{3}\operatorname{tr}\Theta^3\bar{P}_f(\delta^2)\right\}$$
$$+o(1/\sqrt{n})$$

where $\bar{P}_f(\delta^2)=\mathrm{P}\{\chi^2_f(\delta^2)\geqq x\}$ and $\chi^2_f(\delta^2)$ is a non-central chi-squared random variable with $f=p(p+1)/2$ degrees of freedom and a non-centrality parameter $\delta^2=(1/4)\,\mathrm{tr}\,\Theta^2$. The power function (18) was derived by Sugiura [13] and (20) by Nagao [9]. Nagao [9] compared the powers of $T_0$ and $T_2$ numerically and showed that there is no uniform superiority. Noting $\bar{P}_a(\delta^2)\geqq\bar{P}_b(\delta^2)$ for $a>b$, we have the following power comparison in the sence of order $1/\sqrt{n}$,

$$P_2>P_0>P_1 \quad \text{for } \mathrm{tr}\,\Theta>0 \text{ and } \mathrm{tr}\,\Theta^3>0$$

$$P_2=P_0=P_1 \quad \text{for } \mathrm{tr}\,\Theta=0 \text{ and } \mathrm{tr}\,\Theta^3=0$$

$$P_2<P_0<P_1 \quad \text{for } \mathrm{tr}\,\Theta<0 \text{ and } \mathrm{tr}\,\Theta^3<0\,.$$

This agrees with the numerical results of Nagao [9].

## Acknowledgment

The authors wish to express their gratitudes to the referee for his careful reading the manuscript and useful comments.

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