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ON THE RATE OF CONVERGENCE TO NORMALITY FOR GENERALIZED LINEAR RANK STATISTICS

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Summary

A generalized linear rank statistic is introduced to include, as special cases, both signed as well as unsigned linear rank statistics. For this statistic, the rate of convergence to asymptotic normality is investigated. It is shown that this rate is of order $O(N^{-1/2}\log N)$ if the score generating function ϕ is twice differentiable, and it is of order $O(N^{-1/2})$ if the second derivative of ϕ satisfies Lipschitz's condition of order $\geqq 1/2$. The results obtained extend as well as generalize most of the earlier results obtained in this direction.

1. Introduction

Let $X_{N1}, X_{N2}, \cdots, X_{NN}$, $N\geqq 1$ be independent r.v.'s (random variables) and let g be a real valued measurable function such that $X^g_{Nj}=g(X_{Nj})$, $1\leqq j\leqq N$, has a continuous c.d.f. (cumulative distribution function) G_{Nj}. We introduce the generalized linear rank statistic

$$(1.1)\qquad T_N=\sum_{j=1}^{N} c_{Nj}a_{NR^g_{Nj}}(X_{Nj})$$

where $\{c_{Nj}\colon 1\leqq j\leqq N\}$ is an array of regression constants; $\{a_{Nj}(\cdot)\colon 1\leqq j\leqq N\}$ is an array of real valued functions (called scores); and R^g_{Nj}, $1\leqq j\leqq N$, is the rank of X^g_{Nj} among $\{X^g_{Nk}\colon 1\leqq k\leqq N\}$. We assume that the $a_{Nj}(\cdot)$, $1\leqq j\leqq N$, are generated by a known (nonconstant) function (called a score generating function) $\phi(s,t)$, $0<s<1$, $-\infty<t<\infty$, in either of following two ways:

$$(1.2)\qquad a_{Nj}(t)=\phi(\mathrm{E}\,U_{N:j}, t)\,,\qquad 1\leqq j\leqq N \text{ (approximate scores)}$$

$$(1.3)\qquad a_{Nj}(t)=\mathrm{E}\,\phi(U_{N:j}, t)\,,\qquad 1\leqq j\leqq N \text{ (exact scores)}$$

where $U_{N:j}$ denotes the jth order statistic in a random sample of size

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N from the uniform distribution on (0, 1). To exclude trivialities, we assume that $\sum_{j=1}^{N}|c_{Nj}|>0$.

When $g(x)=x$ and $\phi(s,t)=\tilde{\phi}(s)$, the statistic (1.1) reduces to the simple linear rank statistic

$$T_N=\sum_{j=1}^{N} c_{Nj}a_{NR_{Nj}} \tag{1.4}$$

where a_{Nj}'s are usual scores of constants and R_{Nj}, $1\leqq j\leqq N$, is the rank of X_{Nj} among $\{X_{Nk}: 1\leqq k\leqq N\}$. For this statistic, the asymptotic normality was obtained by Hájek [6], [7]; the rate of convergence was investigated by Jurečková and Puri [11], Bergström and Puri [1] and Hušková [9], [10]; and Does [2] studied the rate of convergence as well as asymptotic expansions.

On the other hand, when $g(x)=|x|$ and $\phi(s,t)=\tilde{\phi}(s)\operatorname{sgn} t$, where $\operatorname{sgn} t=1$ or -1 according as $t\geqq 0$ or <0, the statistic (1.1) reduces to the signed linear rank statistic

$$T_N=\sum_{j=1}^{N} c_{Nj}a_{NR^+_{Nj}}\operatorname{sgn} X_{Nj} \tag{1.5}$$

where R^+_{Nj} is the rank of $|X_{Nj}|$ among $\{|X_{Nk}|: 1\leqq k\leqq N\}$. For this statistic, the reader is referred to Hájek [6] and Hušková [8] (for the asymptotic normality); Puri and Wu [16] and Puri and Seoh [13] (for the rate of convergence); and Puri and Seoh [14], [15] (for the Edgeworth expansions).

The aim of this paper is to investigate the problem of the rate of convergence for the statistics (1.1) which include the signed as well as (unsigned) simple linear rank statistics. The results obtained improve or include as special cases some of the results of Jurečková and Puri [11], von Bahr [18], Bergström and Puri [1], Hušková [9], [10] and Does [3] among others.

2. Assumptions and main theorems

We introduce some notations. For any function f, with the domain $D(f)$, $\|f\|$ denotes the supremum norm of f, i.e.,

$$\|f\|=\sup_{x\in D(f)}|f(x)| \qquad \text{or} \qquad \sup_{(x,y)\in D(f)}|f(x,y)|$$

depending on the number of variables. We also use S°, for a r.v. S, centered at its expectation, i.e., $S^\circ=S-ES$. Noting that $R^g_{Nj}=\sum_{k=1}^{N} u(X^g_{Nj}-X^g_{Nk})$, where $u(x)=(1+\operatorname{sgn} x)/2$, we define r.v.'s ρ_{Nj} and ρ_{Njj}, $1\leqq j\leqq N$, as

$$\rho_{Nj}=R^g_{Nj}/(N+1)=(N+1)^{-1}\left\{1+\sum_{k\neq j}^{N}u(X^g_{Nj}-X^g_{Nk})\right\},$$
(2.1)
$$\rho_{Njj}=\mathrm{E}\,(\rho_{Nj}\,|\,X_{Nj})=(N+1)^{-1}\left\{1+\sum_{k\neq j}^{N}G_{Nk}(X^g_{Nj})\right\}.$$

ASSUMPTION (A_l): The score generating function $\phi(s,t)$ is l-times differentiable with respect to its first argument s such that its lth partial derivatives $\phi_1^{(l)}(s,t)=\partial^l\phi(s,t)/(\partial s)^l$ satisfy Lipschitz's condition of order δ, $0<\delta\leq 1$, with respect to s, i.e.,

$$\sup_{-\infty<t<\infty}|\phi_1^{(l)}(x,t)-\phi_1^{(l)}(y,t)|\leq \Delta_l|x-y|^\delta\,,\qquad x,y\in(0,1)\,, \tag{2.2}$$

for some absolute constant Δ_l, and $\phi(s,t)$, $\phi_1^{(1)}(s,t),\cdots,\phi_1^{(l)}(s,t)$ are jointly measurable in s and t.

Remark 2.1. Assumption (A_l) is satisfied with Lipschitz's condition of order $\delta=1/2$ if there exist $(l+1)$th partial derivatives which are square integrable uniformly in t, i.e.,

$$\sup_{-\infty<t<\infty}\int_{-\infty}^{\infty}|\phi_1^{(l+1)}(s,t)|^2ds<\infty\,.$$

Assuming the first partial derivatives, we shall approximate T_N by

$$\hat{S}_N=\sum_{l=1}^{N}\mathrm{E}\,(S_N\,|\,X_l)-(N-1)\,\mathrm{E}\,S_N \tag{2.3}$$

where S_N is the first two terms of Taylor's expansion (with respect to its first argument) of T_N with approximate scores, i.e.,

$$S_N=\sum_{j=1}^{N}c_{Nj}\{\phi(\rho_{Njj},X_{Nj})+(\rho_{Nj}-\rho_{Njj})\phi_1^{(1)}(\rho_{Njj},X_{Nj})\}\,. \tag{2.4}$$

Let $\Phi(x)$ denote the standard normal c.d.f. and put

$$\tau_N^2=\mathrm{Var}\,T_N\,,\quad \sigma_N^2=\mathrm{Var}\,S_N\,,\quad \hat{\sigma}_N^2=\mathrm{Var}\,\hat{S}_N\,,\quad \omega_3=\hat{\sigma}_N^{-3}\sum_{j=1}^{N}|c_{Nj}|^3\,. \tag{2.5}$$

Then our main theorems are as follows:

THEOREM 2.1. *If Assumption* (A_1) *is satisfied, then*

$$\|\mathrm{P}\,(T_N^\circ\leq\hat{\sigma}_N\cdot)-\Phi(\cdot)\|\leq C_1L_N+2eM_N\max\{N^{-1/2},N^{-\delta/2}\}\log N \tag{2.6}$$

where C_1 *is an absolute constant*; L_N *and* M_N *are defined by*

$$L_N=4(2\|\phi\|^3+\|\phi_1^{(1)}\|^3)\omega_3\,,$$
(2.7)
$$M_N=64e\max\{\Delta_1,\|\phi\|,\|\phi_1^{(1)}\|\}\left(\sum_{j=1}^{N}c_{Nj}^2\right)^{1/2}\hat{\sigma}_N^{-1}\,;$$

and δ *and* Δ_1 *are given by* (2.2). *Moreover,*

(2.8) $$|\tau_N-\hat{\sigma}_N|\leqq\hat{\sigma}_N M_N \max\{N^{-1/2}, N^{-\delta/2}\} .$$

THEOREM 2.2. *If the first partial derivative* $\phi_1^{(1)}$ *of the score generating function* ϕ *exists and is bounded on* $(0,1)\times(-\infty,\infty)$, *then*

(2.9) $$\|\mathrm{P}(S_N^\circ\leqq\hat{\sigma}_N\cdot)-\Phi(\cdot)\|=O(\omega_\delta) ,$$

(2.10) $$|\sigma_N-\hat{\sigma}_N|=O(\hat{\sigma}_N\omega_\delta) .$$

THEOREM 2.3. *If Assumption* (A_2) *is satisfied, then*

(2.11) $$\|\mathrm{P}(T_N^\circ\leqq\hat{\sigma}_N\cdot)-\Phi(\cdot)\|=O(\max\{\omega_\delta, \omega_\delta^{2\delta}\}) ,$$

(2.12) $$|\tau_N-\hat{\sigma}_N|=O(\hat{\sigma}_N\omega_\delta) .$$

Remark 2.2. Theorem 2.1 is the improvement, as well as the generalization, of Theorem 1.2 of Bergström and Puri [1]. The latter, dealing with only approximate scores, gives the bound of order $O(N^{-1/2}\log N)$ under stronger condition that $\max_{1\leq j\leq N}|c_{Nj}|=O(N^{-1/2})$ and requires bounded second derivatives, whereas we, dealing with both exact scores and approximate scores, derive the bound under much weaker assumption, namely $\sum_{j=1}^N|c_{Nj}|^3=O(N^{-1/2})$, and require only the first derivatives satisfying Lipschitz's condition of order $\delta=1$.

Remark 2.3. Theorem 2.3 is the generalization of Hušková [9], [10] and also the improvement of Hušková [10]. But, in Hušková [10], Theorem B and proofs of Lemma 2, Lemma 4 and Lemma 6 are incorrect.

From now on, for the ease of convenience, we shall suppress the subscript N in c_{Nj}, a_{Nj}, R_{Nj}^q, etc., whenever it causes no confusion. For notational convenience we also use $\phi_1(x,y)$, instead of $\phi_1^{(1)}(x,y)$, for the first partial derivatives.

3. Preliminaries

In this section we prove several lemmas which are needed in the proofs of main theorems.

In view of (2.1) and (2.4) we can write

(3.1) $$\rho_j-\rho_{jj}=(N+1)^{-1}\sum_{k\neq j}^N H_{jk} ,$$

(3.2) $$S_N=\sum_{j=1}^N c_j\left\{\phi(\rho_{jj}, X_j)+(N+1)^{-1}\sum_{k\neq j}^N H_{jk}\phi_1(\rho_{jj}, X_j)\right\} ,$$

where r.v.'s H_{jk}, $1\leqq j$, $k\leqq N$, are defined by

$$H_{jk}=u(X_j^g-X_k^g)-G_k(X_j^g)\ . \tag{3.3}$$

For each j, conditionally given X_j^g, $\rho_j-\rho_{jj}$ is the sum of independent r.v.'s with zero means. Thus we may apply Lemma A.1 (in the appendix) to obtain for any integer $r\geqq 1$ and $1\leqq j\leqq N$, that

$$\mathrm{E}\,(\rho_j-\rho_{jj})^{2r}\leqq(4e)^r r^r N^{-r}\ . \tag{3.4}$$

We now estimate $\mathrm{E}\,(S_N^\circ-\hat{S}_N^\circ)^{2r}$ where r is an integer $\geqq 1$. It follows by elementary computations that for $1\leqq l\leqq N$,

$$\begin{aligned}\mathrm{E}\,(S_N\,|\,X_l)=&\,c_l\phi(\rho_{ll},X_l)+\mathrm{E}\,S_N-c_l\,\mathrm{E}\,\phi(\rho_{ll},X_l)\\&+(N+1)^{-1}\sum_{j\neq l}^{N}c_j\,\mathrm{E}\,\{H_{jl}\phi_1(\rho_{jj},X_j)\,|\,X_l\}\ .\end{aligned} \tag{3.5}$$

From (3.5), (2.3) and (3.2), we obtain

$$S_N^\circ-\hat{S}_N^\circ=(N+1)^{-1}\sum_{j=1}^{N}\sum_{k\neq j}^{N}c_j V_{jk} \tag{3.6}$$

where

$$V_{jk}=H_{jk}\phi_1(\rho_{jj},X_j)-\mathrm{E}\,\{H_{jk}\phi_1(\rho_{jj},X_j)\,|\,X_k\}\ . \tag{3.7}$$

Since $\mathrm{E}\,\{V_{jk}|X_j\}=\mathrm{E}\,\{V_{jk}|X_k\}=0$ for $j\neq k$, an application of Lemma A.2 (in the appendix) yields that for any positive integer r

$$\mathrm{E}\,(S_N^\circ-\hat{S}_N^\circ)^{2r}\leqq(8e\|\phi_1\|)^{2r}\left(\sum_{j=1}^{N}c_j^2\right)^r(2r)^{2r}N^{-r}\ . \tag{3.8}$$

Remark 3.1. It may be noted that the results (3.4) and (3.8) are generalizations, as well as improvements, of Lemma 2.1 and Lemma 2.2 of Bergström and Puri [1].

We now define r.v.'s I_k for $0\leqq k\leqq l$ as

$$I_k=\sum_{j=1}^{N}c_j\frac{(\rho_j-\rho_{jj})^k}{k!}\phi_1^{(k)}(\rho_{jj},X_j) \tag{3.9}$$

so that $S_N=I_0+I_1$. Also by Taylor's expansion for any x, y and t,

$$\phi(x,t)=\sum_{k=0}^{l-1}\frac{(x-y)^k}{k!}\phi_1^{(k)}(y,t)+\frac{(x-y)^l}{l!}\phi_1^{(l)}(\lambda x+(1-\lambda)y,t)$$

where $0\leqq\lambda\leqq 1$ depends on x, y and t. Thus, under Assumption (A_l)

$$\left|\phi(x,t)-\sum_{k=0}^{l}\frac{(x-y)^k}{k!}\phi_1^{(k)}(y,t)\right|\leqq\frac{A_l}{l!}|x-y|^{l+\delta}\ . \tag{3.10}$$

LEMMA 3.1. *Let $\tilde{T}_N$ denote the statistic* (1.1) *with exact scores (given by* (1.3)) *to distinguish it from the statistic T_N with approximate scores and let $r>0$ be an integer. Then under Assumption* (A_1)

$$\mathrm{E}(\tilde{T}_N^\circ - T_N^\circ)^{2r} \leqq (2\Delta_1)^{2r}\Big(\sum_{j=1}^N c_j^2\Big)^r N^{-\delta r} \tag{3.11}$$

and under Assumption (A_l), $l\geqq 2$,

$$\mathrm{E}(\tilde{T}_N^\circ - T_N^\circ)^{2r} \leqq \|\phi_1^{(2)}\|^{2r}\Big(\sum_{j=1}^N c_j^2\Big)^r N^{-r}\,. \tag{3.12}$$

PROOF. Put $\tilde{a}_j(t) = \mathrm{E}\,\phi(U_{N:j}, t)$, $1\leqq j\leqq N$. Then

$$\begin{aligned}\mathrm{E}(\tilde{T}_N^\circ - T_N^\circ)^{2r} &\leqq 2^{2r}\,\mathrm{E}(\tilde{T}_N - T_N)^{2r} \\ &\leqq 2^{2r}\Big(\sum_{j=1}^N c_j^2\Big)^r \mathrm{E}\Big[\sum_{j=1}^N \Big\{\tilde{a}_{R_j^0}(X_j) - \phi\Big(\frac{R_j^0}{N+1}, X_j\Big)\Big\}^2\Big]^r.\end{aligned} \tag{3.13}$$

Under Assumption (A_1), (3.10) ensures that for $1\leqq j\leqq N$ and any t

$$\Big|\tilde{a}_j(t) - \phi\Big(\frac{j}{N+1}, t\Big)\Big| \leqq \Delta_1\,\mathrm{E}\,\Big|U_{N:j} - \frac{j}{N+1}\Big|^{1+\delta} \leqq \Delta_1 N^{-(1+\delta)/2}\,. \tag{3.14}$$

(3.11) follows from (3.13) and (3.14).

On the other hand, under Assumption (A_l), $l\geqq 2$, there are second partial derivatives, bounded, such that

$$\Big|\tilde{a}_j(t) - \phi\Big(\frac{j}{N+1}, t\Big)\Big| \leqq \frac{\|\phi_1^{(2)}\|}{2}\,\mathrm{E}\Big(U_{N:j} - \frac{j}{N+1}\Big)^2 \leqq \frac{\|\phi_1^{(2)}\|}{2N}\,.$$

This inequality and (3.13) ensure (3.12) and the proof is complete.

LEMMA 3.2. *Let T_N be the statistic* (1.1) *with approximate scores (given by* (1.2)) *and suppose that Assumption (A_l), $l\geqq 1$, is satisfied. Then for any integer $r>0$,*

$$\mathrm{E}\Big(T_N^\circ - \sum_{k=0}^l I_k^\circ\Big)^{2r} \leqq \Big(\frac{2\Delta_l}{l!}\Big)^{2r}\Big(\sum_{j=1}^N c_j^2\Big)^r (4e)^{r(l+1)}(r(l+1))^{r(l+1)} N^{-r(l+\delta-1)}$$

where I_k, $1\leqq k\leqq l$, are defined by (3.9).

PROOF. It follows from (3.10) and Hölder's inequality that

$$\begin{aligned}\mathrm{E}\Big(T_N^\circ - \sum_{k=0}^l I_k^\circ\Big)^{2r} &\leqq 2^{2r}\,\mathrm{E}\Big(\sum_{j=1}^N c_j\Big\{\phi(\rho_j, X_j) - \sum_{k=0}^l \frac{(\rho_j - \rho_{jj})^k}{k!}\phi_1^{(k)}(\rho_{jj}, X_j)\Big\}\Big)^{2r} \\ &\leqq 2^{2r}\Big(\sum_{j=1}^N c_j^2\Big)^r\Big(\frac{\Delta_l}{l!}\Big)^{2r} N^{r-1}\sum_{j=1}^N \mathrm{E}\,|\rho_j - \rho_{jj}|^{2r(l+\delta)}\end{aligned}$$

which, together with (3.4), completes the proof.

Define for any positive integer r

$$\omega_r = \sum_{j=1}^N |c_j^*|^r \tag{3.15}$$

where $c_j^* = \hat{\sigma}_N^{-1} c_j$, $1 \leqq j \leqq N$. Also consider the r.v.'s $\hat{S}_N^{(j)}$, $1 \leqq j \leqq N$, defined as

$$(3.16) \quad \hat{S}_N^{(j)} = c_j^* \{\phi(\rho_{jj}, X_j) - \mathrm{E}\,\phi(\rho_{jj}, X_j)\} + (N+1)^{-1} \sum_{k \neq j}^{N} c_k^* \,\mathrm{E}\,\{[u(X_k^g - X_j^g) - G_j(X_k^g)]\phi_1(\rho_{kk}, X_k) \,|\, X_j\} .$$

Then, in view of (2.3) and (3.5), we have

$$(3.17) \quad \hat{\sigma}_N^{-1} \hat{S}_N^\circ = \sum_{j=1}^{N} \hat{S}_N^{(j)} .$$

It is easy to check that for $1 \leqq j \leqq N$

$$(3.18) \quad \begin{aligned} &\mathrm{E}\,\hat{S}_N^{(j)} = 0 , \qquad \sum_{j=1}^{N} \mathrm{E}\,(\hat{S}_N^{(j)})^2 = 1 , \\ &\mathrm{E}\,(\hat{S}_N^{(j)})^2 \leqq 2\{(c_j^*)^2 \|\phi\|^2 + N^{-1}\omega_2 \|\phi_1\|^2\} , \\ &\mathrm{E}\,|\hat{S}_N^{(j)}|^3 \leqq 4\{2|c_j^*|^3 \|\phi\|^3 + N^{-1}\omega_3 \|\phi_1\|^3\} . \end{aligned}$$

As $\hat{\sigma}_N^{-1} \hat{S}_N^\circ$ is the sum of independent r.v.'s with zero means and satisfying $\sum_{j=1}^{N} \mathrm{E}\,|\hat{S}_N^{(j)}|^3 \leqq L_N$, ($L_N$ defined by (2.7)), Lemma V.2.1 and Theorem V.2.3 of Petrov [12] ensure

LEMMA 3.3. *For* $|t| \leqq (4L_N)^{-1}$

$$(3.19) \quad |\mathrm{E}\exp(it\hat{\sigma}_N^{-1}\hat{S}_N^\circ) - \exp(-t^2/2)| \leqq 16 L_N |t|^3 \exp(-t^2/3) .$$

Moreover, there is an absolute constant C_1 *such that*

$$(3.20) \quad \|\mathrm{P}\,(\hat{\sigma}_N^{-1}\hat{S}_N^\circ \leqq \cdot) - \Phi(\cdot)\| \leqq C_1 L_N .$$

In the rest of this section, we assume that

$$(3.21) \quad \omega_3 = \sum_{j=1}^{N} |c_j^*|^3 \leqq 1 , \qquad \max\{\|\phi\|, \|\phi_1\|, \|\phi_1^{(2)}\|\} \leqq 1/6 .$$

Thus we have

$$(3.22) \quad \mathrm{E}\,(\hat{S}_N^{(j)})^2 \leqq ((c_j^*)^2 + N^{-1}\omega_2)/18 , \qquad \mathrm{E}\,|\hat{S}_N^{(j)}|^3 \leqq (2|c_j^*|^3 + N^{-1}\omega_3)/54 .$$

Since $1 = \sum_{j=1}^{N} \mathrm{E}\,(\hat{S}_N^{(j)})^2 \leqq 9^{-1}\omega_2$, Hölder's inequality yields

$$(3.23) \quad N^{-1/2} \leqq 27^{-1}\omega_3 .$$

The inequality (3.23) plays an important role in the following proofs.

Let $\mu_j(t)$ denote the characteristic function of $\hat{S}_N^{(j)}$, i.e.,

$$(3.24) \quad \mu_j(t) = \mathrm{E}\exp(it\hat{S}_N^{(j)}) .$$

Then it follows from Petrov ([12], p. 110) that

$$|\mu_j(t)|\leqq\exp\left\{-\frac{t^2}{2}\mathrm{E}(\hat{S}_N^{(j)})^2+\frac{2}{3}|t|^3\mathrm{E}|\hat{S}_N^{(j)}|^3\right\}$$

which implies that for $|t|\leqq\omega_3^{-1}$

$$(3.25)\qquad \prod_{j\notin J}|\mu_j(t)|\leqq\exp\left\{-\frac{t^2}{2}+\frac{t^2}{2}\sum_{j\in J}\mathrm{E}(\hat{S}_N^{(j)})^2+\frac{2}{3}|t|^3\sum_{j=1}^{N}\mathrm{E}|\hat{S}_N^{(j)}|^3\right\}$$
$$\leqq\exp\left(-\frac{4t^2}{9}+\frac{t^2}{18}\#J\max_{1\leq j\leq N}|c_j^*|\right)$$

where J is a subset of $\{1,2,\cdots,N\}$ and $\#J$ is its cardinality.

Now we shall estimate $\mathrm{E}\exp(it\hat{\sigma}_N^{-1}S_N^\circ)$ and $\mathrm{E}\exp(it\hat{\sigma}_N^{-1}(S_N^\circ+I_2^\circ))$. To this end we need the following two lemmas:

Recalling that the r.v.'s H_{jk}, $1\leqq j,\ k\leqq N$ are given by (3.3), we define

$$(3.26)\qquad \begin{aligned} I_{21}&=(2(N+1)^2)^{-1}\sum_{(j,k)\neq}\sum c_jH_{jk}^2\phi_1^{(2)}(\rho_{jj},X_j)\,,\\ I_{22}&=(2(N+1)^2)^{-1}\sum\sum_{(j,k,l)\neq}\sum c_jH_{jk}H_{jl}\phi_1^{(2)}(\rho_{jj},X_j)\,,\end{aligned}$$

where $(j,k)\neq$ and $(j,k,l)\neq$ mean that summations are taken for distinct indices j, k and l over the set $\{1,2,\cdots,N\}$. Then it follows from (3.1), (3.9) and (3.26) that

$$(3.27)\qquad I_2^\circ=I_{21}^\circ+I_{22}^\circ\,.$$

LEMMA 3.4. *Suppose the condition* (3.21) *is satisfied and let* I_{21} *and* I_{22} *be defined by* (3.26). *Then for all* $N\geqq3$ *we have*

$$(3.28)\qquad \mathrm{E}(\hat{\sigma}_N^{-1}I_{21}^\circ)^2\leqq3^{-12}\omega_3^4\,,$$

$$(3.29)\qquad \mathrm{E}(\hat{\sigma}_N^{-1}I_{22}^\circ)^{2r}\leqq\left(\frac{2}{9}r\omega_3\right)^{2r}\,,$$

where ω_3 *is given by* (3.15) *and* $r\geqq0$ *is an integer.*

PROOF. Set $\tilde{H}_{jk}=H_{jk}^2\phi_1^{(2)}(\rho_{jj},X_j)-\mathrm{E}\{H_{jk}^2\phi_1^{(2)}(\rho_{jj},X_j)\}$, $1\leqq j,\ k\leqq N$, and then direct expansion yields that

$$(3.30)\qquad \begin{aligned}4(N+1)^4\mathrm{E}(\hat{\sigma}_N^{-1}I_{21}^\circ)^2=&\sum_{(j,k)\neq}\sum\mathrm{E}\{(c_j^*)^2\tilde{H}_{jk}^2+c_j^*c_k^*\tilde{H}_{jk}\tilde{H}_{kj}\}\\ &+\sum\sum_{(j,k,l)\neq}\sum\mathrm{E}\{(c_j^*)^2\tilde{H}_{jk}\tilde{H}_{jl}+c_j^*c_k^*\tilde{H}_{jk}\tilde{H}_{kl}\\ &\quad+c_j^*c_l^*\tilde{H}_{jk}\tilde{H}_{lj}+c_j^*c_l^*\tilde{H}_{jk}\tilde{H}_{lk}\}\\ &+\sum\sum_{(j,k,l,m)\neq}\sum\sum\mathrm{E}\{c_j^*c_l^*\tilde{H}_{jk}\tilde{H}_{lm}\}\,.\end{aligned}$$

Since $(\mathrm{E}\tilde{H}_{jk}^2)^{1/2}\leqq\|\phi_1^{(2)}\|\leqq1/6$, $1\leqq j,\ k\leqq N$, and expectations in the last sum of (3.30) vanish, we have

$$\mathrm{E}\,(\hat{\sigma}_N^{-1}I_{21}^\circ)^2 \leqq (144N^4)^{-1}\{N\omega_2+\omega_1^2+N^2\omega_2+3N\omega_1^2\} \leqq 3^{-12}\omega_3^4$$

where the second inequality follows by (3.23) and facts

$$\omega_1 \leqq N^{2/3}\omega_3^{1/3}\,, \qquad \omega_2 \leqq N^{1/3}\omega_3^{2/3}\,. \tag{3.31}$$

To prove (3.29), we set $\hat{\sigma}_N^{-1}I_{22}^\circ=(2(N+1))^{-1}\sum_{j=1}^N c_j^*\hat{H}_j$, where

$$\hat{H}_j=(N+1)^{-1}\sum_{k\neq j}^N \sum_{l\neq j,k}^N H_{jk}H_{jl}\phi_1^{(2)}(\rho_{jj}, X_j)\,.$$

Then, for any fixed j, $1\leqq j\leqq N$, conditionally given X_j, summands of $\hat{H}_j$ satisfy the assumptions of Lemma A.2. Thus an application of this lemma yields that for $1\leqq j\leqq N$,

$$\mathrm{E}\,\hat{H}_j^{2r} \leqq \mathrm{E}\,[\mathrm{E}\,(\hat{H}_j^{2r}\,|\,X_j)] \leqq (4e\|\phi_1^{(2)}\|)^{2r}(2r)^{2r}$$

which, together with (3.23), (3.31) and Hölder's inequality, ensures (3.29) to complete the proof.

Finally we need the following lemma, the proof of which is given in the appendix.

LEMMA 3.5. *Suppose the condition* (3.21) *is satisfied and let* $\hat{S}_N$, S_N *and* I_{22} *be defined by* (2.3), (2.4) *and* (3.26) *respectively. Then there is an absolute constant* K *such that for* $|t|\leqq\omega_3^{-1}$, *any* $N\geqq 3$ *and a positive integer* r *where* $r\leqq N/3$,

$$\begin{aligned}&|\mathrm{E}\exp(it\hat{\sigma}_N^{-1}\hat{S}_N^\circ)\{\hat{\sigma}_N^{-1}(S_N^\circ-\hat{S}_N^\circ)\}^r|\\&\quad\leqq K^r r^{2r}\omega_3(1+|t|+t^2)^r\exp\left(-\frac{4t^2}{9}+\frac{t^2 r}{9}\max_{1\leqq j\leqq N}|c_j^*|\right),\end{aligned}\tag{3.32}$$

$$\begin{aligned}&|\mathrm{E}\exp(it\hat{\sigma}_N^{-1}\hat{S}_N^\circ)\{\hat{\sigma}_N^{-1}(S_N^\circ-\hat{S}_N^\circ+I_{22}^\circ)\}^r|\\&\quad\leqq K^r r^{3r}\omega_3(1+|t|+t^2)^r\exp\left(-\frac{4t^2}{9}+\frac{t^2 r}{6}\max_{1\leqq j\leqq N}|c_j^*|\right).\end{aligned}\tag{3.33}$$

Furthermore, if $\omega_3\leqq e^{-2}$ *and* $2r\leqq\log\omega_3^{-1}$, *then*

$$\begin{aligned}&|\mathrm{E}\exp(it\hat{\sigma}_N^{-1}\hat{S}_N^\circ)\{\hat{\sigma}_N^{-1}(S_N^\circ-\hat{S}_N^\circ)\}^r|\\&\quad\leqq K^r r^{2r}\omega_3(1+|t|+t^2)^r\exp(-t^2/3)\,,\end{aligned}\tag{3.34}$$

$$\begin{aligned}&|\mathrm{E}\exp(it\hat{\sigma}_N^{-1}\hat{S}_N^\circ)\{\hat{\sigma}_N^{-1}(S_N^\circ-\hat{S}_N^\circ+I_{22}^\circ)\}^r|\\&\quad\leqq K^r r^{3r}\omega_3(1+|t|+t^2)^r\exp(-t^2/4)\,.\end{aligned}\tag{3.35}$$

4. Proofs of main theorems

We now prove the main theorems stated in the Section 2.

PROOF OF THEOREM 2.1. By standard arguments, we have for any $\varepsilon>0$

$$\text{(4.1)}\qquad \begin{aligned} &\mathrm{P}\,(T_N^\circ \leqq \hat\sigma_N x) \leqq \mathrm{P}\,(\hat S_N^\circ \leqq \hat\sigma_N(x+\varepsilon)) + \mathrm{P}\,(|T_N^\circ - \hat S_N^\circ| \geqq \hat\sigma_N\varepsilon)\ , \\ &\mathrm{P}\,(T_N^\circ \leqq \hat\sigma_N x) \geqq \mathrm{P}\,(\hat S_N^\circ \leqq \hat\sigma_N(x-\varepsilon)) - \mathrm{P}\,(|T_N^\circ - \hat S_N^\circ| \geqq \hat\sigma_N\varepsilon)\ . \end{aligned}$$

Recalling the well known inequalities

$$\Phi(x+\varepsilon) \leqq \Phi(x)+\varepsilon\ , \qquad \Phi(x-\varepsilon) \geqq \Phi(x)-\varepsilon\ ,$$

we obtain, using (3.20), (4.1) and Chebychev's inequality, that for any $\varepsilon>0$ and positive integer r,

$$\text{(4.2)}\qquad \|\mathrm{P}\,(T_N^\circ \leqq \hat\sigma_N \cdot) - \Phi(\cdot)\| \leqq C_1 L_N + \varepsilon + (\hat\sigma_N\varepsilon)^{-2r}\,\mathrm{E}\,(T_N^\circ - \hat S_N^\circ)^{2r}$$

both for the approximate as well as exact scores.

To complete the proof it remains to estimate the last quantity in (4.2). For the statistic T_N with approximate scores, Lemma 3.2 ensures that for any positive integer r,

$$\text{(4.3)}\qquad \mathrm{E}\,(T_N^\circ - S_N^\circ)^{2r} \leqq (2\Delta_1)^{2r}\left(\sum_{j=1}^N c_j^2\right)^r (4e)^{2r}(2r)^{2r}N^{-\delta r}\ .$$

For exact scores, it follows from (3.11), (4.3) and Minkowski's inequality that for any positive integer r,

$$\text{(4.4)}\qquad \mathrm{E}\,(T_N^\circ - S_N^\circ)^{2r} \leqq (16\Delta_1 e)^{2r}\left(\sum_{j=1}^N c_j^2\right)^r (2r)^{2r}N^{-\delta r}$$

and this inequality holds for approximate scores as well. Now set

$$B_N^{(r)} = \left[32e(\Delta_1 + \|\phi_1\|)\left(\sum_{j=1}^N c_j^2\right)^{1/2} r\max\{N^{-\delta/2}, N^{-1/2}\}\right]^{2r}\ .$$

It then follows from (3.8) and (4.4) that for any positive integer r,

$$\text{(4.5)}\qquad \mathrm{E}\,(T_N^\circ - \hat S_N^\circ)^{2r} \leqq B_N^{(r)}\ .$$

Substitution of (4.5) into (4.2) yields

$$\text{(4.6)}\qquad \|\mathrm{P}\,(T_N^\circ \leqq \hat\sigma_N \cdot) - \Phi(\cdot)\| \leqq C_1 L_N + \varepsilon + (\hat\sigma_N\varepsilon)^{-2r} B_N^{(r)}\ .$$

Next we choose ε such that $\varepsilon = (\hat\sigma_N\varepsilon)^{-2r}B_N^{(r)}$, i.e.,

$$\text{(4.7)}\qquad \begin{aligned} \varepsilon &= \left[32e(\Delta_1 + \|\phi_1\|)\left(\sum_{j=1}^N c_j^2\right)^{1/2} \hat\sigma_N^{-1} r\max\{N^{-\delta/2}, N^{-1/2}\}\right]^{2r/(2r+1)} \\ &\leqq [M_N r\max\{N^{-\delta/2}, N^{-1/2}\}]^{2r/(2r+1)} \end{aligned}$$

where M_N is given by (2.7). In view of (3.18) we have

$$1=\mathrm{Var}\,(\hat{\sigma}_N^{-1}\hat{S}_N)\leqq 2(\|\phi\|^2+\|\phi_1\|^2)\sum_{j=1}^{N}(c_j^*)^2\leqq M_N^2 \tag{4.8}$$

and hence for any positive integer r,

$$1\leqq M_N^{2r/(2r+1)}\leqq M_N\,. \tag{4.9}$$

Taking $r=[\log N]$, where $[\cdot]$ denotes the integer part, we have

$$(\max\{N^{-1/2},\,N^{-\delta/2}\})^{2r/(2r+1)}\leqq e\max\{N^{-1/2},\,N^{-\delta/2}\}\,. \tag{4.10}$$

Finally (2.6) follows by combining (4.6) through (4.10), and (2.8) follows by (4.5). The proof of Theorem 2.1 is complete.

To prove Theorems 2.2 and 2.3, we shall invoke Esseen's smoothing lemma (see e.g. Feller [5], p. 538), which implies that for all $\omega>0$

$$\|\mathrm{P}\,(S\leqq\cdot)-\Phi(\cdot)\|\leqq\frac{1}{\pi}\int_{|t|\leqq\omega}|t|^{-1}|\eta(t)-e^{-t^2/2}|dt+O(\omega^{-1}) \tag{4.11}$$

where S is a r.v. such that $\mathrm{E}\,S=0$ and $\eta(t)=\mathrm{E}\exp(itS)$.

Note that both Theorems 2.2 and 2.3 are trivial if $\omega_3\geqq e^{-2}$. Thus we consider the case when

$$\omega_3<e^{-2}\,. \tag{4.12}$$

Furthermore, we may, without loss of generality, assume that

$$\max\{\|\phi\|,\,\|\phi_1\|,\,\|\phi_1^{(2)}\|\}\leqq 1/6\,. \tag{4.13}$$

PROOF OF THEOREM 2.2. It follows by (3.19) and (4.11) that, in order to prove (2.9), it suffices to show that for some ε_1, $0<\varepsilon_1\leqq 1$,

$$\int_{|t|\leqq\varepsilon_1\omega_3^{-1}}|t|^{-1}|\mathrm{E}\exp(it\hat{\sigma}_N^{-1}S_N^\circ)-\mathrm{E}\exp(it\hat{\sigma}_N^{-1}\hat{S}_N^\circ)|dt=O(\omega_3)\,. \tag{4.14}$$

We now estimate the integral (4.14). For any positive integer k, Taylor's expansion yields

$$\begin{aligned} &|t|^{-1}|\mathrm{E}\exp(it\hat{\sigma}_N^{-1}S_N^\circ)-\mathrm{E}\exp(it\hat{\sigma}_N^{-1}\hat{S}_N^\circ)| \\ &\quad\leqq\sum_{r=1}^{2k-1}\frac{|t|^{r-1}}{r!}|\mathrm{E}\,[\{\hat{\sigma}_N^{-1}(S_N^\circ-\hat{S}_N^\circ)\}^r\exp(it\hat{\sigma}_N^{-1}\hat{S}_N^\circ)]| \\ &\quad\quad+O\Big(\frac{|t|^{2k-1}}{(2k)!}\mathrm{E}\,(\hat{\sigma}_N^{-1}(S_N^\circ-\hat{S}_N^\circ))^{2k}\Big)\,. \end{aligned} \tag{4.15}$$

By taking $k=1$ and using (3.8), (3.31) and (3.34), we have for any $\varepsilon_1>0$

$$\int_{|t|\leqq\varepsilon_1\omega_3^{-1/2}}|t|^{-1}|\mathrm{E}\exp(it\hat{\sigma}_N^{-1}S_N^\circ)-\mathrm{E}\exp(it\hat{\sigma}_N^{-1}\hat{S}_N^\circ)|=O(\omega_3)\,. \tag{4.16}$$

We next take $2k=[\log \omega_3^{-1}]$. Then (3.34) insures that for any ε_1, $0<\varepsilon_1 \leqq 1$,

$$(4.17)\quad \int_{\varepsilon_1\omega_3^{-1/2}\leqq|t|\leqq\varepsilon_1\omega_3^{-1}} \sum_{r=1}^{2k-1} \frac{|t|^{r-1}}{r!} |\mathrm{E}\{\hat{\sigma}_N^{-1}(S_N^\circ-\hat{S}_N^\circ)\}^r \exp(it\hat{\sigma}_N^{-1}\hat{S}_N^\circ)|dt$$
$$\leqq \omega_3 \exp\left(-\frac{\varepsilon_1^2\omega_3^{-1}}{3}\right)(2k)^{4k} \sum_{r=1}^{2k-1} \frac{K^r}{r!} \int_{|t|\leqq\omega_3^{-1}} |t|^{r-1}(1+|t|+t^2)^r dt$$
$$\leqq \omega_3 \exp\left(-\frac{\varepsilon_1^2\omega_3^{-1}}{3}\right)(2k)^{4k}\omega_3^{-6k}e^{3K}=O(\omega_3)\ .$$

It follows from (3.8), (3.23) and (3.31) that

$$(4.18)\quad \int_{|t|\leqq\varepsilon_1\omega_3^{-1}} \frac{|t|^{2k-1}}{2k!} \mathrm{E}(\hat{\sigma}_N^{-1}(S_N^\circ-\hat{S}_N^\circ))^{2k}dt$$
$$\leqq \frac{(8e\|\phi_1\|)^{2k}(2k)^{2k}\omega_3^{2k}}{(2k)!}(\varepsilon_1\omega_3^{-1})^{2k}$$
$$\leqq (2e\varepsilon_1)^{2k}(2k)^{2k}[(2k)!]^{-1} \leqq (2e^2\varepsilon_1)^{2k}$$

where the last inequality follows by Stirling's formula (see, e.g. Feller [4]). By taking $\varepsilon_1=1/(2e^3)$ and combining (4.15), (4.17) and (4.18), we obtain that

$$(4.19)\quad \int_{\varepsilon_1\omega_3^{-1/2}\leqq|t|\leqq\varepsilon_1\omega_3^{-1}} |t|^{-1}|\mathrm{E}\exp(it\hat{\sigma}_N^{-1}S_N^\circ)-\mathrm{E}\exp(it\hat{\sigma}_N^{-1}\hat{S}_N^\circ)|dt=O(\omega_3)$$

which, together with (4.16), ensures (4.14). Clearly (2.10) follows from (3.8), (3.23) and (3.31). The proof follows.

PROOF OF THEOREM 2.3. By standard arguments, to prove (2.11), it suffices to show that

$$(4.20)\quad \mathrm{P}(|T_N^\circ-S_N^\circ-I_2^\circ+I_{21}^\circ|\geqq\hat{\sigma}_N\omega_3)=O(\max\{\omega_3,\omega_3^{2\delta}\})$$

and that

$$(4.21)\quad \|\mathrm{P}(S_N^\circ+I_{22}^\circ\leqq\hat{\sigma}_N\cdot)-\Phi(\cdot)\|=O(\max\{\omega_3,\omega_3^{2\delta}\})\ .$$

Applying Lemma 3.2 with $r=1$ and $l=2$ and then using (3.23) and (3.31), we obtain that

$$\mathrm{E}(\hat{\sigma}_N^{-1}(T_N^\circ-S_N^\circ-I_2^\circ))^2=O(\omega_3^{2+2\delta})$$

which, together with (3.28) and Chebychev's inequality, ensures (4.20). Similarly, like the proof of Theorem 2.2, (4.21) follows from (3.8), (3.29) and (3.35). Since Assumption (A_1) is satisfied with $\delta=1$ under Assumption (A_2), (2.8), (3.23) and (3.31) ensure (2.12). The proof is complete.

5. Appendix

PROOF OF LEMMA 3.5. Since $\log \omega_3^{-1} \leqq 2(\max_{1\leqq j\leqq N} |c_j^*|)^{-1}$ when $\omega_3 \leqq 1$, (3.34) and (3.35) are immediate consequences of (3.32) and (3.33) respectively. Also the proof of (3.32) is very similar to that of (3.33). Thus we omit it and prove only (3.33).

Recalling that H_{jk} and V_{jk}, $1\leqq j,\ k\leqq N$, are given by (3.3) and (3.7) respectively, we define r.v.'s for $1\leqq j, k, l\leqq N$

$$\tilde{V}_{jkl}=c_j^*\left\{\frac{N+1}{N-2}V_{jk}+(1/2)H_{jk}H_{jl}\phi_1^{(2)}(\rho_{jj},X_j)\right\}. \tag{A.1}$$

Then it follows from (3.6) and (3.26) that

$$\hat{\sigma}_N^{-1}(S_N^\circ-\hat{S}_N^\circ+I_{22}^\circ)=(N+1)^{-2}\sum\sum_{(j,k,l)\neq}\sum \tilde{V}_{jkl}. \tag{A.2}$$

Note that for any distinct indices j, k and l

$$\mathrm{E}\,(\tilde{V}_{jkl}\,|\,X_j)=\mathrm{E}\,(\tilde{V}_{jkl}\,|\,X_k)=\mathrm{E}\,(\tilde{V}_{jkl}\,|\,X_l)=0. \tag{A.3}$$

In view of (3.16) and (3.17), $\hat{\sigma}_N^{-1}\hat{S}_N^\circ$ is a sum of independent r.v.'s. This fact and (A.2) ensure that

$$\begin{aligned}&\mathrm{E}\,\{\exp(it\hat{\sigma}_N^{-1}\hat{S}_N^\circ)\hat{\sigma}_N^{-1}(S_N^\circ-\hat{S}_N^\circ+I_{22}^\circ)\}\\&=(N+1)^{-2}\sum\sum_{(j,k,l)\neq}\sum\Big(\prod_{\nu\neq j,k,l}\mu_\nu(t)\Big)\,\mathrm{E}\,\{\tilde{V}_{jkl}\exp(it(\hat{S}_N^{(j)}+\hat{S}_N^{(k)}+\hat{S}_N^{(l)}))\}\end{aligned} \tag{A.4}$$

where $\mu_\nu(t)$ is the characteristic function of the r.v. $\hat{S}_N^{(\nu)}$, defined by (3.24). It follows by Lemma XV.4.1 of Feller [5] and (A.3) that

$$\begin{aligned}&\mathrm{E}\,\{\tilde{V}_{jkl}\exp(it(\hat{S}_N^{(j)}+\hat{S}_N^{(k)}+\hat{S}_N^{(l)}))\}\\&=O(t^2|c_j^*|\{(c_j^*)^2+(c_k^*)^2+(c_l^*)^2+N^{-1}\omega_2\}).\end{aligned} \tag{A.5}$$

Hence for $r=1$ (3.33) follows from (3.25), (A.4) and (A.5).

We next consider the expansion of $\{\hat{\sigma}_N^{-1}(S_N^\circ-\hat{S}_N^\circ+I_{22}^\circ)\}^r$. Expanding it directly, by the method leading to (3.30), we find that

$$\begin{aligned}\{\hat{\sigma}_N^{-1}(S_N^\circ-\hat{S}_N^\circ+I_{22}^\circ)\}^r&=(N+1)^{-2r}\Big(\sum\sum_{(j,k,l)\neq}\sum\tilde{V}_{jkl}\Big)^r\\&=(N+1)^{-2r}\sum_{b=3}^{3r}\sum_{\nu}q_{b\nu}^{(r)}\end{aligned} \tag{A.6}$$

where $q_{b\nu}^{(r)}$ is of the form

$$q_{b\nu}^{(r)}=\sum\sum_{(j_1,j_2,\cdots,j_b)\neq}\cdots\sum\tilde{V}_{\partial\partial\partial_1}\tilde{V}_{\partial\partial\partial_2}\cdots\tilde{V}_{\partial\partial\partial_r} \tag{A.7}$$

and each "∂" is one of indices $j_1, j_2, \cdots, j_b$. Thus we have

$$|\mathrm{E}\exp(it\hat{\sigma}_N^{-1}\hat{S}_N^\circ)\{\hat{\sigma}_N^{-1}(S_N^\circ-\hat{S}_N^\circ+I_{22}^\circ)\}^r| \tag{A.8}$$

$$
\begin{aligned}
&= \left| (N+1)^{-2r} \sum_{b=3}^{3r} \Big(\prod_{j \neq j_1, \cdots, j_b} \mu_j(t) \Big) \sum_{\nu} \mathrm{E} \left\{ q_{b\nu}^{(r)} \prod_{k=1}^{b} \exp(it\hat{S}_N^{(j_k)}) \right\} \right| \\
&\leq \exp\left(-\frac{4t^2}{9} + \frac{t^2}{6} r \max_{1 \leq j \leq N} |c_j^*| \right) \\
&\quad \times \sum_{b=3}^{3r} \sum_{\nu} N^{-2r} \left| \mathrm{E} \left\{ q_{b\nu}^{(r)} \prod_{k=1}^{b} \exp(it\hat{S}_N^{(j_k)}) \right\} \right|
\end{aligned}
$$

in view of (3.25).

Let $Q_b^{(r)}$, $3 \leq b \leq 3r$, be the cardinality of the collection of all different terms $q_{b\cdot}^{(r)}$ and put $\Omega_r = \sum_{b=3}^{3r} Q_b^{(r)}$, which is the total number of terms $q_{b\cdot}^{(r)}$ in the expansion (A.6). Then clearly $\Omega_1 = 1$. To estimate Ω_r, we first note the recursive relation

$$\Omega_r \leq \Omega_{r-1} \{ (\alpha^2 - \alpha + 1) + \alpha^2 + \alpha + 1 \} \leq 2\alpha^3 \Omega_{r-1}$$

where $\alpha = 3(r-1)$. Thus there is a constant K_1 (independent of N and r) such that for any positive integer r,

$$\Omega_r \leq (54)^{r-1} ((r-1)!)^3 \leq K_1^r r^{3r} \tag{A.9}$$

where the second inequality follows by Stirling's formula (see e.g. Feller [4]).

To complete the proof, it remains to show that for any positive integer r and b, $3 \leq b \leq 3r$,

$$N^{-2r} \left| \mathrm{E}\, q_{b\cdot}^{(r)} \prod_{k=1}^{b} \exp(it\hat{S}_N^{(j_k)}) \right| \leq K_2^r \omega_3 (1 + |t| + t^2)^r \tag{A.10}$$

where K_2 is an absolute constant. We shall prove this by induction on r. Note that (A.10) holds for $r=1$ in view of (A.5). We now suppose that (A.10) holds for $r \leq m-1$ and let $r = m \geq 2$.

If $b \leq 2m = 2r$, then (A.10) is trivial. Thus we consider only terms $q_{b\cdot}^{(m)}$, $b \geq 2m+1$. Pick any s, $1 \leq s \leq m$, and let $b = 2m+s$. Let γ denote the number of indices j's (in (A.7)) which appear at least twice as a subscript of one of $\tilde{V}$-terms and then we have that $3m \geq 2\gamma + 2m + s - \gamma$, i.e.

$$0 \leq \gamma \leq m - s\,. \tag{A.11}$$

Hence at least $m+2s$ indices among $j_1, j_2, \cdots, j_{2m+s}$ appear exactly once. If one of $\tilde{V}$-terms has three subscripts appeared only once, then the expectation in (A.10) is split into expectations of two groups to ensure (A.10) immediately by the induction hypothesis. This is the case when $s > m/2$ or $\gamma < s$.

Next we consider the remaining case when

(A.12) $$1\leqq s\leqq m/2\ ,\qquad s\leqq\gamma\leqq m-s$$

and none of $\tilde{V}$-terms in (A.7) satisfy that every one of its subscripts appears only once. Since at least $m+2s$ indices among $j_1, j_2, \cdots, j_{2m+s}$ appear exactly once, we must have at least $2s$ $\tilde{V}$-terms, of which exactly two subscripts appear only once. Thus, by rearranging the order of summations, we can write the typical form of $q_{b}^{(\tau)}\prod_{k=1}^{b}\exp(it\hat{S}_N^{(j_k)})$ as follows:

$$\begin{aligned}
&N^{-2m}(\sum_{k_1}\sum_{l_1}\cdots\sum_{k_\alpha}\sum_{l_\alpha})(\sum_{m_1}\cdots\sum_{m_\beta})(\sum_{j_1}\cdots\sum_{j_\gamma})\\
&\times\tilde{V}_{\partial k_1 l_1}\cdots\tilde{V}_{\partial k_\delta l_\delta}\tilde{V}_{k_{\delta+1}\partial l_{\delta+1}}\cdots\tilde{V}_{k_\alpha l_\alpha\partial} &\text{(A)}\\
\text{(A.14)}\quad&\times\tilde{V}_{\partial\partial m_1}\cdots\tilde{V}_{\partial\partial m_\varepsilon}\tilde{V}_{\partial m_{\varepsilon+1}\partial}\cdots\tilde{V}_{m_\beta\partial\partial} &\text{(B)}\\
&\times\tilde{V}_{\partial\partial\partial_1}\cdots\tilde{V}_{\partial\partial\partial_\eta} &\text{(C)}\\
&\times\prod_{\nu=1}^{\alpha}\exp(it(\hat{S}_N^{(k_\nu)}+\hat{S}_N^{(l_\nu)}))\prod_{\nu=1}^{\beta}\exp(it\hat{S}_N^{(m_\nu)})\prod_{\nu=1}^{\gamma}\exp(it\hat{S}_N^{(j_\nu)})
\end{aligned}$$

where summations are taken over distinct indices; indices k's, l's and m's appear only once; j's appear at least twice among places marked by "∂"; α, β, γ, δ, ε and η are nonnegative integers such that

$$\begin{aligned}
&1\leqq s\leqq\alpha/2\leqq m/2\ , &&s\leqq\gamma\leqq m-s\ ,\\
\text{(A.15)}\quad&0\leqq\delta\leqq\alpha\leqq m\ , &&0\leqq\varepsilon\leqq\beta\leqq m\ ,\\
&2\alpha+\beta+\gamma=2m+s\ , &&\alpha+\beta+\gamma=m\ .
\end{aligned}$$

We next consider the conditional expectation of (A.14) given r.v.'s $X_{j_1}, X_{j_2}, \cdots, X_{j_\gamma}$. To this end, applying Lemma XV.4.1 of Feller [5] and then taking conditional expectations, we obtain that for distinct indices j, k and l

$$\begin{aligned}
&\sum_{(k,l)\neq}\sum \mathrm{E}(\tilde{V}_{jkl}\tilde{e}_{kl}\,|\,X_j)\leqq\sum_{(k,l)\neq}\sum|t|\{|c_k^*|+|c_l^*|+N^{-1}\omega_1\}\leqq 3|t|N^{5/3}\omega_3^{1/3}\ ,\\
\text{(A.16)}\quad&\sum_{(k,l)\neq}\sum \mathrm{E}(\tilde{V}_{kjl}\tilde{e}_{kl}\,|\,X_j)\leqq\sum_{(k,l)\neq}\sum|t||c_k^*|\{|c_k^*|+|c_l^*|+N^{-1}\omega_1\}\leqq 3|t|N^{4/3}\omega_3^{2/3}\ ,\\
&\sum_{(k,l)\neq}\sum \mathrm{E}(V_{klj}\tilde{e}_{kl}\,|\,X_j)\leqq 3|t|N^{4/3}\omega_3^{2/3}
\end{aligned}$$

where $\tilde{e}_{kl}=\exp(it(\hat{S}_N^{(k)}+\hat{S}_N^{(l)}))$.

We finally split the proof into two cases when $\alpha=m$ or $\alpha<m$. We first prove (A.10) when $\alpha=m$ and then when $\alpha<m$.

When $\alpha=m$, we have only group (A) in (A.14). If $s\geqq 2$, then the expectation of (A.14) is split into two groups and the induction hypothesis ensures (A.10) immediately. If $s=1$ and $\delta=0$, then it follows by substitution of (A.16) into (A.14) that the expectation of (A.14) is bounded by

$$N^{-2m}(3|t|N^{4/3}\omega_3^{2/3})^m N \leqq (3|t|)^m \omega_3 .$$

If $s=1$ and $\delta>0$ then the expectation is bounded by

$$N^{-2m}(3|t|N^{5/3}\omega_3^{1/3})^m \omega_1 \leqq (3|t|)^m \omega_3 .$$

This completes the proof for the case when $\alpha=m$.

We now suppose that $\alpha<m$. If some of indices j's appear only in group (A) of (A.14), then we can again split the expectation into two groups. Thus it remains to show (A.10) when all indices j's in group (A) appear again in group (B) or (C). Note that $\beta+\gamma$ is the total number of indices m's and j's which appear in group (B) or (C) and that the number of indices m's and j's, which appear as one of the second or the third subscripts of $\tilde{V}$-terms in group (B) or (C), is at most $2(\eta+\beta)$. Moreover,

$$\beta+\gamma-2(\eta+\beta)=\gamma-2(m-\alpha-\beta)-\beta=s$$

in view of (A.15). Thus at least s indices among $m_1, m_2, \cdots, m_\beta, j_1, j_2, \cdots, j_\gamma$ must appear as the first subscript of one of $\tilde{V}$-terms in group (B) or (C). Utilizing this fact and substituting (A.16) into (A.14), we find that the expectation of (A.14) is bounded by

$$\begin{aligned} &N^{-2m}(3|t|N^{5/3}\omega_3^{1/3})^\delta(3|t|N^{4/3}\omega_3^{2/3})^{\alpha-\delta}\omega_1^s N^{2m+s-(2\alpha+s)} \\ &\leqq (3|t|)^\alpha N^{-2\alpha+\delta/3+4\alpha/3+2s/3}\omega_3^{2\alpha/3-\delta/3+s/3} \\ &\leqq (3(1+|t|))^m N^{-(\alpha-2s)/3}\omega_3^{(\alpha+s)/3} \leqq (3(1+|t|))^m \omega_3 \end{aligned}$$

because $2s\leqq\alpha$ and $\delta\leqq\alpha$. This completes the proof of the case when $\alpha<m$. The proof follows.

Let $\{Y_j\}_{j=1}^\infty$ be a sequence of independent r.v.'s and $\{d_j\}_{j=1}^\infty$ a sequence of real numbers. Then we have the following lemmas.

LEMMA A.1. *Let Z_j, $j\geqq 1$, be r.v.'s of the form $Z_j=g_j(Y_1, Y_2, \cdots, Y_j)$ such that for $j\geqq 2$*

$$\mathrm{E}(Z_j|Y_1, Y_2, \cdots, Y_{j-1})=0 . \tag{A.17}$$

If the sequence $\{d_j\}$ is non-increasing in absolute values, then for any positive integers r and l

$$\mathrm{E}\left(\sum_{j=1}^{l} d_j Z_j\right)^{2r} \leqq (4e)^r\left(\sum_{j=1}^{l} d_j^2\right)^r r^r m_l^{(r)} \tag{A.18}$$

where $m_l^{(r)}=\max_{1\leqq j\leqq l} \mathrm{E}\, Z_j^{2r}$.

LEMMA A.2. *Let $\tilde{V}_{jk}$ be r.v.'s of the form $\tilde{V}_{jk}=g_{jk}(Y_j, Y_k)$, $1\leqq j$, $k<\infty$, such that for any j and k, $j\neq k$,*

$$\mathrm{E}\,(\tilde{V}_{jk}|Y_j)=\mathrm{E}\,(\tilde{V}_{jk}|Y_k)=0\ . \tag{A.19}$$

Then for any positive integers l and r

$$\mathrm{E}\,\hat{V}_l^{2r}\leqq(4e)^{2r}\Big(\sum_{j=1}^{l} d_j^2\Big)^r(2r)^{2r}l^{-r}\tilde{m}_l^{(r)} \tag{A.20}$$

where

$$\hat{V}_l=\frac{1}{l}\sum_{j=1}^{l}\sum_{k\neq j}^{l} d_j\tilde{V}_{jk} \qquad \textit{and} \qquad \tilde{m}_l^{(r)}=\max_{\substack{1\leq j,k\leq l\\ j\neq k}}\mathrm{E}\,\tilde{V}_{jk}^{2r}\ .$$

PROOF OF LEMMA A.1. It follows by Hölder's inequality that

$$\Big(\sum_{j=1}^{l} d_jZ_j\Big)^{2r}\leqq\Big(\sum_{j=1}^{l} d_j^2\Big)^r\Big(\sum_{j=1}^{l} Z_j^2\Big)^r\leqq\Big(\sum_{j=1}^{l} d_j^2\Big)^r l^{r-1}\sum_{j=1}^{l} Z_j^{2r}$$

which insures (A.18) when $r\geqq l$.

For $r\leqq l$, we prove (A.18) by induction on l with r fixed. Define $V_l=\sum_{j=1}^{l} d_jZ_j$ and suppose that (A.18) is true for $l=n\geqq r$. Then it follows by (A.17) and Hölder's inequality that

$$\begin{aligned}\mathrm{E}\,V_{n+1}^{2r}&\leqq\mathrm{E}\,V_n^{2r}+\sum_{v=2}^{2r}\binom{2r}{v}(\mathrm{E}\,V_n^{2r})^{(2r-v)/2r}\,\mathrm{E}\,(d_{n+1}^{2r}Z_{n+1}^{2r})^{v/2r}\\ &\leqq(4e)^r\Big(\sum_{j=1}^{n} d_j^2\Big)^r r^r\Bigg[1+\frac{4r^2d_{n+1}^2}{4er\sum_{j=1}^{n} d_j^2}\Bigg(1+\frac{|d_{n+1}|}{\Big(4er\sum_{j=1}^{n} d_j^2\Big)^{1/2}}\Bigg)^{2r-2}\Bigg]m_{n+1}^{(r)}\ .\end{aligned} \tag{A.21}$$

Since $r\leqq n$ and $|d_j|$'s are decreasing, we have

$$1+|d_{n+1}|\Big(4er\sum_{j=1}^{n} d_j^2\Big)^{-1/2}\leqq1+(2r)^{-1}\leqq e^{1/(2r)}$$

which, together with (A.21), ensures (A.18) for $l=n+1$ to complete the proof.

PROOF OF LEMMA A.2. Since the assumption and the conclusion of this lemma are invariant under simultaneous permutation of d_j's and Y_j's, we may, without loss of generality, assume that $|d_1|\geqq|d_2|\geqq\cdots\geqq|d_l|$.

Define $Z_1=\tilde{Z}_1=0$ and

$$Z_j=\sum_{k=1}^{j-1} V_{jk}\ ,\qquad \tilde{Z}_j=\sum_{k=1}^{j-1} d_kV_{kj}\ ,\qquad 2\leqq j\leqq l$$

so that $\hat{V}_l=\frac{1}{l}\Big(\sum_{j=1}^{l} d_jZ_j+\sum_{j=1}^{l}\tilde{Z}_j\Big)$. It follows by (A.19) that collections

$\{Z_j\}_{j=1}^{l}$ and $\{\tilde{Z}_j\}_{j=1}^{l}$ satisfy the assumptions of Lemma A.1 and hence

$$\text{(A.22)}\qquad \mathrm{E}\,\hat{V}_l^{2r} \leqq l^{-2r} 2^{2r-1}\left\{\mathrm{E}\left(\sum_{j=1}^{l} d_j Z_j\right)^{2r} + \mathrm{E}\left(\sum_{j=1}^{l} \tilde{Z}_j\right)^{2r}\right\}$$

$$\leqq l^{-2r} 2^{2r-1}(4e)^r r^r \left\{\left(\sum_{j=1}^{l} d_j^2\right)^r \max_{1\leq j\leq l} \mathrm{E}\, Z_j^{2r} + l^r \max_{1\leq j\leq l} \mathrm{E}\, \tilde{Z}_j^{2r}\right\} .$$

Because of (A.19) and the definition of V_{jk}'s, it follows that for each j, conditionally given Y_j, Z_j and $\tilde{Z}_j$ are sums of independent r.v.'s with zero means. Thus we may again apply Lemma A.1 to obtain that for $1\leqq j\leqq l$

$$\text{(A.23)}\qquad \begin{aligned} \mathrm{E}\, Z_j^{2r} &\leqq (4e)^r (j-1)^r r^r \max_{\substack{1\leq k\leq l\\ k\neq j}} \mathrm{E}\, V_{jk}^{2r} \\ \mathrm{E}\, \tilde{Z}_j^{2r} &\leqq (4e)^r \left(\sum_{k=1}^{j-1} d_k^2\right)^r r^r \max_{\substack{1\leq k\leq l\\ k\neq j}} \mathrm{E}\, V_{kj}^{2r} . \end{aligned}$$

The proof follows by combining (A.22) and (A.23).

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