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LIMIT THEOREMS FOR THE MEDIAN DEVIATION

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Summary

In this paper, we obtain a strong law and central limit theorem for the median deviation under only very mild smoothness conditions on the underlying distribution. Under an additional condition implied by symmetry, we derive a weak Bahadur representation for the median deviation and establish the asymptotic equivalence of the median deviation and the semi-interquartile range.

1. Introduction

Let $X_1, \cdots, X_n$ be a sample of independent random variables distributed as X, where X has distribution function F. Let M_n denote the sample median of $X_1, \cdots, X_n$. Put $W_i \equiv W_i(M_n) = |X_i - M_n|$, $1 \leq i \leq n$, and let $S_n(M_n)$ be the sample median of $W_1, \cdots, W_n$. Following Hampel [3], we call $S_n(M_n)$ the median deviation.

Hampel [3] observed that the median deviation is the natural nonparametric estimator of the "probable error" of a single observation and is the scale counterpart of the median, being the "most robust" estimate of scale. The median deviation was shown by Hampel [3] to be an M-estimate of scale. This formulation facilitates calculation of the influence curve (see for example, Huber [5], p. 137).

Our first objective in this paper is to obtain limit theorems for the median deviation under only very mild smoothness conditions on F, and without any symmetry assumptions. We achieve this end by adopting a direct sample quantile approach.

An alternative quick, robust scale estimator is given by $Q_n = (\hat{\xi}_{3/4} - \hat{\xi}_{1/4})/2$, where $\hat{\xi}_p$ is the sample pth quantile, $0 < p < 1$. The estimator Q_n is an L-estimator and is called the semi-interquartile range. The influence curve is easy to calculate (Huber [5], p. 111) and the asymptotic theory follows immediately from the asymptotic results for quantiles (see for example Serfling [7], p. 86). Hampel [3] points out that

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the median deviation (with a breakdown point of 1/2) is superior to the semi-interquartile range (with a breakdown point of 1/4). He (and others) assert that if F is symmetric, the estimators are asymptotically equivalent. Our second objective is to establish asymptotic equivalence under only mild smoothness conditions and a weak condition implied by symmetry. We establish this result by deriving a weak Bahadur representation for the median deviation and showing that this representation also holds for the semi-interquartile range.

Consistency and asymptotic normality could also be proved using results of Huber [4] or Rivest [6]. However, arguing via [6] requires superfluous smoothness or symmetry assumptions. Huber's conditions are somewhat cumbersome and inexplicit, and it requires more effort to check them than to pursue our argument. Furthermore, our attempts at reducing Huber's constraints to a practical form have all resulted in unnecessarily severe conditions. For example, condition (N-3) (ii) ([4], p. 227), when applied to the first component

$$\psi_1(x, \theta_1, \theta_2) = \mathrm{sgn}\,\{(x-\theta_1)/\theta_2\}$$

of the bivariate ψ function, appears to require a restriction on F in a neighbourhood of m_0, such as that F' be bounded in a neighbourhood. This is considerably more restrictive than condition (i) in our Theorem 2.

Throughout this paper, the notation $\xrightarrow{D}$ and $\xrightarrow{P}$ will denote convergence in distribution and convergence in probability respectively.

2. Asymptotic behaviour of the median deviation

Suppose there is a unique point $m_0 \in R$ such that $F(m_0)=1/2$. Define $W=|X-m_0|$ so

$$G(x, m_0) \equiv \mathrm{P}\,(W \leqq x) = F(m_0+x) - F(m_0-x-)\,, \qquad x>0\,.$$

Suppose there is a unique point $s_0>0$ such that

$$G(s_0, m_0) = 1/2\,.$$

Defining $W_1, \cdots, W_n$ as in Section 1, let $G_n(x, M_n)$ and $F_n(x)$ denote the empirical distribution functions of the W_i's and the X_i's respectively, so that

$$G_n(x, M_n) = F_n(M_n+x) - F_n(M_n-x-)\,, \qquad x>0\,.$$

We obtain the following strong law for the median deviation.

THEOREM 1. *Suppose*

i) m_0 *and* s_0 *are unique.*

ii) F *is continuous in neighbourhoods of* $m_0 \pm s_0$.

Then $S_n(M_n)\xrightarrow{\text{a.s.}} s_0$.

PROOF. Let $\varepsilon>0$. By i), we have

$$G(s_0-\varepsilon, m_0)<1/2<G(s_0+\varepsilon, m_0) .$$

For fixed $s=s_0\pm\varepsilon$,

$$|G_n(s, M_n)-G(s, m_0)|\leqq|F_n(M_n+s)-F(m_0+s)| \\ +|F_n(M_n-s-)-F(m_0-s-)| ,$$

and for $t=\pm s$,

$$|F_n(M_n+t)-F(m_0+t)|\leqq \sup_{-\infty<y<\infty}|F_n(y)-F(y)| \\ +|F(M_n+t)-F(m_0+t)|\xrightarrow{\text{a.s.}}0 , \qquad \text{as } n\rightarrow\infty ,$$

by the Glivenko-Cantelli Theorem and the strong law for quantiles (Serfling [7], p. 75). Hence, as $n\rightarrow\infty$,

$$|G_n(s, M_n)-G(s, m_0)|\xrightarrow{\text{a.s.}}0 ,$$

and the result follows from a standard argument.

Remark 1. If m_0 is known, we can replace M_n by m_0 everywhere and omit ii). The condition ii) is thus due to the need to estimate $m_0\pm s_0$.

Remark 2. If the population quantiles $\xi_{3/4}$ and $\xi_{1/4}$ are unique, then $Q_n\xrightarrow{\text{a.s.}}(\xi_{3/4}-\xi_{1/4})/2$, as $n\rightarrow\infty$, by the strong law for quantiles.

The central limit theorem (Theorem 2) is more difficult to derive. Our proof depends on a conditioning argument. The general principle is as follows: If A_n-B_n, conditional on B_n, is asymptotically normal $N(0, \sigma_1^2)$, and if B_n is asymptotically normal $N(0, \sigma_2^2)$, then A_n is asymptotically normal $N(0, \sigma_1^2+\sigma_2^2)$. We suggest that this technique could be used to derive large sample normality for other adaptive estimators, provided (as in the case here) the adaptive component is a reasonably simple function of sample values.

For convenience, take $n=2k+1$. Then exactly k observations, $X_1, \cdots, X_k$, say, are less than M_n and exactly k observations, $X_{k+2},\cdots, X_{2k+1}$, say, are greater than M_n. (We assume that within the sets $\{1, \cdots, k\}$ and $\{k+2,\cdots, 2k+1\}$, the indices are randomly ordered.) The sample of W_i's consists of zero and two k-samples of positive random variables given by

$$Y_i\equiv Y_i(M_n)=M_n-X_i , \qquad 1\leqq i\leqq k ,$$

and

$$Z_i \equiv Z_i(M_n) = X_{i+k+1} - M_n\,, \qquad 1 \leq i \leq k\,.$$

Conditional on the event $\{M_n = m_n\}$, the Y_i's and Z_i's are all independent with distribution functions

$$F_Y(x\,|\,m_n) = \frac{F(m_n) - F(m_n - x -)}{F(m_n -)}\,, \qquad x > 0\,,$$

and

$$F_Z(x\,|\,m_n) = \frac{F(m_n + x) - F(m_n -)}{1 - F(m_n -)}\,, \qquad x > 0\,,$$

respectively. Of course, if $n = 2k$, the sample of W_i's consists simply of two k-samples of Y_i's and Z_i's.

We introduce some notation. For (x, m) in a neighbourhood of (s_0, m_0), put

$$G(x, m) = \frac{F(m) - F(m - x -)}{2F(m)} + \frac{F(m + x) - F(m -)}{2\{1 - F(m -)\}} \tag{2.1}$$

and

$$\Gamma^2(x, m) = \frac{\{F(m) - F(m - x -)\}\,F(m - x -)}{2F^2(m)} + \frac{\{F(m + x) - F(m -)\}\,\{1 - F(m + x)\}}{2\{1 - F(m -)\}^2}\,.$$

Assume that $F'(m_0)$ exists and that $F'(m_0 \pm s_0 + x)$ exists for x in a neighbourhood of the origin. Then

$$g(x, m_0) \equiv \partial G(x, m_0)/\partial x = F'(m_0 + x) + F'(m_0 - x)$$

and

$$\begin{aligned} \gamma(x, m_0) &\equiv [\partial G(x, m)/\partial m\,|\,m = m_0] \\ &= F'(m_0 + x) - F'(m_0 - x) \\ &\quad - 2F'(m_0)\{1 - F(m_0 + x) - F(m_0 - x)\} \end{aligned} \tag{2.2}$$

are well defined for x in a neighbourhood of s_0.

THEOREM 2. *Suppose*

i) *$F'(m_0)$ exists and is positive.*

ii) *$F'(m_0 \pm s_0 + x)$ exists for x in a neighbourhood of the origin and is continuous at $x = 0$.*

iii) *$g(x, m_0) = F'(m_0 + x) + F'(m_0 - x) >$ for x in a neighbourhood of s_0.*

Then

$$n^{1/2}(S_n(M_n) - s_0) \xrightarrow{\text{D}} N(0, \sigma^2)\,,$$

where $\sigma^2=\sigma_1^2+\sigma_2^2$, $\sigma_1=\Gamma(s_0, m_0)/g(s_0, m_0)$ *and* $\sigma_2=\gamma(s_0, m_0)/2F'(m_0)g(s_0, m_0)$.

PROOF. For m in a neighbourhood of m_0, define $s(m)$ by

$$G(s(m), m)=1/2 . \tag{2.3}$$

Condition iii) ensures that $s(m)$ is unique. Also, $s(m_0)=s_0$. Put $R_n(M_n)=n^{1/2}(S_n(M_n)-s(M_n))$ and $r_n(M_n)=n^{1/2}(s(M_n)-s_0)$. Let $x \in R$ be fixed and put

$$u_n(M_n)=n^{1/2}\{1/2-G(s_0+n^{-1/2}x, M_n)\}/\Gamma(s_0, m_0) .$$

Then

$$\begin{aligned} &|\mathrm{P}\{n^{1/2}(S_n(M_n)-s_0)\leqq x\}-\Phi(x/\sigma)| \\ &=|\mathrm{P}\{R_n(M_n)+r_n(M_n)\leqq x\}-\Phi(x/\sigma)| \\ &=|\mathrm{E}[\mathrm{P}\{R_n(M_n)+r_n(M_n)\leqq x \mid M_n\}]-\Phi(x/\sigma)| \\ &=|\mathrm{E}[1-\mathrm{P}\{R_n(M_n)>x-r_n(M_n) \mid M_n\} \\ &\qquad +\Phi(u_n(M_n))-\Phi(u_n(M_n))-\Phi(x/\sigma)]| \\ &\leqq \mathrm{E}|\mathrm{P}\{R_n(M_n)>x-r_n(M_n) \mid M_n\}-\Phi(u_n(M_n))| \\ &\quad +|\mathrm{E}\{\Phi(-u_n(M_n))-\Phi(x/\sigma)\}| . \end{aligned} \tag{2.4}$$

The result will follow if we can show that both terms on the right hand side of (2.4) converge to zero.

We first consider the second term in (2.4). Under condition ii),

$$\begin{aligned} F(M_n+s_0+n^{-1/2}x)=F(M_n+s(M_n))+n^{-1/2}\{x-r_n(M_n)\} \\ \times\{F'(m_0+s_0)+o_p(1)\} \end{aligned}$$

and

$$\begin{aligned} F(M_n-s_0-n^{-1/2}x)=F(M_n-s(M_n))-n^{-1/2}\{x-r_n(M_n)\} \\ \times\{F'(m_0-s_0)+o_p(1)\} . \end{aligned}$$

Hence, by (2.1), (2.3) and i),

$$G(s_0+n^{-1/2}x, M_n)=1/2+n^{-1/2}\{x-r_n(M_n)\}\{g(s_0, m_0)+o_p(1)\} .$$

It follows from the definition of $u_n(M_n)$ that

$$u_n(M_n)+\{x-r_n(M_n)\}/\sigma_1 \xrightarrow{\mathrm{P}} 0 .$$

Since Φ is continuous and bounded, the second term in (2.4) will converge to zero if we can show that

$$|\mathrm{E}[\Phi\{(x-r_n(M_n))/\sigma_1\}-\Phi(x/\sigma)]|\rightarrow 0 . \tag{2.5}$$

Notice that

$$\lim_{m\to m_0} \frac{G(s(m), m_0)-G(s_0, m_0)}{s(m)-s_0}=g(s_0, m_0) .$$

But, $G(s_0, m_0)=G(s(m), m)=1/2$ so that

$$g(s_0, m_0)=\lim_{m\to m_0} \frac{m_0-m}{s(m)-s_0} \frac{G(s(m), m_0)-G(s(m), m)}{m_0-m}$$
$$=\lim_{m\to m_0} \frac{m_0-m}{s(m)-s_0}\{\gamma(s_0, m_0)+o(1)\} .$$

Thus, $s'(m_0)=\lim_{m\to m_0} \frac{s(m)-s_0}{m-m_0}$ exists and equals $-\gamma(s_0, m_0)/g(s_0, m_0)$. Since $s'(m_0)$ exists, we may write

$$r_n(M_n)=n^{1/2}(M_n-m_0)\{s'(m_0)+o_p(1)\} .$$

Under condition i), $2F'(m_0)n^{1/2}(M_n-m_0)\xrightarrow{D} N(0, 1)$ so that

$$r_n(M_n)\xrightarrow{D} N(0, \sigma_2^2) \qquad \text{if } \gamma(s_0, m_0)\neq 0$$

and

$$r_n(M_n)\xrightarrow{P} 0 \qquad \text{if } \gamma(s_0, m_0)=0 .$$

If $\gamma(s_0, m_0)=0$ then $\sigma=\sigma_1$ and $(x-r_n(M_n))/\sigma_1\xrightarrow{P} x/\sigma_1$.
If $\gamma(s_0, m_0)\neq 0$, then $(x-r_n(M_n))/\sigma_1\xrightarrow{D} N(x/\sigma_1, \sigma_2^2/\sigma_1^2)$.
In either case, since Φ is continuous and bounded.

$$\mathrm{E}\left\{\Phi\left(\frac{x-r_n(M_n)}{\sigma_1}\right)-\Phi\left(\frac{x-\sigma_2 N}{\sigma_1}\right)\right\}\to 0 ,$$

where N is distributed as $N(0, 1)$. However, if N' is distributed as $N(0, 1)$ independent of N,

$$\mathrm{E}\,\Phi\{(x-\sigma_2 N)/\sigma_1\}=\mathrm{P}\{N'\leqq(x-\sigma_2 N)/\sigma_1\}=\mathrm{P}\{\sigma_1 N'+\sigma_2 N\leqq x\}=\Phi(x/\sigma) ,$$

so (2.5) holds.

We now consider the first term in (2.4). Observe that for any $t\in R$,

$$\mathrm{P}\{R_n(M_n)>t\,|\,M_n\}$$
$$=\mathrm{P}\{G_n(s(M_n)+n^{-1/2}t, M_n)<1/2\,|\,M_n\}$$
$$=\mathrm{P}\left\{\frac{n^{1/2}\{G_n(s(M_n)+n^{-1/2}t, M_n)-G(s(M_n)+n^{-1/2}t, M_n)\}}{\Gamma(s_0, m_0)}\right.$$
$$\left.<\frac{n^{1/2}\{1/2-G(s(M_n)+n^{-1/2}t, M_n)\}}{\Gamma(s_0, m_0)}\,\middle|\,M_n\right\} .$$

Consequently,

$$\begin{aligned}
&\mathrm{E}\,|\mathrm{P}\,\{R_n(M_n)>x-r(M_n)\,|\,M_n\}-\Phi(u_n(M_n))| \\
&\quad=\mathrm{E}\left|\,\mathrm{P}\left\{\frac{n^{1/2}\{G_n(s_0+n^{-1/2}x,\,M_n)-G(s_0+n^{-1/2}x,\,M_n)\}}{\Gamma(s_0,\,m_0)}\right.\right. \\
&\qquad\qquad\left.\left.<u_n(M_n)\,|\,M_n\right\}-\Phi(u_n(M_n))\right| \\
&\quad\leqq \sup_{-\infty<y<\infty}\mathrm{E}\left|\,\mathrm{P}\left\{\frac{n^{1/2}\{G_n(s_0+n^{-1/2}x,\,M_n)-G(s_0+n^{-1/2}x,\,M_n)\}}{\Gamma(s_0,\,m_0)}\right.\right. \\
&\qquad\qquad\left.\left.<y\,|\,M_n\right\}-\Phi(y)\right| .
\end{aligned}$$

Now for n sufficiently large, we can find a $\lambda>0$ (which may be arbitrarily large) such that $n^{1/2}|M_n-m_0|\leqq\lambda$ with probability arbitrarily close to one. Thus

$$\begin{aligned}
&\sup_{-\infty<y<\infty}\mathrm{E}\left|\,\mathrm{P}\left[\frac{n^{1/2}\{G_n(s_0+n^{-1/2}x,\,M_n)-G(s_0+n^{-1/2}x,\,M_n)\}}{\Gamma(s_0,\,m_0)}\right.\right. \\
&\qquad\left.\left.<y\,|\,M_n\right]-\Phi(y)\right| \\
&\leqq \sup_{-\infty<y<\infty}\ \sup_{n^{1/2}|m_n-m_0|\leq\lambda}\left|\,\mathrm{P}\left[\frac{n^{1/2}\{G_n(s_0+n^{-1/2}x,\,m_n)-G(s_0+n^{-1/2}x,\,m_n)\}}{\Gamma(s_0,\,m_0)}\right.\right. \\
&\qquad\left.\left.<y\right]-\Phi(y)\right|+\mathrm{P}\,\{n^{1/2}|M_n-m_0|>\lambda\} \\
&\leqq \sup_{n^{1/2}|m_n-m_0|\leq\lambda} C_0[nG(s_0+n^{-1/2}x,\,m_n)\{1-G(s_0+n^{-1/2}x,\,m_n)\}]^{-1/2} \\
&\qquad+\mathrm{P}\,\{n^{1/2}|M_n-m_0|>\lambda\}\ ,
\end{aligned}$$

by the Berry-Esseen Theorem. The first term is $O(n^{-1/2})$ and the result obtains because $\lambda>0$ is arbitrarily large.

Remark 3. If m_0 is known, the W_i's are simply a sample of independent random variables with common distribution function $G(x,\,m_0)$, $x>0$. Thus, provided $F'(m_0+s_0)+F'(m_0-s_0)=g(s_0,\,m_0)$ exists and is positive, by the central limit theorem for the median,

$$n^{1/2}(S_n(m_0)-s_0)\xrightarrow{\mathrm{D}}N(0,\,1/\{4g^2(s_0,\,m_0)\})\ .$$

Remark 4. If $F'(\xi_{3/4})$ and $F'(\xi_{1/4})$ exist and are positive, by the central limit theorem for quantiles,

$$\begin{aligned}
&n^{1/2}\{Q_n-(\xi_{3/4}-\xi_{1/4})/2\} \\
&\quad\xrightarrow{\mathrm{D}}N\left(0,\,\frac{1}{64}\left\{\frac{3}{F'(\xi_{3/4})^2}-\frac{2}{F'(\xi_{3/4})F'(\xi_{1/4})}+\frac{3}{F'(\xi_{1/4})^2}\right\}\right)\ .
\end{aligned}$$

3. Asymptotic equivalence results

The weak Bahadur representation for the median deviation and the asymptotic equivalence of the median deviation and the semi-interquartile range follow from the next theorem. Essentially, this theorem gives weak conditions under which there is no loss of information for estimating s_0 if we have to estimate m_0 also.

THEOREM 3. *Suppose*

i) $F'(m_0)$ *exists and is positive.*

ii) $F'(m_0 \pm s_0 + x)$ *exists for* x *in a neighbourhood of the origin and is continuous at* $x=0$.

iii) $g(x, m_0)=F'(m_0+x)+F'(m_0-x)>0$ *for* x *in a neighbourhood of* s_0.

iv) $F(m_0+s_0)=1-F(m_0-s_0)$ *and* $F'(m_0+s_0)=F'(m_0-s_0)$.

Then

$$n^{1/2}\{S_n(M_n)-S_n(m_0)\} \xrightarrow{P} 0 .$$

PROOF. Let $T_n=n^{1/2}(S_n(m_0)-s_0)$ and let $R_n(M_n)$ be defined as in the proof of Theorem 2. Condition iv) ensures that $s'(m_0)=0$, whence $n^{1/2}\cdot(S_n(M_n)-s_0)-R_n(M_n) \xrightarrow{P} 0$, so it suffices to show that

$$T_n-R_n(M_n) \xrightarrow{P} 0 . \tag{3.1}$$

By Remark 3, $T_n=O_p(1)$ so that (3.1) will follow from a lemma of Ghosh [1] if we can show that for all $y \in R$ and all $\varepsilon>0$,

$$\text{(3.2) and} \quad \begin{aligned} &\mathrm{P}\{T_n \leqq y,\ R_n(M_n) \geqq y+\varepsilon\} \to 0 \\ &\mathrm{P}\{T_n \geqq y+\varepsilon,\ R_n(M_n) \leqq y\} \to 0 . \end{aligned}$$

Let $y \in R$ be any fixed real number. Then

$$\{T_n \leqq y\} \Longleftrightarrow \{\tau_n \leqq y_n\} ,$$

where $\tau_n=n^{1/2}\{G(s_0+n^{-1/2}y, m_0)-G_n(s_0+n^{-1/2}y, m_0)\}/g(s_0, m_0)$ and $y_n=n^{1/2}\{G(s_0+n^{-1/2}y, m_0)-1/2\}/g(s_0, m_0)$. Similarly,

$$\{R_n(M_n) \leqq y\} \Longleftrightarrow \{\rho_n(M_n) \leqq y_n\} ,$$

where $\rho_n(M_n)=n^{1/2}\{G(s_0+n^{-1/2}y, m_0)-G_n(s(M_n)+n^{-1/2}y, M_n)\}/g(s_0, m_0)$. Under condition ii), we may write

$$G(s_0+n^{-1/2}y, m_0)=1/2+n^{-1/2}y\{g(s_0, m_0)+o(1)\} ,$$

so that clearly $y_n \to y$. Let $\varepsilon>0$ be given. Then it follows that

$$\mathrm{P}\{T_n \leqq y,\ R_n(M_n) \geqq y+\varepsilon\} = \mathrm{P}\{\tau_n \leqq y_n,\ \rho_n(M_n) \geqq x_n\}$$

and

$$\mathrm{P}\{T_n \geqq y+\varepsilon,\ R_n(M_n) \leqq y\} = \mathrm{P}\{\tau_n \geqq x_n,\ \rho_n(M_n) \leqq y_n\}\ ,$$

where $y_n \to y$ and $x_n \to y+\varepsilon$. Hence (3.2) will follow if we can show that $\tau_n - \rho_n(M_n) \xrightarrow{\mathrm{P}} 0$.

Let $\delta > 0$ be given. Then

$$\begin{aligned}\mathrm{P}\{|\tau_n - \rho_n(M_n)| > \delta\} &\leqq \delta^{-2}\,\mathrm{E}\,(\tau_n - \rho_n(M_n))^2 \\ &= \delta^{-2}\,\mathrm{E}\,[\mathrm{E}\,\{(\tau_n - \rho_n(M_n))^2 \,|\, M_n\}] \\ &= \mathrm{E}\,\{\zeta_1(M_n) + \zeta_2(M_n)\}/\delta^2 g^2(s_0, m_0)\ ,\end{aligned}$$

where

$$\begin{aligned}\zeta_1(M_n) = {} & G(s_0 + n^{-1/2}y,\ m_0)\{1 - G(s_0 + n^{-1/2}y,\ m_0)\} \\ & + G(s(M_n) + n^{-1/2}y,\ M_n)\{1 - G(s(M_n) + n^{-1/2}y,\ M_n)\} \\ & - 2\{G(s_0 + n^{-1/2}y,\ m_0) \Lambda G(s(M_n) + n^{-1/2}y,\ M_n) \\ & - G(s_0 + n^{-1/2}y,\ m_0) G(s(M_n) + n^{-1/2}y,\ M_n)\}\end{aligned}$$

and

$$\zeta_2(M_n) = n\{G(s_0 + n^{-1/2}y,\ m_0) - G(s(M_n) + n^{-1/2}y,\ M_n)\}^2\ .$$

Now $\zeta_1(M_n)$ is a bounded random variable converging in probability to zero so $\mathrm{E}\,\zeta_1(M_n) \to 0$. Also

$$G(s(M_n) + n^{-1/2}y,\ M_n) = 1/2 + n^{-1/2}y\{g(s_0, m_0) + o_p(1)\}$$

and

$$G(s_0 + n^{-1/2}y,\ m_0) = 1/2 + n^{-1/2}y\{g(s_0, m_0) + o(1)\}\ ,$$

so that $\mathrm{E}\,\zeta_2(M_n) = o(1)$ and the result obtains.

COROLLARY 3.1. *Under the conditions of Theorem* 3,

$$S_n(M_n) - s_0 = \frac{\{F(m_0+s_0) - F_n(m_0+s_0) - F(m_0-s_0) + F_n(m_0-s_0)\}}{2F'(m_0+s_0)} + U_n\ ,$$

where $n^{1/2}U_n \xrightarrow{\mathrm{P}} 0$, *as* $n \to 0$.

PROOF. By Theorem 3, it suffices to show that the representation holds for $S_n(m_0)$ and this follows immediately from Theorem 1 of Ghosh [1].

COROLLARY 3.1.1. *Under the conditions of Theorem* 3,

$$n^{1/2}\{S_n(M_n) - Q_n\} \xrightarrow{\mathrm{P}} 0\ .$$

PROOF. By Theorem 1 of Ghosh [1],

$$Q_n - \frac{(\xi_{3/4} - \xi_{1/4})}{2} = \frac{F(\xi_{3/4}) - F_n(\xi_{3/4})}{2F'(\xi_{3/4})} - \frac{F(\xi_{1/4}) - F_n(\xi_{1/4})}{2F'(\xi_{1/4})} + U_n',$$

where $n^{1/2}U_n' \xrightarrow{P} 0$. Under condition iv) of Theorem 3, we may solve $F(m_0+s_0)-F(m_0-s_0)=1/2$ to obtain $s_0=\xi_{3/4}-m_0$ and $s_0=m_0-\xi_{1/4}$. Hence $(\xi_{3/4}-\xi_{1/4})/2=s_0$ and

$$F'(\xi_{3/4}) = F'(m_0+s_0) = F'(m_0-s_0) = F'(\xi_{1/4})$$

so the result follows from Corollary 3.1.

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