Ann. Inst. Statist. Math.
37 (1985), Part A, 17-26

ESTIMATION OF A COMMON PARAMETER FOR POOLED SAMPLES FROM THE UNIFORM DISTRIBUTIONS

MASAFUMI AKAHIRA AND KEI TAKEUCHI

(Received Sept. 6, 1983; revised Mar. 12, 1984)

Summary

The problem to estimate a common parameter for the pooled sample from the uniform distributions is discussed in the presence of nuisance parameters. The maximum likelihood estimator (MLE) and others are compared and it is shown that the MLE based on the pooled sample is not (asymptotically) efficient.

1. Introduction

In regular cases the asymptotic deficiencies of asymptotically efficient estimators were calculated in pooled sample from the same distribution (Akahira [5]) and in the presence of nuisance parameters (Akahira and Takeuchi [7]). In non-regular cases the asymptotic optimality of estimators was discussed in Akahira [4], Akahira and Takeuchi [6], Ibragimov and Has'minskii [10], Jurěcková [11] and others, and also recently a Monte Carlo study on the estimator considered in Akahira [1]-[3] has been done by Antoch [9].

In this paper we consider the problem to estimate an unknown real-valued parameter θ based on m samples of size n from the uniform distributions on the interval $(\theta-\xi_i, \theta+\xi_i)$ $(i=1,\cdots,m)$ with different nuisance parameters which is treated as a typical example in non-regular cases. In some cases the MLE and other estimators will be compared and it will be shown that the MLE based on the pooled sample is not better for both a sample of a fixed size and a large sample. Related results can be found in Akai [8].

2. Results

Suppose that it is required to estimate an unknown real-valued

Key words and phrases: Maximum likelihood estimator, weighted estimator, uniform distributions.

parameter θ based on m samples of size n whose values are X_{ij} ($i=1, \cdots, m$; $j=1, \cdots, n$) from the uniform distributions on the interval $(\theta-\xi_i, \theta+\xi_i)$ with different nuisance parameters ξ_i ($i=1, \cdots, m$). For each i let $X_{i(1)}<X_{i(2)}<\cdots<X_{i(n)}$ be order statistics from $X_{i1}, X_{i2}, \cdots, X_{in}$. We consider some cases.

Case I. $\xi_i=\xi$ ($i=1, \cdots, m$) *are unknown.*

The MLE $\hat{\theta}_{ML}$ of θ based on the pooled sample $\{X_{ij}\}$ is given by

$$\hat{\theta}_{ML}=\frac{1}{2}(\min_{1\leqq i\leqq m} X_{i(1)}+\max_{1\leqq i\leqq m} X_{i(n)}) .$$

An estimator $\hat{\theta}_n$ of θ based on the sample of size n is called to be (asymptotically) median unbiased if

$$\Pr\{\hat{\theta}_n\leqq\theta\}=\Pr\{\hat{\theta}_n\geqq\theta\}=\frac{1}{2}$$

$\left(\lim_{n\to\infty}\left|\Pr\{\hat{\theta}_n\leqq\theta\}-\frac{1}{2}\right|=\lim_{n\to\infty}\left|\Pr\{\hat{\theta}_n\geqq\theta\}-\frac{1}{2}\right|=0\right.$ uniformly in some neighborhood of $\left.\theta\right)$ (e.g. see Akahira and Takeuchi [6]). Then it is shown that $\hat{\theta}_{ML}$ is one-sided asymptotically efficient in the sense that for any asymptotically median unbiased estimator $\hat{\theta}_n$

$$\varliminf_{n\to\infty}[\Pr\{n(\hat{\theta}_{ML}-\theta)\leqq t\}-\Pr\{n(\hat{\theta}_n-\theta)\leqq t\}]\geqq 0 \qquad \text{for all } t>0 ;$$

or

$$\varlimsup_{n\to\infty}[\Pr\{n(\hat{\theta}_{ML}-\theta)\leqq t\}-\Pr\{n(\hat{\theta}_n-\theta)\leqq t\}]\leqq 0 \qquad \text{for all } t<0 ,$$

since in this case $\hat{\theta}_{ML}$ is actually the MLE from the sample of size mn (see Akahira and Takeuchi [6]).

Case II. ξ_i ($i=1, \cdots, m$) *are known.*

The MLE $\hat{\theta}^*_{ML}$ of θ based on the pooled sample $\{X_{ij}\}$ is given by

$$\hat{\theta}^*_{ML}=\frac{1}{2}\{\max_{1\leqq i\leqq m}(X_{i(n)}-\xi_i)+\min_{1\leqq i\leqq m}(X_{i(1)}+\xi_i)\} .$$

Then it can be shown in a similar way as in [4] and [6] that $\hat{\theta}^*_{ML}$ is two-sided asymptotically efficient in the sense that for any asymptotically median unbiased estimator $\hat{\theta}_n$

$$\varliminf_{n\to\infty}[\Pr\{n|\hat{\theta}^*_{ML}-\theta|\leqq t\}-\Pr\{n|\hat{\theta}_n-\theta|\leqq t\}]\geqq 0 \qquad \text{for all } t>0 .$$

(For the definition see Akahira and Takeuchi [6], page 72 and Akahira [4]).

Case III. ξ_i $(i=1,\cdots,m)$ *are unknown.*

For each i the MLE's $\hat{\theta}_i$ and $\hat{\xi}_i$ of θ and ξ_i based on the sample $X_{i1},\cdots,X_{in}$ is given by

$$\hat{\theta}_i=\frac{1}{2}(X_{i(1)}+X_{i(n)}) ; \qquad \hat{\xi}_i=\frac{1}{2}(X_{i(n)}-X_{i(1)}) ,$$

respectively. Let $\hat{\theta}_{ML}$ be the MLE of θ based on the pooled sample $\{X_{ij}\}$. Then it will be shown that $\hat{\theta}_{ML}$ is not two-sided asymptotically efficient. We define

$$\hat{\theta}_n^*=\frac{1}{2}\{\max_{1\leqq i\leqq m}(\hat{\theta}_i+\hat{\xi}_i-\xi_i^0)+\min_{1\leqq i\leqq m}(\hat{\theta}_i-\hat{\xi}_i+\xi_i^0)\} .$$

Then it can be shown from Case II that $\hat{\theta}_n^*$ is asymptotically locally best estimator of θ at $\xi_i=\xi_i^0$ $(i=1,\cdots,m)$ in the sense that for any asymptotically median unbiased estimator $\hat{\theta}_n$

$$\varliminf_{n\to\infty}[P_{\theta,\xi_1^0,\cdots,\xi_m^0}\{n|\hat{\theta}_n^*-\theta|\leqq t\}-P_{\theta,\xi_1^0,\cdots,\xi_m^0}\{n|\hat{\theta}_n-\theta|\leqq t\}]\geqq 0$$

for all $t>0$.

First we shall obtain the MLE $\hat{\theta}_{ML}$ based on the pooled sample $\{X_{ij}\}$. Let $f_i(x,\theta)$ be a density function of the uniform distribution on the interval $(\theta-\xi_i,\theta+\xi_i)$. Since the likelihood function $L(\theta;\xi_1,\cdots,\xi_m)$ is given by

$$L(\theta;\xi_1,\cdots,\xi_m)=\prod_{i=1}^m\prod_{j=1}^n f_i(x_j,\theta)$$
$$=\begin{cases}\dfrac{1}{2^n}\left(\dfrac{1}{\prod_{i=1}^m\xi_i}\right)^n & \text{for } x_{i(n)}-\xi_i\leqq\theta\leqq x_{i(1)}+\xi_i \\ & \qquad (i=1,\cdots,m) ; \\ 0 & \text{otherwise} .\end{cases}$$

In order to obtain the MLE $\hat{\theta}_{ML}$ it is enough to find θ minimizing $\prod_{i=1}^m\xi_i$ under the condition $\xi_i\geqq\max\{\theta-x_{i(1)},x_{i(n)}-\theta\}$ for all i. Let $\hat{\theta}^*$ be some estimator of θ based on the pooled sample $\{X_{ij}\}$. For each i we put $\xi_i^*=\max\{\hat{\theta}^*-x_{i(1)},x_{i(n)}-\hat{\theta}^*\}$. Then we have for each i

$$\xi_i^*=\max\{\hat{\theta}^*-\hat{\theta}_i+\hat{\xi}_i,\hat{\theta}_i+\hat{\xi}_i-\hat{\theta}^*\}=\hat{\xi}_i+|\hat{\theta}_i-\hat{\theta}^*| .$$

Hence the MLE $\hat{\theta}_{ML}$ is given by $\hat{\theta}^*$ minimizing

$$\prod_{i=1}^m(\hat{\xi}_i+|\hat{\theta}_i-\hat{\theta}^*|) ,$$

that is, such an estimator $\hat{\theta}^*$ is given by either of the estimators $\hat{\theta}_i$ $(i=1,\cdots,m)$. Since

$$\prod_{i=1}^{m}(\hat{\xi}_i+|\hat{\theta}_i-\hat{\theta}^*|)=\prod_{i=1}^{m}\hat{\xi}_i\prod_{i=1}^{m}\left(1+\frac{|\hat{\theta}_i-\hat{\theta}^*|}{\hat{\xi}_i}\right),$$

for sufficiently large n it is asymptotically equivalent to

$$\prod_{i=1}^{m}\hat{\xi}_i\left(1+\sum_{i=1}^{m}\frac{|\hat{\theta}_i-\hat{\theta}^*|}{\hat{\xi}_i}\right).$$

Hence it is seen that for sufficiently large n the MLE $\hat{\theta}_{ML}$ is asymptotically equivalent to a weighted median by the weights $1/\hat{\xi}_i$ $(i=1,\cdots,m)$.

Next we shall discuss the comparison among $\hat{\theta}_{ML}$, the weighted estimator and other estimators. We consider the case when $m=2$. Then for each $i=1,2$, $X_{i1},\cdots,X_{in}$ are independently, identically and uniformly distributed random variables on $(\theta-\xi_i, \theta+\xi_i)$. Then for each $i=1,2$, the joint density function $f_n(x,y;\theta,\xi_i)$ of $\hat{\theta}_i$ and $\hat{\xi}_i$ is given by

$$(2.1)\qquad f_n(x,y;\theta,\xi_i)=\begin{cases}\dfrac{n(n-1)}{2\xi_i^n}y^{n-2} & \text{for } 0\leqq y\leqq\xi_i \text{ and } \theta-\xi_i+y\leqq x\leqq\theta+\xi_i-y\,;\\ 0 & \text{otherwise}.\end{cases}$$

For each $i=1,2$ the density function $f_n(x;\theta,\xi_i)$ of $\hat{\theta}_i$ is given by

$$(2.2)\qquad f_n(x;\theta,\xi_i)=\begin{cases}\dfrac{n}{2\xi_i^n}(\xi_i-|x-\theta|)^{n-1} & \text{for } \theta-\xi_i<x<\theta+\xi_i\,;\\ 0 & \text{otherwise}.\end{cases}$$

Also for each $i=1,2$ the conditional density function $f_n(x|y;\theta,\xi_i)$ of $\hat{\theta}_i$ given $\hat{\xi}_i$ is given by

$$(2.3)\qquad f_n(x|y;\theta,\xi_i)=\begin{cases}\dfrac{1}{2(\xi_i-y)} & \text{for } \theta-\xi_i+y\leqq x\leqq\theta+\xi_i-y\,;\\ 0 & \text{otherwise},\end{cases}$$

that is, the conditional distribution of $\hat{\theta}_i$ given $\hat{\xi}_i$ is uniform distribution on the interval $(\theta-(\xi_i-\hat{\xi}_i), \theta+(\xi_i-\hat{\xi}_i))$.

For two (asymptotically) median unbiased estimators $\hat{\theta}^1$ and $\hat{\theta}^2$ of θ, $\hat{\theta}^1$ is called to be (asymptotically) better than $\hat{\theta}^2$ if

$$\Pr\{n|\hat{\theta}^1-\theta|\leqq t\,|\,\hat{\xi}_1,\hat{\xi}_2\}\geqq\Pr\{n|\hat{\theta}^2-\theta|\leqq t\,|\,\hat{\xi}_1,\hat{\xi}_2\}\ \text{a.e.}\qquad\text{for all } t>0$$

($\varliminf_{n\to\infty}[\Pr\{n|\hat{\theta}^1-\theta|\leqq t\}-\Pr\{n|\hat{\theta}^2-\theta|\leqq t\}]\geqq 0$ for all $t>0$), and then for simplicity we denote it symbolically by $\hat{\theta}^1\succ\hat{\theta}^2$ ($\hat{\theta}^1\underset{\text{as.}}{\succ}\hat{\theta}^2$), where $\Pr\{A|\hat{\xi}_1,\hat{\xi}_2\}$ denotes the conditional probability of A given $\hat{\xi}_1$ and $\hat{\xi}_2$. From (2.3) we see that the conditional density of $\hat{\theta}_1-\theta$ and $\hat{\theta}_2-\theta$ given $\hat{\xi}_1$ and $\hat{\xi}_2$

is given by

$$f_n(x_1, x_2 | \hat{\xi}_1, \hat{\xi}_2) = \begin{cases} \dfrac{1}{4\tau_1\tau_2} & \text{for } |x_1| < \tau_1 \text{ and } |x_2| < \tau_2 ; \\ 0 & \text{otherwise ,} \end{cases}$$

where $\tau_i = \xi_i - \hat{\xi}_i$ $(i=1, 2)$. If $c_1\tau_1 > c_2\tau_2$, then the conditional density function $f_n(y | \hat{\xi}_1, \hat{\xi}_2)$ of $\hat{\theta}_0 - \theta$ with $\hat{\theta}_0 = c_1\hat{\theta}_1 + c_2\hat{\theta}_2$ given $\hat{\xi}_1$ and $\hat{\xi}_2$ is given by

$$(2.4) \qquad f_n(y | \hat{\xi}_1, \hat{\xi}_2) = \begin{cases} \dfrac{1}{4c_1c_2\tau_1\tau_2}(c_1\tau_1 + c_2\tau_2 + y) & \text{for } y < -c_1\tau_1 + c_2\tau_2 ; \\ \dfrac{1}{2c_1\tau_1} & \text{for } |y| < c_1\tau_1 - c_2\tau_2 ; \\ \dfrac{1}{4c_1c_2\tau_1\tau_2}(c_1\tau_1 + c_2\tau_2 - y) & \text{for } y > c_1\tau_1 - c_2\tau_2 . \end{cases}$$

If $c_1\tau_1 < c_2\tau_2$, then the conditional density function is given by

$$(2.5) \qquad f_n(y | \hat{\xi}_1, \hat{\xi}_2) = \begin{cases} \dfrac{1}{4c_1c_2\tau_1\tau_2}(c_1\tau_1 + c_2\tau_2 - y) & \text{for } y > -c_1\tau_1 + c_2\tau_2 ; \\ \dfrac{1}{2c_2\tau_2} & \text{for } |y| < -c_1\tau_1 + c_2\tau_2 ; \\ \dfrac{1}{4c_1c_2\tau_1\tau_2}(c_1\tau_1 + c_2\tau_2 + y) & \text{for } y < c_1\tau_1 - c_2\tau_2 . \end{cases}$$

If

$$c_i = \hat{\xi}_j / (\hat{\xi}_1 + \hat{\xi}_2) = c_i' \text{ (say)} \qquad (i \neq j ;\ i, j = 1, 2) ,$$

then

$$(2.6) \qquad \frac{1}{c_1'\tau_1} = \frac{\hat{\xi}_1 + \hat{\xi}_2}{\hat{\xi}_2(\xi_1 - \hat{\xi}_1)} ; \qquad \frac{1}{c_2'\tau_2} = \frac{\hat{\xi}_1 + \hat{\xi}_2}{\hat{\xi}_1(\xi_2 - \hat{\xi}_2)} .$$

If

$$c_i = \hat{\xi}_j^2 / (\hat{\xi}_1^2 + \hat{\xi}_2^2) = c_i'' \text{ (say)} \qquad (i \neq j ;\ i, j = 1, 2) ,$$

then

$$(2.7) \qquad \frac{1}{c_1''\tau_1} = \frac{\hat{\xi}_1^2 + \hat{\xi}_2^2}{\hat{\xi}_2^2(\xi_1 - \hat{\xi}_1)} ; \qquad \frac{1}{c_2''\tau_2} = \frac{\hat{\xi}_1^2 + \hat{\xi}_2^2}{\hat{\xi}_1^2(\xi_2 - \hat{\xi}_2)} .$$

Hence we have the following:

$$(2.8) \qquad \hat{\xi}_2 \gtreqless \hat{\xi}_1 \text{ if and only if } \frac{1}{c_1'\tau_1} = \frac{\hat{\xi}_1 + \hat{\xi}_2}{\hat{\xi}_2(\xi_1 - \hat{\xi}_1)} \gtreqless \frac{\hat{\xi}_1^2 + \hat{\xi}_2^2}{\hat{\xi}_2^2(\xi_1 - \hat{\xi}_1)} = \frac{1}{c_1''\tau_1} ;$$

(2.9) $\hat{\xi}_2 \gtreqless \hat{\xi}_1$ if and only if $\dfrac{1}{c_2'\tau_2}=\dfrac{\hat{\xi}_1+\hat{\xi}_2}{\hat{\xi}_1(\xi_2-\hat{\xi}_2)} \lesseqgtr \dfrac{\hat{\xi}_1^2+\hat{\xi}_2^2}{\hat{\xi}_1^2(\xi_2-\hat{\xi}_2)}=\dfrac{1}{c_2''\tau_2}$.

On the other hand as is seen from the above discussion on the MLE, the MLE $\hat{\theta}_{ML}$ based on the pooled sample $\{X_{ij}\}$ is given by

$$\hat{\theta}_{ML}=\begin{cases} \hat{\theta}_1 & \text{if } \hat{\xi}_1<\hat{\xi}_2 ; \\ \hat{\theta}_2 & \text{if } \hat{\xi}_1>\hat{\xi}_2 . \end{cases}$$

The conditional density of the MLE $\hat{\theta}_{ML}$ given $\hat{\xi}_i$ is given by (2.3). Now we consider two cases, i.e., $\xi_1=\xi_2$ and $\xi_1\neq\xi_2$. Note that the first Case III$_1$ was also mentioned in the Case I with another optimum criterion.

Case III$_1$. $\xi_1=\xi_2=\xi$.

In the case when $c_i=c_i'=\hat{\xi}_j/(\hat{\xi}_1+\hat{\xi}_2)$ $(i\neq j;\ i, j=1, 2)$,

$$\hat{\xi}_1\lesseqgtr\hat{\xi}_2 \text{ if and only if } \hat{\xi}_1(\xi-\hat{\xi}_2)\lesseqgtr\hat{\xi}_2(\xi-\hat{\xi}_1) , \quad \text{i.e., } c_2\tau_2\lesseqgtr c_1\tau_1 .$$

Hence we have

(2.10) $$\hat{\xi}_1\lesseqgtr\hat{\xi}_2 \text{ if and only if } c_1'\tau_1+c_2'\tau_2<\begin{cases}\tau_1\\ \tau_2\end{cases} .$$

We also obtain

$$c_1''\tau_1+c_2''\tau_2-(c_1'\tau_1+c_2'\tau_2)=\frac{\hat{\xi}_1\hat{\xi}_2(\hat{\xi}_1-\hat{\xi}_2)^2}{(\hat{\xi}_1+\hat{\xi}_2)(\hat{\xi}_1^2+\hat{\xi}_2^2)}\geqq 0 .$$

From (2.4) to (2.10) we have established the following theorem.

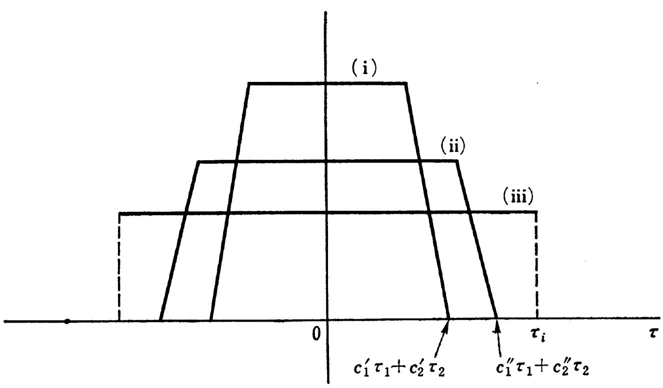

Fig. 2.1. Comparison of the conditional densities of $c_1\hat{\theta}_1+c_2\hat{\theta}_2-\theta$ given $\hat{\xi}_1$ and $\hat{\xi}_2$ with (i) $c_i=c_i'=\hat{\xi}_j/(\hat{\xi}_1+\hat{\xi}_2)$ $(i\neq j;\ i, j=1, 2)$ and (ii) $c_i=c_i''=\hat{\xi}_j^2/(\hat{\xi}_1^2+\hat{\xi}_2^2)$ $(i\neq j;\ i, j=1, 2)$ and of (iii) the MLE $\hat{\theta}_{ML}$ given $\hat{\xi}_i$. They are given by (2.5) and (2.3), respectively.

THEOREM 2.1. *If* $\xi_1=\xi_2$ *and* $\hat{\xi}_1<\hat{\xi}_2$, *then*

$$\frac{\hat{\xi}_2\hat{\theta}_1+\hat{\xi}_1\hat{\theta}_2}{\hat{\xi}_1+\hat{\xi}_2} \succ \frac{\hat{\xi}_2^2\hat{\theta}_1+\hat{\xi}_1^2\hat{\theta}_2}{\hat{\xi}_1^2+\hat{\xi}_2^2} \succ \hat{\theta}_{ML}=\hat{\theta}_1 .$$

If $\xi_1=\xi_2$ *and* $\hat{\xi}_1>\hat{\xi}_2$, *then*

$$\frac{\hat{\xi}_2\hat{\theta}_1+\hat{\xi}_1\hat{\theta}_2}{\hat{\xi}_1+\hat{\xi}_2} \succ \frac{\hat{\xi}_2^2\hat{\theta}_1+\hat{\xi}_1^2\hat{\theta}_2}{\hat{\xi}_1^2+\hat{\xi}_2^2} \succ \hat{\theta}_{ML}=\hat{\theta}_2 .$$

We assume that $c_1=c_2=1/2$ and $\xi_1=\xi_2=\xi$. By (2.2) it follows that for each $i=1, 2$ the asymptotic density of $n(\hat{\theta}_i-\theta)$ is given by

$$f_i(x)=\frac{1}{2\xi}e^{-|x|/\xi} . \tag{2.11}$$

Then the characteristic function $\phi(t)$ of the asymptotic density function of $n[\{(\hat{\theta}_1+\hat{\theta}_2)/2\}-\theta]$ is given by

$$\phi(t)=\frac{1}{(1+\xi^2t^2/4)^2} .$$

We may represent $\phi(t)$ as follows:

$$\phi(t)=\frac{1-\xi^2t^2/4}{2(1+\xi^2t^2/4)^2}+\frac{1}{2(1+\xi^2t^2/4)} . \tag{2.12}$$

Since

$$\int_{-\infty}^{\infty}\frac{1}{\xi}\exp\left(-\frac{2}{\xi}|y|\right)\exp(ity)dy=\frac{1}{2}\int_{-\infty}^{\infty}\exp(-|x|)\exp\left(i\frac{\xi t}{2}x\right)dx$$
$$=\frac{1}{1+\xi^2t^2/4} ,$$

$$\int_{-\infty}^{\infty}\frac{2}{\xi^2}|y|\exp\left(-\frac{2}{\xi}|y|\right)\exp(ity)dy=\frac{1}{2}\int_{-\infty}^{\infty}|x|\exp(-|x|)\exp\left(i\frac{\xi t}{2}x\right)dx$$

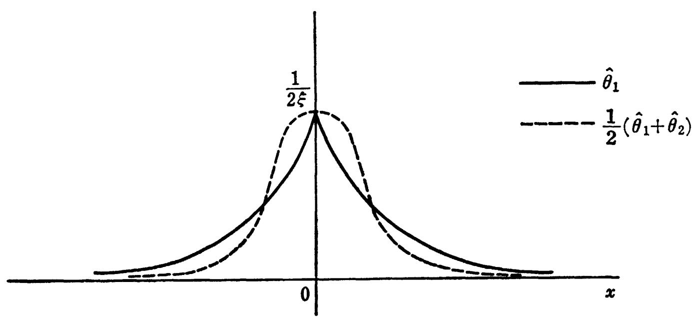

Fig. 2.2. Comparison of the asymptotic densities of $n(\hat{\theta}_1-\theta)$ and $n[\{(\hat{\theta}_1+\hat{\theta}_2)/2\}-\theta]$ given by (2.11) and (2.13), respectively.

$$=\frac{1-\xi^2t^2/4}{(1+\xi^2t^2/4)^2},$$

it follows from (2.12) that the asymptotic density of $n[\{(\hat{\theta}_1+\hat{\theta}_2)/2\}-\theta]$ is given by

$$f(x)=\frac{1}{2\xi}\left(1+2\frac{|x|}{\xi}\right)\exp\left(-\frac{2}{\xi}|x|\right). \tag{2.13}$$

From (2.11) and (2.13) we have established the following theorem.

THEOREM 2.2. *If* $\xi_1=\xi_2=\xi$, *then*

$$\frac{1}{2}(\hat{\theta}_1+\hat{\theta}_2)\underset{\text{as.}}{\succ}\hat{\theta}_{ML}=\begin{cases}\hat{\theta}_1 & \text{if } \hat{\xi}_1<\hat{\xi}_2;\\ \hat{\theta}_2 & \text{if } \hat{\xi}_1>\hat{\xi}_2.\end{cases}$$

Case III$_2$. $\xi_1\neq\xi_2$.

We consider only the case when $\xi_1<\xi_2$. We assume that $c_1\xi_1\neq c_2\xi_2$. By (2.2) it follows that for each $i=1, 2$ the asymptotic density of $n(\hat{\theta}_i-\theta)$ is given by

$$f_i(x)=\frac{1}{2\xi_i}\exp(-|x|/\xi_i), \tag{2.14}$$

and also its characteristic function $\phi_i(t)$ is given by

$$\phi_i(t)=\frac{1}{1+\xi_i^2t^2}.$$

Hence the characteristic function $\phi^*(t)$ of the asymptotic density of $n(\hat{\theta}_0-\theta)=n(c_1\hat{\theta}_1+c_2\hat{\theta}_2-\theta)$ is given by

$$\phi^*(t)=\frac{1}{(1+c_1^2\xi_1^2t^2)(1+c_2^2\xi_2^2t^2)}.$$

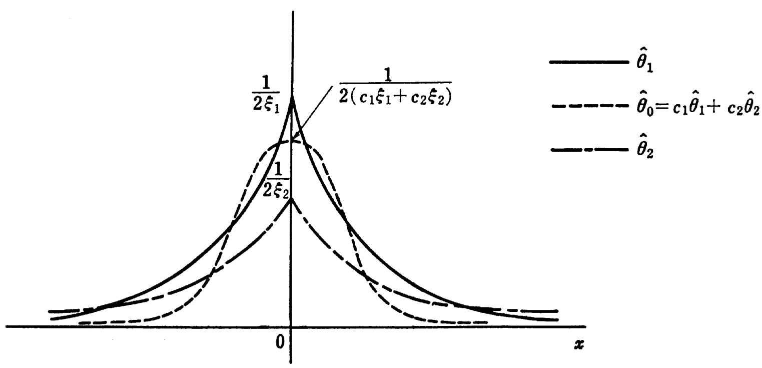

Fig. 2.3. Comparison of the asymptotic densities of $n(\hat{\theta}_1-\theta)$, $n(\hat{\theta}_0-\theta)=n(c_1\hat{\theta}_1+c_2\hat{\theta}_2-\theta)$ and $n(\hat{\theta}_2-\theta)$ given by (2.14), (2.15) and (2.14), respectively, when $\xi_1<\xi_2$.

Since

$$\phi^*(t)=\frac{1}{(c_1^2\xi_1^2-c_2^2\xi_2^2)}\left(\frac{c_1^2\xi_1^2}{1+c_1^2\xi_1^2t^2}-\frac{c_2^2\xi_2^2}{1+c_2^2\xi_2^2t^2}\right),$$

it follows that the asymptotic density $f(x)$ of $n(\hat{\theta}_0-\theta)$ is given by

$$\text{(2.15)} \qquad f(x)=\frac{1}{2(c_1^2\xi_1^2-c_2^2\xi_2^2)}\left\{c_1\xi_1\exp\left(-\frac{|x|}{c_1\xi_1}\right)-c_2\xi_2\exp\left(-\frac{|x|}{c_2\xi_2}\right)\right\}.$$

There exists a positive number t_0 such that

$$\int_0^{t_0} f_1(x)dx=\int_0^{t_0} f(x)dx ,$$

where $f_1(x)$ and $f(x)$ are given by (2.14) and (2.15), respectively. Hence we have established the following theorem.

THEOREM 2.3. *Suppose that* $\xi_1<\xi_2$ *and* $c_1\xi_1\neq c_2\xi_2$. *Then*

$$\hat{\theta}_{ML}=\hat{\theta}_1 \underset{\text{a.s.}}{\succ} \hat{\theta}_0$$

in some neighborhood of θ *in the sense that*

$$\varliminf_{n\to\infty}[\Pr\{n|\hat{\theta}_{ML}-\theta|\leqq t\}-\Pr\{n|\hat{\theta}_0-\theta|\leqq t\}]\geqq 0 \qquad \textit{for all } t\leqq t_0 .$$

Further

$$\hat{\theta}_0 \underset{\text{a.s.}}{\succ} \hat{\theta}_{ML}=\hat{\theta}_1$$

in far away from θ *in the sense that*

$$\varliminf_{n\to\infty}[\Pr\{n|\hat{\theta}_0-\theta|\leqq t\}-\Pr\{n|\hat{\theta}_{ML}-\theta|\leqq t\}]\geqq 0 \qquad \textit{for all } t>t_0 .$$

Remark. From (2.14) and (2.15) it is easily seen that

$$\hat{\theta}_0 \underset{\text{a.s.}}{\succ} \hat{\theta}_2 \qquad \text{and} \qquad \hat{\theta}_{ML}=\hat{\theta}_1 \underset{\text{a.s.}}{\succ} \hat{\theta}_2 .$$

The case $\xi_1>\xi_2$ may be treated quite similarly.

UNIVERSITY OF ELECTRO-COMMUNICATIONS
UNIVERSITY OF TOKYO

REFERENCES

[1] Akahira, M. (1975). Asymptotic theory for estimation of location in non-regular cases, I: Order of convergence of consistent estimators, *Rep. Statist. Appl. Res., JUSE*, **22**, 8-26.

[2] Akahira, M. (1975). Asymptotic theory for estimation of location in non-regular cases, II: Bounds of asymptotic distributions of consistent estimators, *Rep. Statist. Appl. Res., JUSE*, **22**, 99-115.

[3] Akahira, M. (1976). A remark on asymptotic sufficiency of statistics in non-regular cases, *Rep. Univ. Electro-Comm.*, **27**, 125-128.

[4] Akahira, M. (1982). Asymptotic optimality of estimators in non-regular cases, *Ann. Inst. Statist. Math.*, A, **34**, 69-82.

[5] Akahira, M. (1982). Asymptotic deficiencies of estimators for pooled samples from the same distribution, Probability Theory and Mathematical Statistics, *Lecture Notes in Mathematics*, **1021**, 6-14, Springer, Berlin.

[6] Akahira, M. and Takeuchi, K. (1981). Asymptotic Efficiency of Statistical Estimators: Concepts and Higher Order Asymptotic Efficiency, *Lecture Notes in Statistics* 7, Springer, New York.

[7] Akahira, M. and Takeuchi, K. (1982). On asymptotic deficiency of estimators in pooled samples in the presence of nuisance parameters, *Statistics and Decisions*, **1**, 17-38, Akademische Verlagsgesellschaft.

[8] Akai, T. (1982). A combined estimator of a common parameter, *Keio Science and Technology Reports*, **35**, 93-104.

[9] Antoch, J. (1984). Behaviour of estimators of location in non-regular cases: A Monte Carlo study, Asymptotic Statistics 2, *the Proceedings of the 3rd Prague Symposium on Asymptotic Statistics*, 185-195, North-Holland, Amsterdam.

[10] Ibragimov, I. A. and Has'minskii, R. Z. (1981). *Statistical Estimation: Asymptotic Theory*, Springer, New York.

[11] Jurečková, J. (1981). Tail-behaviour of location estimators in non-regular cases, *Commentationes Mathematicae Universitatis Carolinae*, **22**, 365-375.