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ASYMPTOTIC PROPERTIES OF THE MAXIMUM LIKELIHOOD ESTIMATE IN THE FIRST ORDER AUTOREGRESSIVE PROCESS

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Summary

In this paper we obtain an asymptotic expansion of the distribution of the maximum likelihood estimate (MLE) $\hat{\alpha}_{ML}$ based on T observations from the first order Gaussian process up to the term of order T^{-1}. The expansion is used to compare $\hat{\alpha}_{ML}$ with a generalized estimate $\hat{\alpha}_{c_1,c_2}$ including the least square estimate (LSE) $\hat{\alpha}_{LS}$, based on the asymptotic probabilities around the true value of the estimates up to the terms of order T^{-1}. It is shown that $\hat{\alpha}_{ML}$ (or the modified MLE $\hat{\alpha}^*_{ML}$) is better than $\hat{\alpha}_{c_1,c_2}$ (or the modified estimate $\hat{\alpha}^*_{c_1,c_2}$). Further, we note that $\hat{\alpha}^*_{ML}$ does not attain the bound for third order asymptotic median unbiased estimates.

1. Introduction

We consider the first order autoregressive Gaussian process which satisfies the stochastic equation

$$(1.1)\qquad y_t=\alpha y_{t-1}+u_t \qquad (t=\cdots,-1,0,1,\cdots)$$

where α is an unknown parameter, $|\alpha|<1$ and u_t are independent identically distributed as $N(0,\sigma^2)$. Phillips [9], [10] obtained an asymptotic expansion of the distribution of the least square estimate (LSE) $\hat{\alpha}_{LS}$ of α based on T observations from the process (1.1) up to the term of order T^{-1}. Ochi [7] extended the expansion to the case of a generalized estimate $\hat{\alpha}_{c_1,c_2}$ including $\hat{\alpha}_{LS}$.

In this paper we obtain an asymptotic expansion of the distribution of the maximum likelihood estimate (MLE) $\hat{\alpha}_{ML}$ up to the term of order T^{-1}. Using the expansion, we compare $\hat{\alpha}_{ML}$ with $\hat{\alpha}_{c_1,c_2}$ in terms of their probabilities of concentration around of the true value. It is

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shown that $\hat{\alpha}_{ML}$ is better than the estimate $\hat{\alpha}_{c_1,c_2}$ with $c_1+c_2=1$. Such comparisons for a class of asymptotically median unbiased (AMU) estimates was studied by Akahira [1], [2]. He showed that the second AMU estimates $\hat{\alpha}^*_{ML}$ and $\hat{\alpha}^*_{LS}$ are second order asymptotically efficient. The study is based on the asymptotic distributions of $\hat{\alpha}^*_{ML}$ and $\hat{\alpha}^*_{LS}$ up to the term of order $T^{-1/2}$. We note that the difference between $\hat{\alpha}^*_{ML}$ and $\hat{\alpha}^*_{LS}$ (or $\hat{\alpha}^*_{c_1,c_2}$) appears in the T^{-1}-terms of their asymptotic distributions. It is shown that $\hat{\alpha}^*_{ML}$ is better than $\hat{\alpha}^*_{c_1,c_2}$, but $\hat{\alpha}^*_{ML}$ does not attain the bound for third order AMU estimates.

2. Preliminaries

Let $\boldsymbol{y}=(y_1,\cdots,y_T)'$ be a vector of random variables forming the first order Gaussian process (1.1). Then the density of $\boldsymbol{y}$ is written as

$$f(\boldsymbol{y};\alpha,\sigma^2)=(2\pi\sigma^2)^{-T/2}(1-\alpha^2)^{1/2} \cdot \exp\left[-(2\sigma^2)^{-1}\{(1+\alpha^2)TX_2-2\alpha TX_1+X_3\}\right] \tag{2.1}$$

where $X_1=(1/T)\sum_{t=2}^{T} y_t y_{t-1}$, $X_2=(1/T)\sum_{t=2}^{T-1} y_t^2$ and $X_3=y_1^2+y_T^2$. Consider a class of the estimates of α defined by

$$\hat{\alpha}_{c_1,c_2}=X_1/\{c_1y_1^2+c_2y_T^2+X_2\} \tag{2.2}$$

where c_1 and c_2 are constants. Some well known estimates are the special cases $\hat{\alpha}_{c_1,c_2}$ with suitable constants c_1 and c_2. When $c_1=1$ and $c_2=0$, $\hat{\alpha}_{1,0}$ is the LS estimate $\hat{\alpha}_{LS}$. It is known (Ochi [7]) that

$$\begin{aligned}\Pr\{\sqrt{T}(\hat{\alpha}_{c_1,c_2}-\alpha)/(1-\alpha^2)^{1/2}\leqq x\} &=\Phi(x)-\phi(x)\Big[-\frac{\alpha}{\sqrt{T}(1-\alpha^2)^{1/2}}(x^2+1+c)\\ &\quad+\frac{1}{4T(1-\alpha^2)}\{(1-\alpha^2-8\alpha^2(c_1-1)(c_2-1)-2c(2-3c\alpha^2))x\\ &\quad+(4\alpha^2c-1-\alpha^2)x^3+2\alpha^2x^5\}\Big]+o(T^{-1})\end{aligned} \tag{2.3}$$

where $\Phi(x)$ is the distribution function of $N(0,1)$, $\phi(x)$ is its density function and $c=c_1+c_2-1$. Let $\hat{\alpha}_0=\hat{\alpha}_{0,0}=X_1/X_2$. Then, putting $c_1=c_2=0$ in (2.3) we have

$$\Pr\{\sqrt{T}(\hat{\alpha}_0-\alpha)/(1-\alpha^2)^{1/2}\leqq x\}=F_0(x)+o(T^{-1}) \tag{2.4}$$

where

$$F_0(x)=\Phi(x)+\phi(x)\Big[\frac{\alpha}{\sqrt{T}(1-\alpha^2)^{1/2}}x^2 \tag{2.5}$$

$$+\frac{1}{4T(1-\alpha^2)}\{(3\alpha^2-5)x+(5\alpha^2+1)x^3-2\alpha^2x^5\}\Big] .$$

3. Asymptotic expansion of the maximum likelihood estimate

From (2.1) we can see that the MLE of α is given as a solution of

$$\hat{\alpha}_{\mathrm{ML}}^3\left(1-\frac{1}{T}\right)X_2-\hat{\alpha}_{\mathrm{ML}}^2\left(1-\frac{2}{T}\right)X_1-\hat{\alpha}_{\mathrm{ML}}\left\{\left(1+\frac{1}{T}\right)X_2+\frac{1}{T^2}X_3\right\}+X_1=0 . \tag{3.1}$$

Anderson ([4], p. 369) gave an approximation

$$\tilde{\alpha}_{\mathrm{ML}}=(1-T^{-1})(X_1/X_2) \tag{3.2}$$

for $\hat{\alpha}_{\mathrm{ML}}$. Since $\tilde{\alpha}_{\mathrm{ML}}$ is an asymptotic solution of $\hat{\alpha}_{\mathrm{ML}}$ with an error of the term of $O(T^{-2})$, it is conjectured that

$$\begin{aligned}\Pr(\sqrt{T}(\hat{\alpha}_{\mathrm{ML}}-\alpha)/(1-\alpha^2)^{1/2}\leqq x)\\ =\Pr(\sqrt{T}(\tilde{\alpha}_{\mathrm{ML}}-\alpha)/(1-\alpha^2)^{1/2}\leqq x)+o(T^{-1}) .\end{aligned} \tag{3.3}$$

The distribution of $\sqrt{T}(\tilde{\alpha}_{\mathrm{ML}}-\alpha)$ is much simpler than that of $\sqrt{T}(\hat{\alpha}_{\mathrm{ML}}-\alpha)$. It is easily seen that

$$\begin{aligned}\sqrt{T}(\tilde{\alpha}_{\mathrm{ML}}-\alpha)=&\Big\{(1-\alpha^2)\left(1-\frac{1}{T}\right)U_1\\ &-\alpha(1-\alpha^2)U_2-\frac{1}{\sqrt{T}}\alpha\Big\}\left\{1+\frac{1}{\sqrt{T}}(1-\alpha^2)U_2\right\}^{-1}\end{aligned}$$

where $U_1=\sqrt{T}(X_1-\alpha(1-\alpha^2)^{-1})$ and $U_2=\sqrt{T}(X_2-(1-\alpha^2)^{-1})$. Therefore, if the existence of a valid expansion for the distribution of U_1 and U_2 is assured, then the existence of a valid expansion for the distribution of $\sqrt{T}(\tilde{\alpha}_{\mathrm{ML}}-\alpha)$ is also assured. It may be noted that the existence of a valid expansion for the distribution of U_1 and U_2 has been proved by Durbin [6] in the circular case. In the following we shall prove the formula (3.3) under the assumption that the existence of a valid expansion for the distribution of U_1 and U_2 is assured. So, the proof of the existence is left for the non-circular case.

When we treat the distribution of $\hat{\alpha}_{\mathrm{ML}}$, we may assume $\sigma^2=1$ without loss of generality. The MLE $\hat{\alpha}_{\mathrm{ML}}$ can be expressed in terms of U_1, U_2 and $U_3=X_3/\{4(1-\alpha^2)\}$. We use the following Lemmas 1 and 2.

LEMMA 1. *Let J_T be the set of U_1, U_2 and U_3 such that $|U_i|<2\log T$ $(i=1, 2, 3)$. Then*

$$\Pr(J_T^c)=O(T^{-2}) . \tag{3.4}$$

PROOF. It is sufficient to show that $\Pr(|U_i|>2\log T)=O(T^{-2})$, $i=$

1, 2, 3. We use the moment generating function of (U_1, U_2), which is equal to

$$M(\theta_1, \theta_2)=\mathrm{E}[\exp(\theta_1 U_1+\theta_2 U_2)]$$
$$=\exp\left[\frac{1}{2}(1-\alpha^2)^{-3}\{(1+4\alpha^2-\alpha^4)\theta_1^2+8\alpha\theta_1\theta_2+2(1+\alpha^2)\theta_2^2\}\right](1+O(T^{-1/2})) \qquad \text{(cf. Ochi [7])}.$$

Using a Tchebycheff-type inequality we obtain

$$\Pr(U_1>2\log T)\leqq\exp(-2\log T)\,\mathrm{E}[\exp(U_1)]$$
$$=(1/T)^2M(1, 0)=O(T^{-2}).$$

Similarly $\Pr(U_2>2\log T)=O(T^{-2})$. Further we have

$$\Pr(U_3>2\log T)\leqq\Pr(|y_1/(1-\alpha^2)^{1/2}|\geqq 2(\log T)^{1/2})$$
$$+\Pr(|y_2/(1-\alpha^2)^{1/2}|\geqq 2(\log T)^{1/2})$$
$$=O(T^{-2}) \qquad \text{(see Anderson [5])}.$$

This completes the proof.

LEMMA 2. *Let*

$$\sqrt{T}(\hat{\alpha}_{\mathrm{ML}}-\alpha)=\sqrt{T}(\tilde{\alpha}_{\mathrm{ML}}-\alpha)+T^{-3/2}l.$$

Then for any fixed δ *such that* $0<\delta<1/2$,

$$\Pr(|T^{-3/2}l|>T^{-1-\delta})=O(T^{-2}).$$

PROOF. Considering a Taylor expansion of $\hat{\alpha}_{\mathrm{ML}}$ about $X_1=\alpha/(1-\alpha^2)$, $X_2=1/(1-\alpha^2)$ and $X_3=0$ we can write

$$\sqrt{T}(\hat{\alpha}_{\mathrm{ML}}-\alpha)=\sum_{j=1}^{3}T^{-(j-1)/2}\alpha^{(j)}+T^{-3/2}l_1 \tag{3.5}$$

where each of $\alpha^{(j)}$ is a polynomial of degree j in U_1, U_2 and U_3 and $T^{-2}l_1$ is the usual remainder term in the Taylor expansion plus T^{-2} times a polynomial of $T^{-\beta/2}U_i$ ($\beta=0, 1, \cdots, 4$, $i=1, 2, 3$). Then using Lemma 1 it is seen that for any fixed δ such that $0<\delta<1/2$,

$$\Pr(|T^{-3/2}l_1|>T^{-1-\delta})=O(T^{-2}). \tag{3.6}$$

By substituting (3.5), $X_1=\alpha(1-\alpha^2)^{-1}+(1/\sqrt{T})U_1$, $X_2=(1-\alpha^2)^{-1}+(1/\sqrt{T})U_2$, and $X_3=4(1-\alpha^2)U_3$ to (3.1) we can find that

$$\alpha^{(1)}=(1-\alpha^2)(U_1-\alpha U_2), \qquad \alpha^{(2)}=-\alpha-(1-\alpha^2)\alpha^{(1)}U_2, \tag{3.7}$$
$$\alpha^{(3)}=-\alpha^{(1)}+(1-\alpha^2)^2U_2^2\alpha^{(1)}.$$

Applying the same method to $\tilde{\alpha}_{\mathrm{ML}}$, it can be also seen that

$$\sqrt{T}(\tilde{\alpha}_{\mathrm{ML}}-\alpha)=\sum_{j=0}^{3} T^{-(j-1)/2}\alpha^{(j)}+T^{-3/2}l_2 \tag{3.8}$$

where $\alpha^{(j)}$ are the same ones as in (3.7) and l_2 is a remainder term satisfying the same properties as the l_1 in (3.5). This implies Lemma 2.

In order to obtain the formula (3.3) we use a well known inequality for any random variables Y_1 and Y_2

$$\begin{aligned}&\Pr(Y_1\leqq a-h)-\Pr(|Y_2|>h)\\&\leqq\Pr(Y_1+Y_2\leqq a)\leqq\Pr(Y_1\leqq a+h)+\Pr(|Y_2|>h)\end{aligned} \tag{3.9}$$

where a is any number and h is any positive number. Let $Y_1=\sqrt{T}\cdot(\tilde{\alpha}_{\mathrm{ML}}-\alpha)$, $Y_2=T^{-3/2}l$ and $h=T^{-1-\delta}$ in (3.9). Then, assuming that Y_1 has a valid expansion as in (2.3), we have $\Pr(Y_1\leqq a\pm h)=\Pr(Y_1\leqq a)+o(T^{-1})$. Therefore we have (3.3).

From (3.3) we have

$$\begin{aligned}&\Pr(\sqrt{T}(\hat{\alpha}_{\mathrm{ML}}-\alpha)/(1-\alpha^2)^{1/2}\leqq x)\\&=F_0((1-T^{-1})^{-1}\{x+\alpha/(\sqrt{T}(1-\alpha^2)^{1/2})\})+o(T^{-1})\,.\end{aligned} \tag{3.10}$$

Simplifying the last expression of (3.10) we have the following theorem.

THEOREM 1. *Let* $\hat{\alpha}_{\mathrm{ML}}$ *be the* MLE *given by a solution of* (3.1). *Then it holds that*

$$\Pr(\sqrt{T}(\hat{\alpha}_{\mathrm{ML}}-\alpha)/(1-\alpha^2)^{1/2}\leqq x)=F_{\mathrm{ML}}(x)+o(T^{-1}) \tag{3.11}$$

where

$$\begin{aligned}F_{\mathrm{ML}}(x)=\Phi(x)+\phi(x)\Big[&\frac{\alpha}{\sqrt{T}(1-\alpha^2)^{1/2}}(x^2+1)+\frac{1}{4T(1-\alpha^2)}\\&\cdot\{(5\alpha^2-1)x+(\alpha^2+1)x^3-2\alpha^2x^5\}\Big]\,.\end{aligned} \tag{3.12}$$

When $c_1+c_2=1$, we can write the formula (2.3) as

$$\begin{aligned}&\Pr(\sqrt{T}(\hat{\alpha}_{c_1,c_2}-\alpha)/(1-\alpha^2)^{1/2}\leqq x)\\&=F_{\mathrm{ML}}(x)-\frac{1}{T(1-\alpha^2)}\alpha^2\{2c_1(c_1-1)+1\}x\phi(x)+o(T^{-1})\,.\end{aligned} \tag{3.13}$$

This implies the following theorem.

THEOREM 2. *Let* $\hat{\alpha}_{c_1,c_2}$ *be the estimate of* α *defined by* (2.2). *Assume that* $c_1+c_2=1$. *Then, for any positive numbers* a *and* b

$$\lim_{T\to\infty}(\sqrt{T})^{s-1}[\Pr(-a<\sqrt{T}(\hat{\alpha}_{\mathrm{ML}}-\alpha)/(1-\alpha^2)^{1/2}<b) \tag{3.14}$$

$$-\Pr(-a<\sqrt{T}(\hat{\alpha}_{c_1,c_2}-\alpha)/(1-\alpha^2)^{1/2}<b)]$$
$$=\begin{cases} 0, & 1\leqq s<3, \\ \alpha^2(1-\alpha^2)^{-1}\left\{2\left(c_1-\frac{1}{2}\right)^2+\frac{1}{2}\right\}\{a\phi(a)+b\phi(b)\}, & s=3. \end{cases}$$

From Theorem 2 we can say that if we use the asymptotic probabilities around the true value of the estimates up to order T^{-1} as a criterion for comparing estimates, $\hat{\alpha}_{ML}$ is better than $\hat{\alpha}_{LS}$.

4. The modified maximum likelihood estimate

In general, if an estimate $\hat{\alpha}$ satisfies

$$(4.1)\qquad \lim_{T\to\infty}(\sqrt{T})^{s-1}\left|\Pr(\hat{\alpha}\leqq\alpha)-\frac{1}{2}\right|=\lim_{T\to\infty}(\sqrt{T})^{s-1}\left|\Pr(\hat{\alpha}\geqq\alpha)-\frac{1}{2}\right|=0.$$

Then $\hat{\alpha}$ is called an sth order AMU estimate. Akahira [1], [2] considered the second order AMU estimates $\hat{\alpha}^*_{ML}$ and $\hat{\alpha}^*_{LS}$ defined by

$$(4.2)\qquad \hat{\alpha}^*_{ML}=(1+T^{-1})\hat{\alpha}_{ML} \quad \text{and} \quad \hat{\alpha}^*_{LS}=(1+T^{-1})\hat{\alpha}_{LS}$$

respectively. He showed that for any positive numbers a and b

$$(4.3)\qquad \lim_{T\to\infty}\sqrt{T}[\Pr(-a<\sqrt{T}(\hat{\alpha}^*_{ML}-\alpha)<b)-\Pr(-a<\sqrt{T}(\hat{\alpha}^*_{LS}-\alpha)<b)]=0.$$

In order to clear the difference between $\hat{\alpha}^*_{ML}$ and $\hat{\alpha}^*_{LS}$ we have to study the T^{-1}-terms of their asymptotic distributions. First we note that $\hat{\alpha}^*_{ML}$ is a third order AMU estimate. This result follows from that

$$(4.4)\qquad \begin{aligned} &\Pr(\sqrt{T}(\hat{\alpha}^*_{ML}-\alpha)/(1-\alpha^2)^{1/2}\leqq x) \\ &=F_{ML}((1+T^{-1})^{-1}\{x-\alpha/(\sqrt{T}(1-\alpha^2)^{1/2})\})+o(T^{-1}) \\ &=F_0(x)+o(T^{-1}). \end{aligned}$$

and $F_0(0)=1/2+o(T^{-1})$. The function $F_0((1-\alpha^2)^{1/2}x)$ is called the third order asymptotic distribution of $\hat{\alpha}^*_{ML}$. Similarly we may define the third order AMU estimate of $\hat{\alpha}_{c_1,c_2}$ by

$$(4.5)\qquad \hat{\alpha}^*_{c_1,c_2}=\{1+(c_1+c_2)/T\}\hat{\alpha}_{c_1,c_2}.$$

Using (2.3) we have

$$(4.6)\qquad \begin{aligned} &\Pr(\sqrt{T}(\hat{\alpha}^*_{c_1,c_2}-\alpha)/(1-\alpha^2)^{1/2}\leqq x) \\ &=\Pr(\sqrt{T}(\hat{\alpha}_{c_1,c_2}-\alpha)/(1-\alpha^2)^{1/2}\leqq\{1+(c_1+c_2)/T\}^{-1} \\ &\qquad\cdot\{x-(c_1+c_2)\alpha/(\sqrt{T}(1-\alpha^2)^{1/2})\}) \\ &=F_0(x)-\frac{1}{T(1-\alpha^2)}(c_1^2+c_2^2)\alpha^2x\phi(x)+o(T^{-1}). \end{aligned}$$

From (4.4) and (4.6) we have the following theorem.

THEOREM 3. *Let $\hat{\alpha}^*_{ML}$ and $\hat{\alpha}^*_{c_1,c_2}$ be the third order AMU estimates defined by* (4.2) *and* (4.5), *respectively. Then it holds that for any positive numbers a and b*

$$(4.7)\qquad \lim_{T\to\infty} (\sqrt{T})^{s-1}[\Pr(-a<\sqrt{T}(\hat{\alpha}^*_{ML}-\alpha)<b)] \\ -[\Pr(-a<\sqrt{T}(\hat{\alpha}^*_{c_1,c_2}-\alpha)<b)] \\ =\begin{cases} 0\,, & 1\leqq s<3 \\ \alpha^2(1-\alpha^2)^{-1}(c_1^2+c_1^2)\{\tilde{a}\phi(\tilde{a})+\tilde{b}\phi(\tilde{b})\}\geqq 0\,, & s=3\,. \end{cases}$$

where $\tilde{a}=a/(1-\alpha^2)^{1/2}$ *and* $\tilde{a}=b/(1-\alpha^2)^{1/2}$.

As a special case of (4.7) we have

$$(4.8)\qquad \lim_{T\to\infty} T[\Pr(-a<\sqrt{T}(\hat{\alpha}^*_{ML}-\alpha)<b)-\Pr(-a<\sqrt{T}(\hat{\alpha}^*_{LS}-\alpha)<b)] \\ =\alpha^2(1-\alpha^2)^{-1}\{\tilde{a}\phi(\tilde{a})+\tilde{b}\phi(\tilde{b})\}>0\,.$$

From (4.7) it is easy to see that the best one among the estimates $\hat{\alpha}^*_{c_1,c_2}$ is not $\hat{\alpha}^*_{LS}$, but $\hat{\alpha}_0=X_1/X_2$ and $\hat{\alpha}_0$ has the same asymptotic concentration probability as $\hat{\alpha}^*_{ML}$ up to order T^{-1}.

An estimate is sometimes so modified as to satisfy a higher order asymptotic unbiasedness. Let

$$(4.9)\qquad \hat{\alpha}^{**}_{ML}=(1+2T^{-1})\hat{\alpha}_{ML} \quad\text{and}\quad \hat{\alpha}^{**}_{c_1,c_2}=(1+(c_1+c_2+1)T^{-1})\hat{\alpha}_{c_1,c_2}\,.$$

Then

$$(4.10)\qquad \mathrm{E}[\hat{\alpha}^{**}_{ML}]=\alpha+o(T^{-3/2}) \quad\text{and}\quad \mathrm{E}[\hat{\alpha}^{**}_{c_1,c_2}]=\alpha+o(T^{-3/2})\,.$$

By the same argument as in the case of AMU estimates we have the following theorem.

THEOREM 4. *Let $\hat{\alpha}^{**}_{ML}$ and $\hat{\alpha}^{**}_{c_1,c_2}$ be the estimates of α defined by* (4.9). *Then it holds that for any positive numbers a, b and s* ($1\leqq s\leqq 3$)

$$(4.11)\qquad \lim_{T\to\infty} (\sqrt{T})^{s-1}[\Pr(-a<\sqrt{T}(\hat{\alpha}^{**}_{ML}-\alpha)<b) \\ -\Pr(-a<\sqrt{T}(\alpha^{**}_{c_1,c_2}-\alpha)<b)]$$

is equal to the right-hand side of (4.7).

5. The bound for third order AMU estimates

An estimate is called to be third order asymptotically efficient if the third order asymptotic distribution of it attains uniformly the bound $F_\lambda(x:\alpha)$ for third order AMU estimates. Following a general theory for higher order asymptotic efficiency (for a summary, see Akahira and Takeuchi [3], Pfanzagl [8]), the bound may be defined as follows. Let α_0 be arbitrary but fixed in $(-1, 1)$. Consider the problem of testing $H: \alpha=\alpha_0+x/\sqrt{T}$ $(=\alpha_1,\ x>0)$ against alternative $K: \alpha=\alpha_0$ with significance level $1/2+o(T^{-1})$. Then the best test procedure is to reject H when $\lambda=f(\boldsymbol{y}: \alpha_0, \sigma^2)/f(\boldsymbol{y}: \alpha_1, \sigma^2)>\lambda_0$. The test procedure is equivalent to reject H when

$$R=\sqrt{T}\{X_1-(\alpha_0+x/(2\sqrt{T}))X_2\}\leqq d \tag{5.1}$$

where d is so determined as to satisfy

$$\lim_{T\to\infty} T\left[\Pr(R\leqq d\,|\,H)-\frac{1}{2}\right]=0\ . \tag{5.2}$$

The bound $F_\lambda(x: \alpha)$ for $x>0$ is defined by

$$\Pr(R\leqq d\,|\,K)=F_\lambda(x: \alpha_0)+o(T^{-1})\ . \tag{5.3}$$

The bound $F_\lambda(x: \alpha)$ for $x<0$ may be also defined by the left-hand side of (5.3). Then the bound satisfies

$$\begin{aligned} &\lim_{T\to\infty} T[F_\lambda(x: \alpha)-\Pr(\sqrt{T}(\hat{\alpha}-\alpha)\leqq x)]\geqq 0 \qquad \text{for } x>0\ ,\\ &\lim_{T\to\infty} T[F_\lambda(x: \alpha)-\Pr(\sqrt{T}(\hat{\alpha}-\alpha)\leqq x)]\leqq 0 \qquad \text{for } x<0 \end{aligned} \tag{5.4}$$

where $\hat{\alpha}$ is any third order AMU estimate having a third order asymptotic distribution.

When we obtain $F_\lambda(x: \alpha)$, we may assume $\sigma^2=1$ without loss of generality. The bound $F_\lambda(x: \alpha)$ is obtained by considering asymptotic expansions of the distribution of R up to the term of order T^{-1} when $\alpha=\alpha_1$ and α_0. If the existence of a valid expansion for the distribution of R is assured, the expansion may be obtained by formally inverting the characteristic function of $R=\sqrt{T}[X_1-\{\alpha_1+(-x/2)/\sqrt{T}\}X_2]$. Here we shall find the asymptotic expansion by formally inverting the characteristic function of R obtained from

$$\begin{aligned} &\log \mathrm{E}[\exp(itR)\,|\,\alpha=\alpha_1]\\ &\quad=\frac{1}{2(1-\alpha_1^2)}\{itx+(it)^2\}+\frac{\alpha_1}{\sqrt{T}(1-\alpha_1^2)^2}\{(1-\alpha_1^2)it+x(it)^2+(it)^3\}\\ &\quad\quad+\frac{1}{T(1-\alpha_1^2)^3}\Big[-(1-\alpha_1^2)xit+\Big\{\frac{1}{2}(3\alpha_1^2-1)(1-\alpha_1^2)\\ &\quad\quad+\frac{1}{4}(1+\alpha_1^2)x^2\Big\}(it)^2+(1+2\alpha_1^2)x(it)^3+\frac{1}{4}(7\alpha_1^2+3)(it)^4\Big]\\ &\quad\quad+O(T^{-3/2}) \end{aligned} \tag{5.5}$$

which is implicitly given in Ochi [7]. From the asymptotic expansion of the distribution of R under K we obtain

$$d=\left(\frac{1}{2}\tilde{x}+\Delta\right)\Big/(1-\alpha_0^2)^{1/2} \tag{5.6}$$

where $\tilde{x}=x/(1-\alpha_0^2)^{1/2}$ and $\Delta=\alpha_0\tilde{x}^2/\{\sqrt{T}(1-\alpha_0^2)^{1/2}\}+\{(\alpha_0^2-2)\tilde{x}+(1/2)(3\alpha_0^2+1)\tilde{x}^3\}/\{T(1-\alpha_0^2)\}$. In order to evaluate the left-hand side of (5.3) we consider the distribution of R under $\alpha=\alpha_0$. Noting that the log-characteristic function of R under $\alpha=\alpha_0$ is obtained from (5.5) by substituting $-x$ and α_0 to x and α_1, respectively, it is shown that

$$\Pr\left((1-\alpha_0^2)^{1/2}R+\frac{1}{2}\tilde{x}\leqq z\,|\,\alpha=\alpha_0\right)=F_\beta(z)+o(T^{-1}) \tag{5.7}$$

where

$$\begin{aligned} F_\beta(z)=\Phi(z)+\phi(z)\Big[&\frac{\alpha_0}{\sqrt{T}(1-\alpha_0^2)^{1/2}}(\tilde{x}z-z^2)+\frac{1}{T(1-\alpha_0^2)} \\ &\cdot\Big\{(\alpha_0^2-2)\tilde{x}-\left(\frac{5}{4}\alpha_0^2-\frac{11}{4}\right)z+\left(\frac{9}{4}\alpha_0^2-\frac{3}{4}\right)z^3+(1-3\alpha_0^2)\tilde{x}z^2 \\ &+\left(\frac{5}{4}\alpha_0^2-\frac{1}{4}\right)\tilde{x}^2z-\frac{1}{2}\alpha_0^2(\tilde{x}^2-2\tilde{x}z+z^2)z^3\Big\}\Big]\,. \end{aligned} \tag{5.8}$$

From (5.7) we have

$$\begin{aligned} \Pr(R\leqq d\,|\,\alpha=\alpha_0)&=F_\beta(\tilde{x}+\Delta)+o(T^{-1}) \\ &=\Phi(\tilde{x})+\phi(\tilde{x})\Big[\frac{\alpha_0}{\sqrt{T}(1-\alpha^2)^{1/2}}\tilde{x}^2+\frac{1}{4T(1-\alpha_0^2)} \\ &\quad\cdot\{(3\alpha_0^2-5)\tilde{x}+(4\alpha_0^2+2)\tilde{x}^3-2\alpha_0^2\tilde{x}^5\}\Big]+o(T^{-1})\,. \end{aligned} \tag{5.9}$$

This implies the following theorem.

THEOREM 5. *The bound $F_\lambda(x:\alpha)$ for third order* AMU *estimates of α is given by*

$$F_\lambda(x:\alpha)=F_0(\tilde{x})+\phi(\tilde{x})\tilde{x}^3/(4T) \tag{5.10}$$

where $\tilde{x}=x/(1-\alpha^2)^{1/2}$ and $F_0(x)$ is defined by (2.5).

It is known (Akahira [1], [2]) that $\hat{\alpha}^*_{\mathrm{ML}}$ and $\hat{\alpha}^*_{\mathrm{LS}}$ is second order asymptotically efficient. Theorem 3 shows that $\hat{\alpha}^*_{\mathrm{ML}}$ is better than $\hat{\alpha}^*_{\mathrm{LS}}$. However, Theorem 5 shows that $\hat{\alpha}^*_{\mathrm{ML}}$ does not attain the bound for third order AMU estimates, since the third order asymptotic distribution of $\hat{\alpha}^{**}_{\mathrm{ML}}$ is $F_0(\tilde{x})$.

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REFERENCES

[1] Akahira, M. (1975). A note on the second order asymptotic efficiency of estimators in an autoregressive process, *Rep. Univ. Electro-comm. Sci. & Tech. Sect.*, **26**, 143-149.

[2] Akahira, M. (1979). On the second order asymptotic optimality of estimators in an autoregressive process, *Rep. Univ. Electro-comm. Sci. & Tech. Sect.*, **29**, 213-218.

[3] Akahira, M. and Takeuchi, K. (1980). Asymptotic Efficiency of Statistical Estimators: Concepts and Higher Order Asymptotic Efficiency, *Lecture Note in Statistics*, Vol. 7, Springer, Deu.

[4] Anderson, T. W. (1971). *The Statistical Analysis of Time Series*, Wiley, New York.

[5] Anderson, T. W. (1974). An asymptotic expansion of the distribution of the limited information maximum likelihood estimate of a coefficient in a simultaneous equation system, *J. Amer. Statist. Ass.*, **69**, 565-573.

[6] Durbin, J. (1980). Approximations for densities of sufficient estimators, *Biometrika*, **67**, 311-333.

[7] Ochi, Y. (1981). Asymptotic expansions for the distribution of an estimator in the first order autoregressive process, *Hiroshima Univ. Stat. Res. Group Tech. Rep.*, No. 42.

[8] Pfanzagl, J. (1973). Asymptotic expansions in parametric statistical theory, in *Developments in Statistics*, Vol 3, (ed. P. R. Krishnaiah), Academic Press, New York, 1-97.

[9] Phillips, P. C. B. (1977). Approximations to some finite sample distributions associated with a first order stochastic difference equation, *Econometrica*, **45**, 463-486.

[10] Phillips, P. C. B. (1978). Edgeworth and saddlepoint approximations in the first-order noncircular autoregression, *Biometrika*, **65**, 91-98.