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# NOTE ON THE CONSTRUCTION OF OPTIMUM CHEMICAL BALANCE WEIGHING DESIGNS

S. KAGEYAMA AND G. M. SAHA

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## Summary

Sufficient conditions for the existence of certain optimum chemical balance weighing designs are investigated.

## 1. Introduction

As relations between BIB designs and optimum weighing designs, Saha [4] has proved constructively the following two theorems.

THEOREM 1. *The existence of a* BIB *design with parameters* $v$, $b$, $r$, $k$, $\lambda$, *satisfying* $b \leqq 4(r-\lambda)$, *implies the existence of an optimum chemical balance weighing design for* $v$ *objects in* $4(r-\lambda)$ *weighings.*

THEOREM 2. *The existence of an affine resolvable* BIB *design with parameters* $v$, $b=2r$, $r$, $k$, $\lambda$, *implies the existence of an optimum chemical balance weighing design for* $r$ *objects in* $v$ *weighings.*

This note investigates a BIB design with parameters $v$, $b$, $r$, $k$, $\lambda$, satisfying $b \leqq 4(r-\lambda)$, and tabulates the parameters (in the practical range) of BIB designs which validate Theorems 1 and 2. Furthermore, some related results are similarly given with illustrations.

## 2. Discussions

Concerning a condition given in Theorem 1, we at first have

PROPOSITION 1. An affine resolvable BIB design with parameters $v$, $b$, $r$, $k$, $\lambda$ satisfies $b \leqq 4(r-\lambda)$ if and only if it has one of the forms: $v=9$, $b=12$, $r=4$, $k=3$, $\lambda=1$; $v=4(t+1)$, $b=2(4t+3)$, $r=4t+3$, $k=2(t+1)$, $\lambda=2t+1$ for a non-negative integer $t$.

PROOF. It is well known (Bose [1]) that the parameters of an affine resolvable BIB design are expressed in terms of only two integral parameters $n\,(\geqq 2)$ and $t\,(\geqq 0)$ as $v=n^2[(n-1)t+1]$, $b=n(n^2t+n+1)$, $r=n^2t+n+1$, $k=n[(n-1)t+1]$, $\lambda=nt+1$. In this case, it follows that $b\leqq 4(r-\lambda)$ is equivalent to $n[(n-2)^2t+n-3]\leqq 0$ which holds if and only if (i) $n=3$ and $t=0$, or (ii) $n=2$ and any $t\geqq 0$. These cases yield the required result.

*Remark* 1. Proposition 1 shows that the existence of an affine resolvable BIB design with parameters $v$, $b=2r$, $r$, $k$, $\lambda$, which is used in Theorem 2, also implies the existence of an optimum chemical balance weighing design for $v$ objects in $4(r-\lambda)$ weighings. In fact, *any* BIB design with $b=2r$ yields an optimum chemical balance weighing design for $v$ objects in $4(r-\lambda)$ weighings, as will be seen in Proposition 2 and Remark 2.

Since the complement of a BIB design with parameters $v$, $b$, $r$, $k$, $\lambda$ is also a BIB design, and further a design and its complement either satisfies or does not satisfy $b\leqq 4(r-\lambda)$, we can assume, without loss of generality, $v\geqq 2k$.

PROPOSITION 2. In a BIB design with parameters $v\,(\geqq 2k)$, $b$, $r$, $k$ and $\lambda$, a relation $b\leqq 4(r-\lambda)$ holds if and only if $(v-\sqrt{v})/2\leqq k\leqq v/2$. In particular, $b=4(r-\lambda)$ is equivalent to $k=(v-\sqrt{v})/2$.

PROOF. $b\leqq 4(r-\lambda)$ yields, from $\lambda=r(k-1)/(v-1)$, $b\leqq 4r(v-k)/(v-1)$, i.e., $bk\leqq 4rk(v-k)/(v-1)$, which also yields $v\leqq 4k(v-k)/(v-1)$ being equivalent to $(v-\sqrt{v})/2\leqq k\leqq (v+\sqrt{v})/2$. Now, from $k\leqq v/2$, the proof is completed. (Converse is obvious.)

*Remark* 2. Theorem 1 can be rewritten as: The existence of a BIB design with parameters $v$, $b$, $r$, $k$, $\lambda$ having $k$ in the range $(v-\sqrt{v})/2\leqq k\leqq (v+\sqrt{v})/2$ implies the existence of an optimum chemical balance weighing design for $v$ objects in $4(r-\lambda)$ weighings. For example, a BIB design satisfying $(v-\sqrt{v})/2\leqq k\leqq v/2$ is characterized by $v=2k+l$ and $k\geqq l(l-1)/2$ for some non-negative integer $l$. This includes necessarily cases in which $v=2k$, $v=2k+1$ and $v=2k+2$. Incidentally, Proposition 2 shows that in Theorem 1 if the number, $v$, of objects to be estimated is a priori given, then there exist at most finite number of BIB designs, up to some duplications of the designs, in which one can utilize the BIB designs for construction of optimum chemical balance weighing designs.

*Remark* 3. Shrikhande [5] characterized the parameters of BIB designs satisfying $b=4(r-\lambda)$ as two series for positive integers $t$ and $m$

$v=[2(m+1)]^2$, $b=4t(m+1)$, $r=t(2m+1)$, $k=(m+1)(2m+1)$, $\lambda=mt$;

$v=(2m+3)^2$, $b=4t(2m+3)$, $r=4t(m+1)$, $k=(m+1)(2m+3)$, $\lambda=t(2m+1)$, which in fact satisfy the condition of attaining the bound in Proposition 2.

As a generalization of Theorem 1 of Saha [4], we can easily present the following by referring to his proof.

PROPOSITION 3. The existence of a pairwise-balanced design with parameters $v$, $b$, $r$ and $\lambda$, satisfying $b \leqq 4(r-\lambda)$, implies the existence of an optimum chemical balance weighing design for $v$ objects in $4(r-\lambda)$ weighings.

*Example.* We can construct a pairwise-balanced design from a BIB design by omitting some treatments. For example, a BIB design (of No. 9 in the tabulation) with parameters $v^*=10$, $b^*=15$, $r^*=6$, $k^*=4$, $\lambda^*=2$ yields a pairwise-balanced design with parameters $v=9$, $b=15$, $r=6$, $\lambda=2$, $k_j=4$ or 3. This design yields an optimum chemical balance weighing design for 9 objects in 16 weighings.

Note that if the pairwise-balanced design has a constant block size, then Proposition 3 yields Theorem 1 of Saha [4].

Similarly, consider a block design of the linked block type with parameters $v$, $b$, $k$, $\mu$, where $\mu$ is the number of treatments common to any two blocks. Then the dual of this design is a pairwise-balanced design with parameters $v^*=b$, $b^*=v$, $r^*=k$ and $\lambda^*=\mu$, in which $b^*-4(r^*-\lambda^*)=v-4(k-\mu)$. Thus, from Proposition 3, we have

PROPOSITION 4. The existence of a block design of the linked block type with parameters $v$, $b$, $r$ and $\mu$, satisfying $v \leqq 4(k-\mu)$, implies the existence of an optimum chemical balance weighing design for $b$ objects in $4(k-\mu)$ weighings.

As applications of Proposition 4, we have the following two results.

PROPOSITION 5. The existence of an affine resolvable BIB design with parameters $v=4(t+1)$, $b=2(4t+3)$, $r=4t+3$, $k=2(t+1)$, $\lambda=2t+1$ for a positive integer $t$ implies the existence of an optimum chemical balance weighing design for $4t+3$ objects in $4(t+1)$ weighings.

PROOF. In an affine resolvable BIB design with parameters $v$, $b$, $r$, $k$, $\lambda$, take from each complete replication set a block containing a given treatment, $\theta$, (say). For this collection of $r$ blocks delete $\theta$ from each block. The resulting design is a linked block design with parameters $v^*=v-1$, $b^*=r$, $r^*=\lambda$, $k^*=k-1$ and $\mu=k^2/v-1$, in which $\mu>0$ if and only if $k^2>v$. From Proposition 4, if $v-1 \leqq 4(k-k^2/v)$, we have an optimum chemical balance weighing design for $r$ objects in $4k(v-k)/v$ weighings. Furthermore, it is obvious that $v-1 \leqq 4(k-k^2/v)$ is equivalent to $(v-\sqrt{v})/2 \leqq k \leqq (v+\sqrt{v})/2$, which, from Propositions 1 and 2, and $k^2>v$, yields the required result.

Note (cf. Sprott [6]) that the affine resolvable BIB design in Proposition 5 always exists provided $4t+3$ is a prime or a prime power.

*Remark* 4. The series of weighing designs of Proposition 5 was also obtained in Saha [4]. But the method of construction used there is different from that indicated here. An advantage of the present method is that this can be generalized to affine resolvable PBIB designs also, as the following proposition shows.

PROPOSITION 6. The existence of an affine resolvable PBIB design with parameters $v$, $b$, $r$, $k$, $\lambda_i$, $n_i$, $p^i_{jl}$ $(i, j, l=1, 2, \cdots, m)$ satisfying

$$\sum_{i|\lambda_i>0} n_i \leqq 4k(v-k)/v$$

implies the existence of an optimum chemical balance weighing design for $r$ objects in $4k(v-k)/v$ weighings, where the summation extends over $i\,(=1, 2, \cdots$, or $m)$ such that $\lambda_i>0$.

PROOF. We can proceed in a manner similar to Proposition 5. But, here the number of treatments in the resulting linked block design is

$$v^* = \sum_{i|\lambda_i>0} n_i\ ,$$

where the summation extends over $i$ such that $\lambda_i>0$ for $i=1, 2, \cdots, m$.

*Example*. From Clatworthy [2], we present three examples.

(i) An affine resolvable semi-regular group divisible PBIB design, SR 71, with parameters $v=12$, $b=20$, $r=10$, $k=6$, $\lambda_1=4$, $\lambda_2=5$, $m=2$, $n=6$, $n_1=5$ and $n_2=6$ yields an optimum chemical balance weighing design for 10 objects in 12 weighings.

(ii) An affine resolvable Latin square type PBIB design, LS 98, with parameters $v=16$, $b=12$, $r=6$, $k=8$, $\lambda_1=4$, $\lambda_2=2$, $n_1=6$ and $n_2=9$ yields an optimum chemical balance weighing design for 6 objects in 16 weighings.

(iii) An affine resolvable Latin square type PBIB design, LS 100, with parameters $v=16$, $b=18$, $r=9$, $k=8$, $\lambda_1=3$, $\lambda_2=5$, $n_1=6$ and $n_2=9$ yields an optimum chemical balance weighing design for 9 objects in 16 weighings. Note that the structure of this design is different from that of the example just after Proposition 3.

*Remark*. A generalization of Propositions 5 and 6 to affine $\alpha$-resolvability is immediate. Most of existing affine resolvable PBIB designs have relatively small values of $r$, which, in Proposition 6, correspond to the number of objects in a weighing design. But, since the complement of an affine resolvable BIB design is in general an affine $\alpha$-resolvable BIB design, in this case we can present useful examples.

*Additional remark.* Let $H_i$ be Hadamard matrices for $i=1, 2, \cdots, l(\geqq 1)$ of order $n$. Then $\Theta=[H_1 : H_2 : \cdots : H_l]$ gives an optimum weighing design for $n$ objects in $ln$ weighings. Indeed, $\Theta\Theta'=lnI$, where $I$ is the identity matrix. If we take two Hadamard matrices, $H_1$ and $H_2$, of order $n$ with all $+1$'s in the first row, then $H$, given by

$$H=\begin{bmatrix} H_1 & H_2 \\ \underline{1}' & -\underline{1}' \end{bmatrix},$$

is an optimum chemical balance weighing design for $n+1$ objects in $2n$ weighings, where $\underline{1}'=(1, 1, \cdots, 1)$. If more weighings are wanted, we can repeat the above matrix $H$ the desired number of times. As another idea, we also can use a balanced orthogonal design to construct an optimum weighing design.

Finally, from a practical point of view such that one wants to use Theorems 1 and 2, and Proposition 5 for constructions of optimum chemical balance weighing designs, we shall tabulate the parameters of existent BIB designs satisfying $b \leqq 4(r-\lambda)$ with a useful range of $v \leqq 50$ and $r \leqq 20$ ($r \leqq 30$ for a symmetric BIB design), and of existent affine resolvable BIB designs with $b=2r$ such that $v \leqq 100$ and $r \leqq 20$. The reference number with asterisk (*) is an affine resolvable BIB design. Solutions for these designs can be found from tables of Takeuchi [7] and Kageyama [3]. Note that designs constructed by taking copies of the original BIB design are not listed.

| No. | $v$ | $k$ | $b$ | $r$ | $\lambda$ | No. | $v$ | $k$ | $b$ | $r$ | $\lambda$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1* | 4 | 2 | 6 | 3 | 1 | 20 | 16 | 6 | 24 | 9 | 3 |
| 2 | 5 | 2 | 10 | 4 | 1 | 21 | 16 | 6 | 40 | 15 | 5 |
| 3 | 6 | 2 | 15 | 5 | 1 | 22* | 16 | 8 | 30 | 15 | 7 |
| 4 | 6 | 3 | 10 | 5 | 2 | 23 | 17 | 8 | 34 | 16 | 7 |
| 5 | 7 | 3 | 7 | 3 | 1 | 24 | 18 | 9 | 34 | 17 | 8 |
| 6* | 8 | 4 | 14 | 7 | 3 | 25 | 19 | 9 | 19 | 9 | 4 |
| 7* | 9 | 3 | 12 | 4 | 1 | 26* | 20 | 10 | 38 | 19 | 9 |
| 8 | 9 | 4 | 18 | 8 | 3 | 27 | 21 | 9 | 35 | 15 | 6 |
| 9 | 10 | 4 | 15 | 6 | 2 | 28 | 21 | 10 | 42 | 20 | 9 |
| 10 | 10 | 5 | 18 | 9 | 4 | 29 | 23 | 11 | 23 | 11 | 5 |
| 11 | 11 | 4 | 55 | 20 | 6 | 30 | 27 | 13 | 27 | 13 | 6 |
| 12 | 11 | 5 | 11 | 5 | 2 | 31 | 31 | 15 | 31 | 15 | 7 |
| 13* | 12 | 6 | 22 | 11 | 5 | 32 | 35 | 17 | 35 | 17 | 8 |
| 14 | 13 | 5 | 39 | 15 | 5 | 33 | 36 | 15 | 36 | 15 | 6 |
| 15 | 13 | 6 | 26 | 12 | 5 | 34 | 36 | 15 | 48 | 20 | 8 |
| 16 | 14 | 7 | 26 | 13 | 6 | 35 | 39 | 19 | 39 | 19 | 9 |
| 17 | 15 | 6 | 35 | 14 | 5 | 36 | 43 | 21 | 43 | 21 | 10 |
| 18 | 15 | 7 | 15 | 7 | 3 | 37 | 47 | 23 | 47 | 23 | 11 |
| 19 | 16 | 6 | 16 | 6 | 2 | | | | | | |

Hiroshima University, Japan
Indian Statistical Institute, India

## References

[1] Bose, R. C. (1942). A note on the resolvability of balanced incomplete block designs, *Sankhyā*, **6**, 105-110.

[2] Clatworthy, W. H. (1973). Tables of Two-Associate-Class Partially Balanced Designs, *NBS Appl. Math. Ser.*, **63**.

[3] Kageyama, S. (1972). A survey of resolvable solutions of balanced incomplete block designs, *Int. Statist. Rev.*, **40**, 269-273.

[4] Saha, G. M. (1975). A note on relations between incomplete block and weighing designs, *Ann. Inst. Statist. Math.*, **27**, 387-390.

[5] Shrikhande, S. S. (1962). On a two-parameter family of balanced incomplete block designs, *Sankhyā*, A, **24**, 33-40.

[6] Sprott, D. A. (1956). Some series of balanced incomplete block designs, *Sankhyā*, **17**, 185-192.

[7] Takeuchi, K. (1962). A table of difference sets generating balanced incomplete block designs, *Rev. Int. Statist. Inst.*, **30**, 361-366.