UNIFORM ASYMPTOTIC JOINT NORMALITY OF A SET OF INCREASING NUMBER OF SAMPLE QUANTILES

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Summary

Type \((B)_d\) asymptotic normality of the joint distribution of sample quantiles is investigated when the number of sample quantiles increases as the sample size increases. This paper aims at a refinement of the original results by Ikeda and Matsunawa [3].

1. Introduction

Let \(X_{n_1} < X_{n_2} < \cdots < X_{n_n}\) be order statistics from a continuous distribution with pdf \(f(x)\) and cdf \(F(x)\). Let \(0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < 1\) be any given spacing of \(k\) probability levels, and take a corresponding set of \(k\) (\(<n\)) sample quantiles \(X_{n(k)} = (X_{n_{i_1}}, X_{n_{i_2}}, \ldots, X_{n_{i_k}})'\), with \(n_i = [(n+1)\lambda_i]\), \(i = 1, 2, \ldots, k\). Let the corresponding population quantiles be \(\xi_i \equiv F^{-1}(\lambda_i), i = 1, 2, \ldots, k\). Mosteller [4] then showed that for fixed \(\lambda_i\)'s and \(k\) the asymptotic joint distribution of \(\sqrt{n}(X_{n_{i_1}} - \xi_i), i = 1, 2, \ldots, k\), is \(k\)-dimensional normal, in a sense of type \((M)_d\) in our terminology, with zero mean vector and covariance matrix \(\Sigma_{(k)} = [\lambda_i(1-\lambda_i)/f(\xi_i)f(\xi_j)]\), \(i \leq j\), provided that \(f(x)\) is differentiable in the neighborhoods of \(\xi_i\) and that \(f(\xi_i) \neq 0\).

Later on, Weiss [5] got a result on asymptotic distribution in a strong sense, of a set of increasing number of sample quantiles, which was extended to a more general situation.

Ikeda and Matsunawa [3] have proved the type \((B)_d\) asymptotic joint normality of increasing number of sample quantiles. In the first step, they proved the result in case of uniform distribution over \((0, 1)\): Let \(U_{n_1} < U_{n_2} < \cdots < U_{n_n}\) be order statistics based on a random sample of size \(n\) drawn from a uniform distribution over \((0, 1)\). Select \(k\) order statistics, \(U_{n_{i_1}} < U_{n_{i_2}} < \cdots < U_{n_{i_k}}\), and put \(U_{n(k)} = (U_{n_{i_1}}, U_{n_{i_2}}, \ldots, U_{n_{i_k}})\)', where \(k\) and \((n_{i_1}, n_{i_2}, \ldots, n_{i_k})\) may depend on \(n\) as \(n \to \infty\). By evaluating the K-L information, they proved that under the condition, \(k/\min (n_{i_1} - n_{i_{l-1}}) \to 0, (n \to \infty), U_{n(k)}\) and \(Z_{n(k)}\) are asymptotically equivalent.
in the sense of type $(B)_d$ as $n \to \infty$, where $Z_{n(k)} = (Z_{n1}, Z_{n2}, \ldots, Z_{nk})'$ stands for a normal random variable with mean vector $\mu_{n(k)} = (\mu_{n1}, \mu_{n2}, \ldots, \mu_{nk})'$, $L_{n(k)} = \begin{bmatrix} l_{n1} & \cdots & l_{nk} \\ \vdots & \ddots & \vdots \\ l_{nk} & \cdots & l_{nn} \end{bmatrix}$, $l_{ni} = \frac{n_i}{n(n+1)}$, $i=1,2,\ldots,k$, and covariance matrix $C_{n(k)} = \begin{bmatrix} c_{n1} & \cdots & c_{nk} \\ \vdots & \ddots & \vdots \\ c_{nk} & \cdots & c_{nn} \end{bmatrix}$, $c_{ij} = \frac{l_{ni}l_{nj}}{(n+2)f_{ni}f_{nj}}$, $i \leq j$. Here we have taken a convention $n_0 = 0$, $n_{k+1} = n+1$.

In the second step, they utilized the above result to get a result in more general situation. Let $X_{n(k)}$ be as before, but in this case $k$ and $(n_1, n_2, \ldots, n_k)$ may depend on $n$ as $n \to \infty$. Also, let $Y_{n(k)} = (Y_{n1}, Y_{n2}, \ldots, Y_{nk})'$ be a normal random variable with mean vector $\mu_{n(k)} = (\mu_{n1}, \mu_{n2}, \ldots, \mu_{nk})'$, with $\mu_{ni} = F^{-1}(l_{ni})$, $i=1,2,\ldots,k$ and covariance matrix $C_{n(k)} = \begin{bmatrix} c_{n1} & \cdots & c_{nk} \\ \vdots & \ddots & \vdots \\ c_{nk} & \cdots & c_{nn} \end{bmatrix}$, $c_{ij} = \frac{l_{ni}l_{nj}}{(n+2)f_{ni}f_{nj}}$, $i \leq j$, with $f_{ni} = f(s_{ni})$. Then, they have proved that, under certain conditions, $X_{n(k)}$ and $Y_{n(k)}$ are asymptotically equivalent in the sense of type $(B)_d$ as $n \to \infty$.

However, the conditions still remain unsatisfactory and are not convenient to practical use. Moreover, in contrast with the case of uniform distribution, there exists a stronger condition for the spacing of $(n_1, n_2, \ldots, n_k)$, i.e., $k^2 \min_{1 \leq i \leq k+1} (n_i - n_{i-1}) \to 0$, $n \to \infty$.

In the present paper, we improve these points and present the refined conditions under which $X_{n(k)} \sim Y_{n(k)}$ $(B)_d$ holds as $n \to \infty$. In the next section, the outline of our proof is stated. In Section 3, we give a new result. A generalization of this result is considered in Section 4.

2. Statement of the outline of the proof

Let $X_{n1} < X_{n2} < \cdots < X_{nn}$ be order statistics based on a random sample of size $n$ drawn from a continuous distribution over the real line, whose pdf and cdf being given by $f(x)$ and $F(x)$, respectively. Choose $k$, $X_{n1} < X_{n2} < \cdots < X_{nk}$, and put

(2.1) \hspace{1cm} X_{n(k)} = (X_{n1}, X_{n2}, \ldots, X_{nk})'.

First, we will make the following assumption.

ASSUMPTION 3.1. The support of $f(x)$ is identical to the entire real line: $D_f = (-\infty, \infty)$.

Let $U_{n(k)}$ and $Z_{n(k)}$ be the same as in Section 1. Then, the transformed variable

(2.2) \hspace{1cm} F(X_{n(k)}) \equiv (F(X_{n1}), F(X_{n2}), \ldots, F(X_{nk}))'

is identically distributed with $U_{n(k)}$, or

(2.3) \hspace{1cm} F^{-1}(U_{n(k)}) \equiv (F^{-1}(U_{n1}), F^{-1}(U_{n2}), \ldots, F^{-1}(U_{nk}))'

is identically distributed with $X_{n(k)}$. 

Let us consider a truncation of $Z_{n(k)}$ over the domain $A_{n(k)} = \{ z_{(k)} | 0 < z_{i} < 1; \ i = 1, 2, \ldots, k \}$, and denote it by $Z_{n(k)}^{*}$. The pdf of $Z_{n(k)}^{*}$, $p_{n}^{*}(z_{(k)})$, say, is given by

\begin{align}
    p_{n}^{*}(z_{(k)}) = \begin{cases} 
        p_{n}(z_{(k)})/r_{n}, & \text{if } z_{(k)} \in A_{n(k)} \\
        0, & \text{otherwise}
    \end{cases}
\end{align}

where we have used the symbol $p_{n}(z_{(k)})$ as the pdf of $Z_{n(k)}$, and we have put

\begin{align}
    r_{n} = P_{n(k)}^{*}(A_{n(k)}).
\end{align}

Then, it is easy to see that $Z_{n(k)}^{*} \sim Z_{n(k)}^{*} (B)_{d}$ holds as $n \to \infty$. Indeed it is evident that $r_{n} \to 1$, $(n \to \infty)$, and

\begin{align}
    I(Z_{n(k)}^{*}; Z_{n(k)}) &= \int_{A_{n(k)}} p_{n}^{*} \log \left( p_{n}^{*}/p_{n} \right) d\mu_{(k)} \\
    &= \int_{A_{n(k)}} p_{n}^{*}(-\log r_{n}) d\mu_{(k)} = -\log r_{n} \to 0, \quad (n \to \infty)
\end{align}

which implies the required result. On the other hand, as we have already mentioned in Section 1, Ikeda and Matsunawa [3] have proved that, under the condition

\begin{align}
    k/ \min_{1 \leq i \leq k+1} (n_{i} - n_{i-1}) \to 0, \quad (n \to \infty),
\end{align}

it holds that $U_{n(k)} \sim Z_{n(k)}^{*} (B)_{d}$, $(n \to \infty)$. Since the notion of asymptotic equivalence is reflexive and transitive in the sense of any given type, under the condition (2.6) it holds also that $U_{n(k)} \sim Z_{n(k)}^{*} (B)_{d}$, $(n \to \infty)$. Further, let us put

\begin{align}
    Y_{n(k)}^{*} = F^{-1}(Z_{n(k)}^{*}) = (F^{-1}(Z_{n1}^{*}), F^{-1}(Z_{n2}^{*}), \ldots, F^{-1}(Z_{nk}^{*}))'.
\end{align}

Then, $X_{n(k)} \sim Y_{n(k)}^{*} (B)_{d}$ holds as $n \to \infty$, under the same condition.

Finally, let $Y_{n(k)} = (Y_{n1}, Y_{n2}, \ldots, Y_{nk})'$ be a normal random variable with mean vector

\begin{align}
    s_{n(k)} = (s_{n1}, s_{n2}, \ldots, s_{nk})', \quad \text{with } s_{ni} = F^{-1}(l_{ni}), \ i = 1, 2, \ldots, k
\end{align}

and covariance matrix

\begin{align}
    S_{n(k)} = \frac{1}{n+2} \begin{bmatrix}
        l_{n1}(1-l_{n1}) & l_{n1}(1-l_{n2}) & \cdots & l_{n1}(1-l_{nk}) \\
        f_{n1}^2 & f_{n1}f_{n2} & \cdots & f_{n1}f_{nk} \\
        l_{n2}(1-l_{n2}) & f_{n2}^2 & \cdots & l_{n2}(1-l_{nk}) \\
        f_{n2}f_{n3} & \cdots & f_{n2}f_{nk} \\
        \vdots & \cdots & \cdots & \cdots
    \end{bmatrix}
\end{align}
where we have put $f_{ni} = f(s_{ni}) = f(F^{-1}(l_{ni}))$, $i = 1, 2, \ldots, k$. Then, if one can show that $Y_{n(k)} \sim Y_{m(k)}^*(B)$, $(n \to \infty)$, with possibly some additional conditions, it holds that $X_{n(k)} \sim Y_{n(k)}^*(B)$ as $n \to \infty$. We will investigate this problem in the next section.

The following diagram indicates the relations among the variables introduced so far.

\[
\begin{array}{ccc}
(B)_d & (B)_d \\
U_{n(k)} \sim Z_{n(k)} \sim Z_{n(k)}^* \\
\end{array}
\]

3. Type $(B)_d$ asymptotic equivalence of $Y_{n(k)}$ and $Y_{n(k)}^*$

The pdf's of $Y_{n(k)}$ and $Y_{n(k)}^*$ are given respectively by

\[
(3.1) \quad q_n(y_{(k)}) = \left(\frac{1}{\sqrt{2\pi}}\right)^k \frac{1}{|S_{n(k)}|^{1/2}} \exp \left[ -\frac{1}{2} (y_{(k)} - s_{n(k)})' S_{n(k)}^{-1} (y_{(k)} - s_{n(k)}) \right] \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad}
-(y_{(k)} - s_{n(k)})' S_{n(k)}^{-1}(y_{(k)} - s_{n(k)})}ight).

Before proceeding to the calculation, we will make the following assumption.

**Assumption 4.1.** \( f(x) \) is twice differentiable and \( f''(x) \) is bounded and continuous over the entire real line.

Then, we get

\[(3.5) \quad \log f(y_i) - \log f_{n_l} = \frac{f'_{n_l}}{f_{n_l}} (y_i - s_{n_l}) + \frac{1}{2} R_{nl}^* \]

and

\[(3.6) \quad (F(y_{(k)}) - F(s_{n(k)}))' L_{n(k)}^{-1}(F(y_{(k)}) - F(s_{n(k)}))
= (y_{(k)} - s_{n(k)})' S_{n(k)}^{-1}(y_{(k)} - s_{n(k)}) + \alpha_{n(k)}' L_{n(k)}^{-1} \beta_{n(k)}
+ \alpha_{n(k)}' L_{n(k)}^{-1} \gamma_{n(k)} + \frac{1}{4} \beta_{n(k)}' L_{n(k)}^{-1} \beta_{n(k)} + \beta_{n(k)}' L_{n(k)}^{-1} \gamma_{n(k)} + \gamma_{n(k)}' L_{n(k)}^{-1} \gamma_{n(k)}
\]

where we have put

\[\alpha_{n(k)} = (\alpha_{n1}, \ldots, \alpha_{nk})', \]
\[\beta_{n(k)} = (\beta_{n1}, \ldots, \beta_{nk})', \]
\[\gamma_{n(k)} = (\gamma_{n1}, \ldots, \gamma_{nk})', \]
\[\alpha_{n_l} = f_{n_l}(y_i - s_{n_l}), \quad i = 1, 2, \ldots, k, \]
\[\beta_{n_l} = f'_{n_l}(y_i - s_{n_l}), \quad i = 1, 2, \ldots, k, \]
\[\gamma_{n_l} = f_{n_l}, \quad i = 1, 2, \ldots, k, \]
\[R_{n_l}^* = R_{n_l}^*(y_i) = \phi(y_i^*)(y_i - s_{n_l}), \quad i = 1, 2, \ldots, k, \]
\[R_{nl} = R_{nl}(y_i) = \frac{1}{6} f''(y_i^*)(y_i - s_{n_l}), \quad i = 1, 2, \ldots, k, \]
\[\phi(x) = \{f(x)f''(x) - (f'(x))^2\} / f^3(x), \]

and \(y_i^*\) and \(y_i^{**}\) are some values between \(y_i\) and \(s_{n_l}\).

Now, we will make the following

**Assumption 4.2.** The function, \( \phi(x) = \{f(x)f''(x) - (f'(x))^2\} / f^3(x) \), is bounded uniformly for all \( x \) in \((-\infty, \infty)\).

Then, from (3.5) it is seen that
\begin{align}
\left| \sum_{i=1}^{k} E \left[ \log f(Y_{nt}) - \log f_{nt} \right] \right| & = \frac{1}{2} \left| \sum_{i=1}^{k} E \left[ R_{nt}^i \right] \right| \leq \frac{M_i}{2(n+2)} \sum_{i=1}^{k} \frac{l_{ni}(1-l_{ni})}{f_{ni}^2},
\end{align}

where \( M_i = \sup_{-\infty < x < \infty} \left| \psi(x) \right| \).

For notational simplicity, let us put

\begin{align}
\eta_i &= 1/(l_{ni+1} - l_{ni}), \quad i = 1, 2, \ldots, k,
\end{align}

and hence

\begin{align}
L_{n(k)}^{-1} = (n+2) \begin{bmatrix}
\eta_1 + \eta_0 & -\eta_1 & 0 \\
-\eta_1 & \eta_2 + \eta_1 & -\eta_2 \\
& \ddots & \ddots & \ddots \\
& & 0 & -\eta_{k-1} & \eta_k + \eta_{k-1}
\end{bmatrix}.
\end{align}

The moments of the RHS of (3.6) will be evaluated as follows. First, since

\begin{align}
\alpha_{n(k)} L_{n(k)}^{-1} \beta_{n(k)} &= (n+2) \left\{ \sum_{i=1}^{k} (\eta_i + \eta_{i-1}) f_{nt} f'_{nt}(y_t - s_{nt})^3 \\
& - \sum_{i=1}^{k-1} \eta_i f_{nt} f'_{nt}(y_t - s_{nt})(y_{t+i} - s_{nt+i})^2 \\
& - \sum_{i=1}^{k-1} \eta_i f_{nt+i} f'_{nt}(y_{t+i} - s_{nt+i})(y_t - s_{nt})^2 \right\},
\end{align}

we have

\begin{align}
E \left[ \alpha_{n(k)} L_{n(k)}^{-1} \beta_{n(k)} \right] &= 0.
\end{align}

Second, since

\begin{align}
\alpha_{n(k)} L_{n(k)}^{-1} \gamma_{n(k)} &= (n+2) \left\{ \sum_{i=1}^{k} (\eta_i + \eta_{i-1}) f_{nt}(y_t - s_{nt}) R_{nt} \\
& - \sum_{i=1}^{k-1} \eta_i f_{nt}(y_t - s_{nt}) R_{nt+i} \\
& - \sum_{i=1}^{k-1} \eta_i f_{nt+i}(y_{t+i} - s_{nt+i}) R_{nt} \right\},
\end{align}

and by the Schwarz inequality

\begin{align}
\left| E \left[ (Y_{nt} - s_{nt}) R_{nt} \right] \right| & \leq \frac{3M_2}{(n+2)^2} \cdot \sigma_{nt}^2,
\end{align}

\begin{align}
\left| E \left[ (Y_{nt} - s_{nt}) R_{nt+1} \right] \right| & \leq \frac{\sqrt{15} M_2}{6(n+2)^2} \cdot \sigma_{nt} \sigma_{nt+1}^2,
\end{align}

\begin{align}
\left| E \left[ (Y_{nt+1} - s_{nt+1}) R_{nt} \right] \right| & \leq \frac{\sqrt{15} M_2}{6(n+2)^2} \cdot \sigma_{nt+1} \sigma_{nt}^2.
\end{align}
where we have put \( M_i = \text{sup}_{-\infty < x < \infty} |f''(x)|, \quad \sigma_{ni}^2 = \lambda_{ni}(1 - \lambda_{ni})/f_n^2, \quad i = 1, 2, \ldots, k, \)
it follows from (3.13) that

\[
\left| \mathbb{E} \left[ \alpha_{n(k)} L_{n(k)} \gamma_{n(k)} \right] \right| \leq \frac{3M_2}{n+2} \sum_{i=1}^{k} (\eta_i + \eta_{i-1}) f_{ni} \sigma_{ni}^2 + \frac{\sqrt{15} M_2}{n+2} \sum_{i=1}^{k-1} \eta_i f_{ni} \sigma_{ni}^3 \sigma_{ni+1}^3 + \frac{\sqrt{15} M_2}{n+2} \sum_{i=1}^{k-1} \eta_i f_{ni+1} \sigma_{ni}^3 \sigma_{ni+1}^3.
\]

Third, since

\[
\frac{1}{4} \beta_{n(k)} L_{n(k)}^{-1} \beta_{n(k)} = \frac{n+2}{4} \left\{ \sum_{i=1}^{k} (\eta_i + \eta_{i-1}) \left(f_n' \right)^2 (y_i - s_{ni})^4 - 2 \sum_{i=1}^{k-1} \eta_i f_{ni} f_{ni+1} (y_i - s_{ni})^3 (y_{i+1} - s_{ni+1}) \right\},
\]

we have

\[
\left| \mathbb{E} \left[ \frac{1}{4} \beta_{n(k)} L_{n(k)}^{-1} \beta_{n(k)} \right] \right| \leq \left| \frac{n+2}{4} \sum_{i=1}^{k} (\eta_i + \eta_{i-1}) \left(f_n' \right)^2 \mathbb{E} (y_i - s_{ni})^4 \right| \leq \frac{3}{4(n+2)} \sum_{i=1}^{k} (\eta_i + \eta_{i-1}) \left(f_n' \right)^2 \sigma_{ni}^4.
\]

Fourth,

\[
\beta_{n(k)} L_{n(k)}^{-1} \beta_{n(k)} = (n+2) \left\{ \sum_{i=1}^{k} (\eta_i + \eta_{i-1}) f_{ni} (y_i - s_{ni})^2 R_{ni} - \sum_{i=1}^{k-1} \eta_i f_{ni}' (y_i - s_{ni})^2 R_{ni+1} - \sum_{i=1}^{k-1} \eta_i f_{ni+1} (y_{i+1} - s_{ni+1})^2 R_{ni} \right\}.
\]

Here, we see that

\[
\left| \mathbb{E} [(y_{ni} - s_{ni})^2 R_{ni}] \right| \leq \frac{\sqrt{105} M_2}{2(n+2)^{3/2}} \cdot \sigma_{ni}^5,
\]

\[
\left| \mathbb{E} [(y_{ni} - s_{ni})^2 R_{ni+1}] \right| \leq \frac{\sqrt{5} M_2}{2(n+2)^{3/2}} \cdot \sigma_{ni}^2 \sigma_{ni+1}^3,
\]

and

\[
\left| \mathbb{E} [(y_{ni+1} - s_{ni+1})^2 R_{ni}] \right| \leq \frac{\sqrt{5} M_2}{2(n+2)^{3/2}} \cdot \sigma_{ni+1}^2 \sigma_{ni}^2.
\]

Hence we get
\begin{align}
|E \left[ \gamma_n^{(k)} \gamma_n^{(k)} \right]| & \leq \frac{\sqrt{105} M_2}{2(n+2)^{3/2}} \sum_{i=1}^{k} (\eta_i + \gamma_{i-1}) f'_n \sigma_n^i \\
& + \frac{\sqrt{5} M_2}{2(n+2)^{3/2}} \sum_{i=1}^{k-1} \gamma_i f'_n \sigma_n^i \sigma_{n+1}^i \\
& + \frac{\sqrt{5} M_2}{2(n+2)^{3/2}} \sum_{i=1}^{k-1} \eta_i f'_n \sigma_n^i \sigma_{n+1}^i.
\end{align}

Finally,

\begin{align}
\gamma_n^{(k)} L_n^{(k)} \gamma_n^{(k)} = (n+2) \left( \sum_{i=1}^{k} (\eta_i + \gamma_{i-1}) R_n^i - 2 \sum_{i=1}^{k-1} \eta_i R_n^i R_{n+1}^i \right).
\end{align}

Here

\begin{align}
E \left[ R_n^i \right] & \leq \frac{5M_2^2}{12(n+2)^3} \sigma_n^i,
\end{align}

\begin{align}
E \left[ R_n^i R_{n+1}^i \right] & \leq \frac{5M_2^2}{12(n+2)^3} \sigma_n^i \sigma_{n+1}^i.
\end{align}

Hence

\begin{align}
|E \left[ \gamma_n^{(k)} L_n^{(k)} \gamma_n^{(k)} \right]| & \leq \frac{5M_2^2}{12(n+2)^3} \left( \sum_{i=1}^{k} (\eta_i + \gamma_{i-1}) \sigma_n^i + 2 \sum_{i=1}^{k-1} \eta_i \sigma_n^i \sigma_{n+1}^i \right).
\end{align}

Thus, summarizing the results so far obtained, it is seen that

\begin{align}
I(Y_n^{(k)}; Y_{n+1}^{(k)}) & \leq \log r_n + \frac{M_2}{2(n+2)} \sum_{i=1}^{k} \sigma_n^i + \frac{3M_2}{n+2} \sum_{i=1}^{k} (\eta_i + \gamma_{i-1}) f'_n \sigma_n^i \\
& + \frac{\sqrt{15} M_2}{4(n+2)^{3/2}} \sum_{i=1}^{k-1} \eta_i f'_n \sigma_n^i \sigma_{n+1}^i + \frac{\sqrt{15} M_2}{n+2} \sum_{i=1}^{k-1} \gamma_i f'_n \sigma_n^i \sigma_{n+1}^i \\
& + \frac{3}{4(n+2)^{3/2}} \sum_{i=1}^{k} (\eta_i + \gamma_{i-1}) f'_n \sigma_n^i + \frac{\sqrt{105} M_2}{2(n+2)^{3/2}} \sum_{i=1}^{k} (\eta_i + \gamma_{i-1}) f'_n \sigma_n^i \\
& + \frac{\sqrt{5} M_2}{2(n+2)^{3/2}} \sum_{i=1}^{k-1} \eta_i f'_n \sigma_n^i \sigma_{n+1}^i + \frac{\sqrt{5} M_2}{2(n+2)^{3/2}} \sum_{i=1}^{k-1} \gamma_i f'_n \sigma_n^i \sigma_{n+1}^i \\
& + \frac{5M_2^2}{12(n+2)^3} \left( \sum_{i=1}^{k} (\eta_i + \gamma_{i-1}) \sigma_n^i + 2 \sum_{i=1}^{k-1} \eta_i \sigma_n^i \sigma_{n+1}^i \right).
\end{align}

Since \( \eta_i = 1/(l_{n+1} - l_n) \) and \( l_n = n/(n+1) \) for each \( i \), it holds that

\begin{align}
\frac{1}{n+2} \sum_{i=0}^{k} \eta_i &= \frac{n+1}{n+2} \sum_{i=0}^{k} \frac{1}{n_{i+1} - n_i} \leq \left( 1 - \frac{1}{n+2} \right) \sum_{i=0}^{k} \frac{1}{n_{i+1} - n_i} \\
& \leq \frac{1}{\sum_{i=0}^{k} n_{i+1} - n_i}.
\end{align}

Let us put \( w_n = \sum_{i=0}^{k} 1/(n_{i+1} - n_i) \), \( \sigma_n = \max \sigma_n^i \), \( M_2 = \sup_{-\infty < x < \infty} f(x) \) and \( M_2 = \sum_{i=0}^{k} \frac{1}{n_{i+1} - n_i} \).
\[ \sup_{-\infty < x < \infty} |f'(x)|, \text{ it then follows that} \]
\[ I(Y_{n(k)} : Y^*_{n(k)}) \leq \log r_n + \frac{k \sigma_n^2 M_1}{2(n+2)} + w_n^* \left( 2(3 + \sqrt{105}) M_2 M_3 + \frac{3}{2} M_1^2 \right) \]
\[ + w_n^* \frac{\sigma_n^2}{\sqrt{n+2}} \left( \sqrt{105} + \sqrt{5} \right) M_2 M_4 + \frac{5 w_n^* \sigma_n^4 M_2}{3(n+2)} . \]

Thus we get the following theorem.

**Theorem 3.1.** Suppose that the assumptions 2.1, 3.1 and 3.2 are fulfilled. Then, in order that \( X_{n(k)} \sim Y_{n(k)} (B)_n, (n \to \infty), \) it is sufficient that the following conditions are satisfied simultaneously:
\[ w_n \equiv \sum_{i=0}^{k} \frac{1}{n_{i+1} - n_i} \to 0, \quad (n \to \infty), \]
and
\[ w_n \cdot \sigma_n^2 \to 0, \quad (n \to \infty), \]
where we have put
\[ \sigma_n^2 = \max_{i \leq k} \sigma_{n_i}^2, \quad \sigma_{n_i}^2 = l_{n_i}(1 - l_{n_i})/f_{n_i}^2, \quad i = 1, 2, \ldots, k. \]

It should be noted that the assumptions 2.1 and 3.1 in the theorem are satisfied for a wider class of probability distributions including normal, Cauchy and Laplace:
\[ \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \pi^{-1}/(1+x^4), \quad \frac{1}{2} e^{-|x|}. \]

4. Extension of Theorem 3.1

Now we generalize the result of Theorem 3.1 to the case where \( D_f, \) the support of \( f(x), \) is not necessarily the entire real line.

**Assumption 4.1.** The support of \( f(x) \) is identical to an open interval: \( D_f = (a, b), \) where \( a \) and \( b \) are extended real.

Let \( U_{n(k)} \), \( Z_{n(k)} \), \( X_{n(k)} \), \( Z_{n(k)}^* \), \( Y_{n(k)}^* \) and \( Y_{n(k)} \) be the same as before. Note that the range of \( Y_{n(k)}^* \) is now restricted to the \( k \)-dimensional open cube: \( C_{(k)} = (a, b)^k \).

Let \( Y_{n(k)} \) be the truncation of \( Y_{n(k)} \) over the set \( C_{(k)} \), and define
\[ \rho_n \equiv P \{ Y_{n(k)} \in C_{(k)} \}. \]
Then, the pdf of \( Y_{n(k)} \) is given by
\[ q_n = \begin{cases} q_n/\rho_n, & \text{on } C_{(k)} \\ 0, & \text{elsewhere} \end{cases} \]
if \( \rho_n \to 1, \quad (n \to \infty) \), then it holds that \( Y_{n(k)} \sim Y_{n(k)}^* \) \( (B)_d, \quad (n \to \infty) \). Hence, we will first derive a condition which satisfies this requirement.

By using the Chebychev inequality, we get

\[
1 - \rho_n = P \left\{ Y_{n(k)} \notin C_{(k)} \right\} \\
\leq k \cdot \max_{1 \leq i < k} P \left\{ |Y_{n(i)} - s_{ni}| \geq \min \left( |a - s_{ni}|, |b - s_{ni}| \right) \right\} \\
\leq \frac{k}{n + 2} \cdot \max_{1 \leq i \leq k} \frac{\sigma_{ni}^2}{\min \left( |a - s_{ni}|^2, |b - s_{ni}|^2 \right)},
\]

the vanishing of which implies our requirement.

We will next find the conditions under which \( \bar{Y}_{n(k)} \sim Y_{n(k)}^* \) \( (B)_d, \quad (n \to \infty) \). For this, we evaluate the K-L information

\[
I(\bar{Y}_{n(k)}; Y_{n(k)}^*) = \int_{\mathcal{S}_{(k)}} q_{n} \log \left( \frac{q_{n}}{q_{n}^*} \right) d\mu_{(k)} \\
= \frac{1}{\rho_n} \int_{C_{(k)}} q_{n} \log \left( \frac{q_{n}}{q_{n}^*} \right) d\mu_{(k)} - \log \rho_n,
\]

the vanishing of which implies the required result. But, since \( \rho_n \to 1, \quad (n \to \infty) \), under the condition that the last member of (5.3) tends to zero as \( n \to \infty \), this is equivalent to the condition

\[
I^*(\bar{Y}_{n(k)}) = I^*(Y_{n(k)}^*) = \int_{C_{(k)}} q_{n} \log \left( \frac{q_{n}}{q_{n}^*} \right) d\mu_{(k)} \to 0, \quad (n \to \infty).
\]

Parallel with the previous section, we will make the following assumptions.

**Assumption 4.2.** \( f(x) \) is twice differentiable and \( f''(x) \) is bounded and continuous over \( (a, b) \).

**Assumption 4.3.** The function, \( \phi(x) = \{f(x)f''(x)-(f''(x))^2\}/f''(x) \), is bounded uniformly for all \( x \) in \( (a, b) \).

Let us designate the integral operator \( \int_{C_{(k)}} \cdot q_{n} d\mu_{(k)} \) by \( E^*[\cdot] \). Then, one can see that the calculation goes similarly to that of deriving the preceding theorem. In fact, since \( E^*[\phi(y_{n(k)})] \leq E[\phi(y_{n(k)})] \) provided \( \phi(y_{n(k)}) \geq 0 \), this is much the same as in the previous section, except for the case \( E^*[\alpha_{n(k)}^* L_{n(k)}^{-1} \beta_{n(k)}^*] \).

In that case, from (3.11) we have

\[
|E [\alpha_{n(k)}^* L_{n(k)}^{-1} \beta_{n(k)}^*] - E^* [\alpha_{n(k)}^* L_{n(k)}^{-1} \beta_{n(k)}^*]| \leq (n + 2) M_2 M_3 \sum_{i=1}^{k} (\eta_{i} + \eta_{i-1}) \int_{y_{i}(a,b)} |y_{i} - s_{ni}|^p dP^{y_{ni}} \\
+ \sum_{i=1}^{k-1} \eta_{i} \int_{y_{i}(a,b)} |y_{i} - s_{ni}| |y_{i+1} - s_{ni+1}|^p dP^{y_{ni}} r_{ni} r_{ni+1}
\]

\[
+ \sum_{i=1}^{k-1} \eta_{i} \int_{y_{i}(a,b)} |y_{i} - s_{ni}| |y_{i+1} - s_{ni+1}|^p dP^{y_{ni}} r_{ni} r_{ni+1}
\]
\[ + \sum_{i=1}^{k-1} \gamma_i \int_{(y_i, y_{i+1}) \in (a, b)^2} |y_i - s_{nt}| |y_{i+1} - s_{nt+1}| dP^{r_{nt+1}} \]

Here, by Schwarz's inequality
\[ \int_{y_i \in (a, b)} |y_i - s_{nt}|^2 dP^{r_{nt}} \leq \left( \int_{|y_i - s_{nt}| \geq \zeta_{nt}} dP^{r_{nt}} \right)^{1/2} \left( \int_{|y_i - s_{nt}| \geq \zeta_{nt}} (y_i - s_{nt})^2 dP^{r_{nt}} \right)^{1/2}, \]
and by Chebychev's inequality
\[ \int_{|y_i - s_{nt}| \geq \zeta_{nt}} dP^{r_{nt}} \leq \frac{\sigma^2_{nt}}{(n+2) \zeta_{nt}^2}, \]
where \( \zeta_{nt} = \min \{ |a - s_{nt}|, |b - s_{nt}| \}. \) Also
\[ \int_{|y_i - s_{nt}| \geq \zeta_{nt}} (y_i - s_{nt})^2 dP^{r_{nt}} \leq \int_{-\infty < y_i < \infty} (y_i - s_{nt})^2 dP^{r_{nt}} = \frac{15 \sigma^6_{nt}}{(n+2)^3}, \]
hence
\[ (4.7) \quad \int_{y_i \in (a, b)} |y_i - s_{nt}|^2 dP^{r_{nt}} \leq \frac{\sqrt{15} \sigma^4_{nt}}{(n+2)^3 \zeta_{nt}}. \]

Also, we have
\[ (4.8) \quad \int_{(y_i, y_{i+1}) \in (a, b)^2} |y_i - s_{nt}| |y_{i+1} - s_{nt+1}|^2 dP^{r_{nt+1}} \]
\[ \leq \left( \int_{(y_i, y_{i+1}) \in (a, b)^2} dP^{r_{nt+1}} \right)^{1/2} \times \left( \int_{(y_i, y_{i+1}) \in (a, b)^2} |y_i - s_{nt}| |y_{i+1} - s_{nt+1}|^2 dP^{r_{nt+1}} \right)^{1/2} \]
\[ \leq \left( \frac{\sigma^2_{nt}}{(n+2) \zeta_{nt}^2} \right)^{1/2} \frac{\sigma^4_{nt+1}}{(n+2)^3} \leq c_i \frac{\sigma^4_{nt} \sigma^4_{nt+1}}{(n+2)^3 \zeta_{nt}^5}, \]
and similarly
\[ (4.9) \quad \int_{(y_i, y_{i+1}) \in (a, b)^2} |y_i - s_{nt}| |y_{i+1} - s_{nt+1}|^2 dP^{r_{nt+1}} \leq c_i \frac{\sigma^4_{nt} \sigma^2_{nt+1}}{(n+2)^3 \zeta_{nt}^5}. \]

From (3.12), (4.6), (4.7), (4.8) and (4.9) it follows that
\[ (4.10) \quad |E^* [\alpha_{n(k)} L_{n(k)}^{-1} \beta_{n(k)}]| \leq c \cdot w_n \cdot \max_{1 < i < k} \left\{ \frac{\sigma^2_{nt}}{\min \{ |a - s_{nt}|, |b - s_{nt}| \}} \right\}, \]
where \( c \) is a constant independent of \( n \).

Summarizing the results thus obtained, we can state the following

**Theorem 4.1.** Under the assumptions 4.1, 4.2 and 4.3, in order that \( X_{n(k)} \sim Y_{n(k)} (B)_d, (n \to \infty) \), it is sufficient that the following conditions are satisfied simultaneously:
\[
(4.11) \quad w_n = \sum_{i=1}^{k} \frac{1}{n_{i+1} - n_i} \to 0, \quad (n \to \infty),
\]

\[
(4.12) \quad w_n \cdot \sigma_n^t \to 0, \quad (n \to \infty),
\]

and

\[
(4.13) \quad w_n \cdot \max_{1 \leq k \leq k} \frac{\sigma_n^t}{\zeta_n^t} \to 0, \quad (n \to \infty).
\]

where \(\zeta_n = \min(|a - s_n|, |b - s_n|)\).

In the case of uniform distribution, this theorem gives an equivalent to Theorem 3.1 as will be seen below.

Suppose the basic distribution be uniform, \(U(a, b)\). Without loss of generality, one can assume that \(a = 0\) and \(b = 1\). Then, since \(\sigma_n^t = l_n(1 - l_n)\) is less than or equal to unity, the condition (4.12) is a consequence of the condition (3.11). Also, since \(s_n = l_n\), we have

\[
\frac{\sigma_n^t}{\zeta_n^t} = \frac{l_n(1 - l_n)}{\min(l_n, (1 - l_n)^t)} = \max(l_n, (1 - l_n)^t) \leq 1.
\]

Therefore, (4.11) implies (4.13). \(f(x) = 1(0 \leq x \leq 1), 0\) (otherwise), satisfies the assumptions of the theorem. Thus, the sole condition (4.11) implies the asymptotic \((B)_d\) normality of \(X_{n(x)}\) as \(n \to \infty\), which is nothing but the result of Ikeda and Matsunawa [3].

If \(a \to -\infty\) and \(b \to +\infty\), then \(\zeta_n^t \to \infty\), whatsoever the spacing \(s_n\)'s should be. Hence the condition (4.13) of the above theorem becomes trivial, which shows that Theorem 4.1 is a generalization of Theorem 3.1.

The followings are the immediate consequences from Theorem 4.1.

**Corollary 4.1.** Suppose that \(0 < M \leq f(x)\) for some \(M\), and \(f'(x), f''(x)\) and \(\phi(x)\) are uniformly bounded over a finite interval \(D_j = (a, b)\). Then the condition (4.11) implies the asymptotic \((B)_d\) normality of \(X_{n(x)}\) as \(n \to \infty\).

**Corollary 4.2.** Suppose that \(0 < M \leq f(x) \leq M'\) for some \(M\) and \(M'\), and \(f'(x)\) and \(f''(x)\) are uniformly bounded over a finite \(D_j = (a, b)\). Then the sole condition (4.11) implies the asymptotic \((B)_d\) normality of \(X_{n(x)}\) as \(n \to \infty\).

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**References**


