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A NOTE ON A QUADRATIC MEASURE OF DEVIATION OF DENSITY ESTIMATES

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Summary

The results of Rosenblatt on quadratic measure of deviations of density estimates have been generalized to a wider class of weight functions. It is pointed out that the proof of Theorem 1 of Rosenblatt is incorrect. A corrected version of the proof is also provided.

1. Introduction

Let $X_1, \cdots, X_n$ be independent and identically distributed random two-dimensional vectors with the common density function $f(x)$, $x=(x^{(1)}, x^{(2)})$. We consider a class of estimators $f_n(x)$ of $f(x)$ determined by a bounded weight function $W(\cdot, \cdot)$ (not necessarily vanishing off a rectangle).

$$(1)\qquad f_n(x^{(1)}, x^{(2)})=(nh^2(n))^{-1}\sum_{j=1}^{n} W\left(\frac{x^{(1)}-X_j^{(1)}}{h(n)}, \frac{x^{(2)}-X_j^{(2)}}{h(n)}\right)$$
$$=(h^2(n))^{-1}\int W\left(\frac{x-u}{h(n)}\right)dF_n(u),$$

where $F_n(u)=F_n(u^{(1)}, u^{(2)})$ is the sample distribution function determined by $X_1, \cdots, X_n$. Here $h(n)$ is the band width such that $h(n)\to 0$ and $nh^2(n)\to\infty$ as $n\to\infty$. Rosenblatt [1] considered similar estimates based on weight functions with finite support. In his paper Rosenblatt used the technique of Poissonization to show the asymptotic normality of some quadratic functional of the deviation of the density estimates. The proof of the main theorem, Theorem 1, in [1] is not correct.

The purpose of this paper is to generalize the results of Rosenblatt [1] to density estimates based on a wider class of weight function. We also provide a correct version of the proof of Theorem 1 of Rosenblatt. In Section 2 the main theorems are stated. The proofs are

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given in Section 3.

2. Main results

The assumptions made in this paper are similar to those in [1] except the first assumption, A1. We restate all the assumptions for the sake of completeness.

(A1) The weight function $W(\cdot, \cdot)$ is bounded and

$$\int W(x^{(1)}, x^{(2)}) dx^{(1)} dx^{(2)} = 1 . \tag{2}$$

(Note: Rosenblatt assumed $W(\cdot, \cdot)$ to be zero outside a rectangle.)

(A2) The probability density function f is bounded and is continuosly differentiable up to second order with bounded derivatives in its domain of positivity.

(A3) The weight function is symmetric ($W(u)=W(-u)$) so that the first moment of $W(\cdot, \cdot)$ and $W^2(\cdot, \cdot)$ are zero and the matrix of second moments of $W(\cdot, \cdot)$

$$\int x^{(i)} x^{(j)} W(x^{(1)}, x^{(2)}) dx^{(1)} dx^{(2)} , \qquad i, j = 1, 2 , \tag{3}$$

is positive definite.

(A4) The function $a(x^{(1)}, x^{(2)})$ used in the definition of $\tilde{T}_n$ below is bounded integrable.

(A5) Let $\{c(n)\} \uparrow \infty$ and $W(\cdot, \cdot)$ be such that

(i) $d(n) = c(n)h(n) \to 0 \quad$ as $n \to \infty$ and

(ii) $\displaystyle\int_{\substack{|t_1| > c(n) \\ \text{or } |t_2| > c(n)}} W^2(t_1, t_2) dt_1 dt_2 = O(h^2(n))$.

(A6) Let $\{c(n)\} \uparrow \infty$ and $W_1(\cdot)$ and $W_2(\cdot)$ be such that

(i) $d(n) = c(n)h(n) \to 0 \quad$ as $n \to \infty$ and

(ii) $\displaystyle\int_{|t| > c(n)} W_i^2(t) = o(h^6(n)) , \qquad i = 1, 2$.

Let

$$\tilde{T}_n = nh^2(n) \int (f_n(x) - f(x))^2 a(x) dx \tag{4}$$

$$\mu = \int a(x) f(x) dx \int W^2(x) dx \tag{5}$$

(6) $$\sigma^2=2W^{(4)}(0)\int a^2(x)f(x)dx\,.$$

THEOREM 1. *Under assumptions* (A1)–(A5)

(7) $$\frac{\tilde{T}_n-\mu}{h(n)\sigma}\xrightarrow{\mathscr{D}}N(0,1)$$

if $nh^2(n)\to\infty$ *and* $h(n)=o(n^{-1/5})$.

Suppose the components of $X_i=(X_i^{(1)}, X_i^{(2)})$ are independent with marginal densities $g_1(\cdot)$ and $g_2(\cdot)$. Let $g_{1n}(x^{(1)})$ and $g_{2n}(x^{(2)})$ denote the estimates of g_1 and g_2 based on weight functions $W_1(\cdot)$, $W_2(\cdot)$ respectively. Let

(8) $$g_{1n}(x^{(1)})=(nh)^{-1}\sum_{i=1}^{n}W_1\left(\frac{x^{(1)}-X_i^{(1)}}{h}\right)$$

(9) $$g_{2n}(x^{(2)})=(nh)^{-1}\sum_{i=1}^{n}W_2\left(\frac{x^{(2)}-X_i^{(2)}}{h}\right)$$

(10) $$\tilde{\mu}_n=\int f(x)a(x)dx+h\int g_1(x^{(1)})g_2^2(x^{(2)})a(x)dx\left[1+\int W_1^2(u_1)W_2(u_2)du_1du_2\right]$$
$$+h\int g_1^2(x^{(1)})g_2(x^{(2)})a(x)dx\left[1+\int W_1(u_1)W_2^2(u_2)du_1du_2\right]$$

(11) $$a(x)=a(x^{(1)}, x^{(2)})$$
$$W(x)=W(x^{(1)}, x^{(2)})=W_1(x^{(1)})W_2(x^{(2)})$$

(12) $$\tilde{\tilde{T}}_n=nh^2\int(f_n(x^{(1)}, x^{(2)})-g_{1n}(x^{(1)})g_{2n}(x^{(2)}))^2a(x^{(1)}, x^{(2)})dx^{(1)}dx^{(2)}\,.$$

THEOREM 2. *Under assumptions* (A1)–(A4) *and* (A6)

(13) $$\frac{\tilde{\tilde{T}}_n-\tilde{\mu}_n}{h\sigma}\xrightarrow{\mathscr{D}}N(0,1)$$

if $nh^2\to\infty$ *and* $h=o(n^{-1/5})$.

3. Proofs of the theorems

We state below two lemmas from [1] which are essential to the proofs of main theorems.

LEMMA 1 (*Rosenblatt* [1]). *Let W be a bounded integrable weight function with*

(i) $\int x^{(i)}W(x^{(1)}, x^{(2)})dx^{(1)}dx^{(2)}=0$ *for* $i=1, 2$

(14) (ii) $\left|\int (t_1^2+t_2^2) W(t_1, t_2)dt_1dt_2\right| < \infty$.

If assumptions (A2) *and* (A4) *are satisfied, then*

(15) $$R(n)=nh^2\int \{(f_n(x)-f(x))^2-(f_n(x)-\mathrm{E}\, f_n(x))^2\}\, a(x)dx$$
$$=o(h(n)) \qquad \text{if } h(n)=o(n^{-1/5})\, .$$

Let N be a Poisson random variable with mean n independent of X_1, $X_2, \cdots$. Set

(16) $$f_n^*=(nh^2)^{-1}\sum_{j=1}^{N} W\left(\frac{x^{(1)}-X_j^{(1)}}{h},\ \frac{x^{(2)}-X_j^{(2)}}{h}\right).$$

LEMMA 2 (*Rosenblatt* [1]). *Under assumptions of Lemma* 1

(17) $$R(n)=nh^2\int \{(f_n^*(x)-\mathrm{E}\, f_n^*(x))^2-(f_n(x)-\mathrm{E}\, f_n(x))^2\}\, a(x)dx$$
$$=o(h(n))$$

if $nh^2\to\infty$ *and* $h(n)\downarrow 0$ *as* $n\to\infty$.

Lemmas 1 and 2 together imply that if we let $h(n)\downarrow 0$ at the proper rate, then

(18) $$S_n=nh^2(n)\int (f_n^*(x)-\mathrm{E}\, f_n^*(x))^2 a(x)dx$$

can be considered instead of

$$nh^2(n)\int (f_n(x)-f(x))^2 a(x)dx$$

with small error as $n\to\infty$ and $h(n)\downarrow 0$. Next lemma shows that if the weight function satisfies certain conditions then in fact we can truncate the weight function and replace $f_n^*(x)$ by a truncated estimator $\tilde{f}_n^*(x)$. Where

(19) $$\tilde{f}_n^*(x)=(nh^2)^{-1}\sum_{j=1}^{N} \tilde{W}\left(\frac{x^{(1)}-X_j^{(1)}}{h},\ \frac{x^{(2)}-X_j^{(2)}}{h}\right)$$

and

(20) $$\tilde{W}(t_1, t_2)= W(t_1, t_2) \qquad \text{if } |t_1|\leqq c(n) \text{ and } |t_2|\leqq c(n)$$
$$=0 \qquad \text{elsewhere}\, .$$

LEMMA 3. *Under the assumption of Lemma* 1 *and* (A5)

(21) $$R(n)=nh^2\int \{(\tilde{f}_n^*(x)-\mathrm{E}\, \tilde{f}_n^*(x))^2-(f_n^*(x)-\mathrm{E}\, f_n^*(x))^2\}\, a(x)dx$$

$$=o(h(n)) .$$

PROOF. Set

$$\tag{22} \begin{aligned} &\tilde{\tilde{W}}(n)=W(x)-\tilde{W}(x) \\ &\tilde{\tilde{f}}_n^*(x^{(1)}, x^{(2)})=(nh^2)^{-1}\sum_{j=1}^{N}\tilde{\tilde{W}}\left(\frac{x^{(1)}-X_j^{(1)}}{h}, \frac{x^{(2)}-X_j^{(2)}}{h}\right). \end{aligned}$$

Clearly

$$\tag{23} f_n^*(x)=\tilde{f}_n^*(x)+\tilde{\tilde{f}}_n^*(x)$$

and hence

$$\tag{24} \begin{aligned} &(f_n^*(x)-\mathrm{E}\,f_n^*(x))^2-(\tilde{f}_n^*(x)-\mathrm{E}\,\tilde{f}_n^*(x))^2 \\ &=(\tilde{\tilde{f}}_n^*(x)-\mathrm{E}\,\tilde{\tilde{f}}_n^*(x))^2+2(\tilde{f}_n^*(x)-\mathrm{E}\,\tilde{f}_n^*(n))(\tilde{\tilde{f}}_n^*(n)-\mathrm{E}\,\tilde{\tilde{f}}_n^*(n)) \\ &=R_{1n}(x)+2R_{2n}(x)\,,\ \text{say.} \end{aligned}$$

Now

$$\tag{25} \begin{aligned} \mathrm{E}\,R_{1n}(x)&=\mathrm{E}\,(\tilde{\tilde{f}}_n^*(x)-\mathrm{E}\,\tilde{\tilde{f}}_n^*(x))^2 \\ &=(nh^4(n))^{-1}\,\mathrm{E}\left[\tilde{\tilde{W}}\left(\frac{x^{(1)}-X^{(1)}}{h}, \frac{x^{(2)}-X^{(2)}}{h}\right)\right]^2 \\ &=(nh^2)^{-1}\int_{\substack{|t_1|>c(n) \\ \text{or } |t_2|>c(n)}} W^2(t)f(x+ht)dt \\ &=(nh^2(n))^{-1}\left\{\int W^2(t)f(x+ht)dt-\int_{\substack{|t_1|\leq c(n) \\ |t_2|\leq c(n)}} W^2(t)f(x+ht)dt\right\} \\ &=(nh^2(n))^{-1}\left\{f(x)\int W^2(t)dt-f(x)\int_{\substack{|t_1|\leq c(n) \\ |t_2|\leq c(n)}} W^2(t)dt+O(h^2)\right\} \\ &=(nh^2(n))^{-1}\left\{f(x)\int_{\substack{|t_1|\geq c(n) \\ \text{or } |t_2|\geq c(n)}} W^2(t)dt+O(h^2(n))\right\} . \end{aligned}$$

Hence

$$\left|nh^2(n)\int \mathrm{E}\,R_{1n}(x)a(x)dx\right|=O(h^2(n)) \qquad \text{if (A5) holds}, $$
$$=o(h(n)) .$$

Let

$$p^*q(x)=\int p(x-y)q(y)dy$$

denote the convolution of p and q at x. Then

$$
\begin{aligned}
(26)\quad & \mathrm{E}\, R_{2n}(x) R_{2n}(x') \\
&= (n^2 h^8(n))^{-1} \Big\{ \mathrm{E}\, \tilde{W}\Big(\frac{x-X}{h}\Big) \tilde{W}\Big(\frac{x'-X}{h}\Big) \mathrm{E}\, \tilde{\tilde{W}}\Big(\frac{x-X}{h}\Big) \tilde{\tilde{W}}\Big(\frac{x'-X}{h}\Big) \\
&\qquad + \mathrm{E}\, \tilde{W}\Big(\frac{x-X}{h}\Big) \tilde{\tilde{W}}\Big(\frac{x'-X}{h}\Big) \mathrm{E}\, \tilde{\tilde{W}}\Big(\frac{x-X}{h}\Big) \tilde{W}\Big(\frac{x'-X}{h}\Big) \Big\} \\
&= (n^2 h^4(n))^{-1} \Big[\Big\{ f(x') \tilde{W} * \tilde{W}\Big(\frac{x-x'}{h}\Big) + O(h^2(n)) \Big\} \\
&\qquad \times \Big\{ f(x') \tilde{\tilde{W}} * \tilde{\tilde{W}}\Big(\frac{x-x'}{h}\Big) + O(h^2(n)) \Big\} \\
&\qquad + \Big\{ f(x') \tilde{W} * \tilde{\tilde{W}}\Big(\frac{x-x'}{h}\Big) + O(h^2(n)) \Big\} \\
&\qquad \times \Big\{ f(x') \tilde{\tilde{W}} * \tilde{W}\Big(\frac{x-x'}{h}\Big) + O(h^2(n)) \Big\} \Big] \\
&= (n^2 h^4(n))^{-1} \Big[f^2(x') \Big\{ \tilde{W} * \tilde{W}\Big(\frac{x-x'}{h}\Big) \tilde{\tilde{W}} * \tilde{\tilde{W}}\Big(\frac{x-x'}{h}\Big) \\
&\qquad + \tilde{W} * \tilde{\tilde{W}}\Big(\frac{x-x'}{h}\Big) \tilde{\tilde{W}} * \tilde{W}\Big(\frac{x-x'}{h}\Big) \Big\} + O\Big(h^2 H\Big(\frac{x-x'}{h}\Big)\Big) + O(h^4(n)) \Big]
\end{aligned}
$$

where

$$
\begin{aligned}
H\Big(\frac{x-x'}{h}\Big) &= \tilde{W} * \tilde{W}\Big(\frac{x-x'}{h}\Big) + \tilde{\tilde{W}} * \tilde{\tilde{W}}\Big(\frac{x-x'}{h}\Big) \\
&\quad + \tilde{W} * \tilde{\tilde{W}}\Big(\frac{x-x'}{h}\Big) + \tilde{\tilde{W}} * \tilde{W}\Big(\frac{x-x'}{h}\Big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
(27)\quad & \mathrm{E}\Big(n h^2(n) \int R_{2n}(x) a(x) dx \Big)^2 \\
&= n^2 h^4(n) \iint \mathrm{E}\, (R_{2n}(x) R_{2n}(x')) a(x) a(x') dx dx' \\
&= \iint f^2(x') \Big(\tilde{W} * \tilde{W}\Big(\frac{x-x'}{h}\Big) \tilde{\tilde{W}} * \tilde{\tilde{W}}\Big(\frac{x-x'}{h}\Big) \\
&\qquad + \tilde{W} * \tilde{\tilde{W}}\Big(\frac{x-x'}{h}\Big) \tilde{\tilde{W}} * \tilde{W}\Big(\frac{x-x'}{h}\Big) \Big) a(x) a(x') dx dx' \\
&\quad + h^2(n) \iint H\Big(\frac{x-x'}{h}\Big) a(x) a(x') dx dx' + O(h^4(n)) \\
&= O(h^2(n)\, \tilde{W} * \tilde{W} * \tilde{\tilde{W}} * \tilde{\tilde{W}}(0)) + O(h^4(n)) \\
&= o(h^2(n)), \qquad \text{since } (\tilde{W} * \tilde{\tilde{W}})^{(2)}(0) \to 0 \text{ as } n \to \infty .
\end{aligned}
$$

This completes the proof.

Lemmas 1, 2 and 3 together imply that if we let $h(n) \downarrow 0$ at a proper rate and if the weight function does not have a heavy tail, then

$$\tilde{S}_n = nh^2 \int (\tilde{f}_n^*(x) - \mathrm{E}\,\tilde{f}_n^*(x))^2 a(x)dx \tag{28}$$

can be considered instead of S_n.

3.1. *Proof of Theorem* 1

Note that

$$\tilde{S}_n = \sum_j \sum_k \tilde{U}_{jk}(n) \tag{29}$$

where

$$\begin{aligned} \tilde{U}_{jk}(n) &= \int_{jd(n)}^{(j+1)d(n)} \int_{kd(n)}^{(k+1)d(n)} nh^2(\tilde{f}_n^*(x) - \mathrm{E}\,\tilde{f}_n^*(x))^2 a(x)dx \\ &= \int_{jd(n)}^{(j+1)d(n)} \int_{kd(n)}^{(k+1)d(n)} \left(\frac{\sqrt{n}}{h(n)} \int \tilde{W}\left(\frac{x^{(1)} - u_1}{h}, \frac{x^{(2)} - u_2}{h} \right) d(F_n^*(u) - F(u)) \right)^2 \\ &\quad \times a(x)dx , \end{aligned} \tag{30}$$

and

$$d(n) = 2h(n)c(n) , \qquad u = (u_1, u_2) \tag{31}$$

$$nF_n^*(x) = \frac{N}{n} F_n(x) . \tag{32}$$

Since $\tilde{W}(\cdot, \cdot)$ vanishes outside $[-c(n), c(n)] \times [-c(n), c(n)]$ and nF_n^* is a Poisson process on the plane, the random variables $\{\tilde{U}_{jk}(n)\}$ are 2×2 independent. Set

$$\tilde{V}_{jk}(n) = \int_{\Delta_j}^{\Delta_{j'}} \int_{\Delta_k}^{\Delta_{k'}} nh^2(n)(\tilde{f}_n^*(x) - \mathrm{E}\,\tilde{f}_n^*(x))^2 a(x)dx \tag{33}$$

where

$$\Delta_j = (j+1)d(n) + \Delta(n) \quad \text{and} \quad \Delta_{j'} = (j+1)(d(n) + \Delta(n))$$

with

$$d(n) = o(\Delta(n)) \quad \text{and} \quad \Delta(n) \downarrow 0 \quad \text{as } n \to \infty .$$

At this stage is should be mentioned that the claim

$$\mathrm{E}\,|V_{jk}(n) - S_n| = O(b\Delta)$$

in [1] is not true. In fact, in the notations Rosenblatt [1],

$$\mathrm{E}\,|V_{jk}(n) - S_n| = O(b/\Delta)$$

and this approximation is not enough to replace S_n by $\sum_j \sum_k V_{jk}(n)$ in the proof of asymptotic normality. A different approach will be given here to show the asymptotic normality. Let $Q(n)$ denote the rectangle $[-q(n), q(n)]\times[-q(n), q(n)]$ with $q(n)\to\infty$ as $n\to\infty$. Set

$$S(Q(n))=nh^2\int_{Q(n)}(\tilde{f}_n^*(x)-\mathrm{E}\,\tilde{f}_n^*(x))^2a(x)dx\,. \tag{34}$$

Let

$$I(j,k)=(\Delta_j, \Delta_{j'})\times(\Delta_k, \Delta_{k'})$$

and let

$$Q(n,\Delta)=\bigcup_{j,k} I(j,k)\,,\qquad \tilde{Q}(n,\Delta)=Q(n)-Q(n,\Delta)\,,$$

where the union is taken over all $I(j, k)$ within $Q(n, \Delta)$. Also let

$$\begin{aligned}&\tilde{Q}(n,\Delta)=Q(n)-Q(n,\Delta)\\ &\tilde{Q}(n)=(-\infty,\infty)\times(-\infty,\infty)-Q(n)\,.\end{aligned} \tag{35}$$

The following lemma shows that we can in fact replace $\tilde{S}_n$ by $\sum_j \sum_k \tilde{V}_{jk}(n)$ for large n.

LEMMA 4. *Under the assumptions of Lemma* 1

(i) $$\frac{\mathrm{Var}\,\{(\tilde{S}_n-\mathrm{E}\,\tilde{S}_n)-(\tilde{S}_n(Q(n))-\mathrm{E}\,\tilde{S}_n(Q(n)))\}}{\mathrm{Var}\,(\tilde{S}_n)}\to 0 \qquad as\ n\to\infty\,,$$

(ii) $$\frac{\mathrm{Var}\,\{(\tilde{S}_n(Q(n))-\mathrm{E}\,\tilde{S}_n(Q(n)))-(\sum_j\sum_k \tilde{V}_{jk}(n)-\sum_j\sum_k \mathrm{E}\,\tilde{V}_{jk}(n))\}}{\mathrm{Var}\,(\tilde{S}_n)}\to 0$$

as $n\to\infty$

and

(iii) $$\frac{\mathrm{Var}\,(\sum_j\sum_k \tilde{V}_{jk}(n))}{\mathrm{Var}\,(\tilde{S}_n)}\to 1 \qquad as\ n\to\infty\,.$$

PROOF. Observe that

$$\tilde{S}_n-\tilde{S}_n(Q(n))=\int_{\tilde{Q}(n)} nh^2(n)(\tilde{f}_n^*(x)-\mathrm{E}\,\tilde{f}_n^*(x))^2a(x)dx\,. \tag{36}$$

Hence

$$\mathrm{Var}\,(\tilde{S}_n-\tilde{S}_n(Q(n)))\approx \tilde{W}^{(4)}(0)\int_{\tilde{Q}(n)} a^2(x)(2h^2(n)f^2(x)+n^{-1}f(x))dx$$

and

$$\mathrm{Var}\,(\tilde{S}_n) \approx W^{(4)}(0) \int_{R^2} a^2(x)(2h^2(n)f^2(x)+n^{-1}f(x))dx$$

where $R^2=(-\infty, \infty)\times(-\infty, \infty)$. Therefore

$$\frac{\mathrm{Var}\,(\tilde{S}_n-\tilde{S}_n(Q(n)))}{\mathrm{Var}\,(\tilde{S}_n)} \approx \frac{\int_{\tilde{Q}(n)} a^2(x)f^2(x)dx}{\mathrm{Var}\,(\tilde{S}_n)} \to 0 \qquad \text{as } n\to\infty . \tag{37}$$

Since $\int_{R^2} a^2(x)f^2(x)<\infty$ and $\tilde{Q}(n)\to\phi$, this completes the proof of (i).

Proceeding as before it can be shown that

$$\frac{\mathrm{Var}\,(\tilde{S}_n(Q(n))-\sum_j\sum_k \tilde{V}_{jk}(n))}{\mathrm{Var}\,(\tilde{S}_n)} \approx \frac{\int_{\tilde{Q}(n,\Delta)} a^2(x)f^2(x)dx}{\int_{R^2} a^2(x)f^2(x)dx} . \tag{38}$$

The numerator in the right hand side of (38) can be expressed as the sum of integrals of the form

$$\tilde{I}_{(j,k)}=\int_{jd(n)}^{(j+1)d(n)}\int_{\Delta_k}^{\Delta_{k'}} a^2(x)f^2(x)dx .$$

But

$$\frac{\tilde{I}(j,k)}{I(j,k)}=O\left(\frac{d(n)}{\Delta(n)}\right) \tag{39}$$

and hence the right hand side of (38) is $\approx O\left(\frac{d(n)}{\Delta(n)}\right)\to 0$ as $n\to\infty$. Since the numerator of (ii) is equal to $\mathrm{Var}\,(\tilde{S}_n(Q(n))-\sum_j\sum_k \tilde{V}_{jk}(n))$, this completes the proof of (ii).

Finally to show (iii) observe that

$$\begin{aligned}\frac{\mathrm{Var}\,(\sum_j\sum_k \tilde{V}_{jk}(n))}{\mathrm{Var}\,(\tilde{S}_n)} &\approx \frac{\int_{Q(n,\Delta)} a^2(x)f^2(x)dx}{\int_{R^2} a^2(x)f^2(x)dx} \\ &=1-\frac{\int_{\tilde{Q}(n)} a^2(x)f^2(x)dx+\int_{\tilde{Q}(n,\Delta)} a^2(x)f^2(x)dx}{\int_{R^2} a^2(x)f^2(x)dx}\to 1 \\ &\qquad \text{as } n\to\infty .\end{aligned} \tag{40}$$

The conclusion above follows from the arguments used in (ii). This completes the proof of the lemma.

The final step in the proof of Theorem 1 is to show that the Liapunov's conditions are satisfied. Following the analysis of Rosenblatt [1] it can be shown that

$$(41)\qquad \mathrm{E}\,|\tilde{V}_{jk}(n)-\mathrm{E}\,\tilde{V}_{jk}(n)|^4=O\Big(h^4(n)\Delta^2(n)\int_{I(j,k)}a^2(x)f^2(x)dx\Big).$$

Hence

$$(42)\qquad \sum_j\sum_k \mathrm{E}\,|\tilde{V}_{jk}(n)-\mathrm{E}\,\tilde{V}_{jk}(n)|^4=O(h^4(n)\Delta^2(n)).$$

This shows that

$$(43)\qquad \frac{\sum_j\sum_k \mathrm{E}\,|\tilde{V}_{jk}(n)-\mathrm{E}\,\tilde{V}_{jk}(n)|^4}{(\mathrm{Var}\,(\sum_j\sum_k \tilde{V}_{jk}(n)))^2}=O(\Delta^2(n))\to 0 \qquad \text{as } n\to\infty .$$

This completes the proof of Theorem 1.

3.2. *Proof of Theorem* 2

To prove Theorem 2 it is enough to show that we can replace $(f_n\cdot(x^{(1)},x^{(2)})-g_{1n}(x^{(1)})g_{2n}(x^{(2)}))$ by $(\tilde{f}_n(x^{(1)},x^{(2)})-\tilde{g}_{1n}(x^{(1)})\tilde{g}_{2n}(x^{(2)}))$ with a small error. The following lemma shows that in fact this can be done.

LEMMA 5. *Under the assumptions* (A1)–(A4) *and* (A6)

$$(40)\qquad nh^2(n)\int\{(f_n(x)-g_{1n}(x^{(1)})g_{2n}(x^{(2)}))^2-(\tilde{f}_n(x)-\tilde{g}_{1n}(x^{(1)})\tilde{g}_{2n}(x^{(2)}))^2\}a(x)d(x)$$
$$=o(h(n)) .$$

PROOF. The proof of this is similar to those given in Lemma 3. The details are omitted.

4. Concluding remarks

The results of this paper provide a generalization of the results of Rosenblatt to wider class of weight functions. Also a correct proof of Theorem 1 of Rosenblatt is given. Theorem 1 and Theorem 2 can be used to test goodness of fit and to test of independence for density estimates based on a wider class of weight functions than those considered by Rosenblatt.

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