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OPTIMAL ALLOCATION OF OBSERVATIONS IN ONE-WAY RANDOM NORMAL MODEL

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1. Introduction and summary

DeGroot [3] has considered the problem of allocation of the observations in one-way random normal model when both variance components are known and number of subclasses is not fixed. He has shown that the allocation in which every observation is made in each individual subclass, is optimal.

This note presents some complements to the result. One of them concerns optimal allocation of the observations when number of subclasses is fixed. It has been shown that the most uniform allocation of the observations in subclasses is optimal.

The second problem deals with the case when one of the variance components is unknown. Some results connected with the choice of an allocation in this situation are given in Section 4. However, an optimal allocation exists.

Section 4 also includes some remarks on allocations of the observations when both variance components are unknown. It appears, that no two essentially different allocations, i.e. such allocations that one of them cannot be obtained from the other by a denumeration of subclasses, are comparable.

2. Definitions, notations and known results

An experiment is defined as a family $\{P_\theta,\ \theta\in\Theta\}$ of probability measures on a measurable space $(\mathcal{X}, \mathcal{A})$. The concept of comparison of experiments is due to Blackwell [1] and [2].

Let $\mathcal{E}=\{\mathcal{X}, \mathcal{A}, P_\theta,\ \theta\in\Theta\}$ and $\mathcal{F}=\{\mathcal{Y}, \mathcal{B}, Q_\theta,\ \theta\in\Theta\}$ be two experiments with the same parameter set Θ.

DEFINITION 2.1. The experiment $\mathcal{E}$ is said to be more informative than the experiment $\mathcal{F}$ if for every finite-decision statistical problem and for every decision function δ_F based on $\mathcal{F}$ exists a decision function δ_E based on $\mathcal{E}$ such that $R(\delta_E, \theta) \leqq R(\delta_F, \theta)$ for every $\theta \in \Theta$.

DEFINITION 2.2. The experiment $\mathcal{E}$ is said to be sufficient for the experiment $\mathcal{F}$ if there exists a stochastic transformation $\pi(\cdot,\cdot)$ from $\mathcal{X}$ into $\mathcal{Y}$ such that

$$Q_\theta(B) = \int_{\mathcal{X}} \pi(B, x) dP_\theta(x)$$

for all $B \in \mathcal{B}$ and $\theta \in \Theta$.

It follows from the Theorem 3 in LeCam [5] that if the family $\{P_\theta, \theta \in \Theta\}$ is dominated, $\mathcal{Y}$, is a Borel subset of a Polish space and $\mathcal{B}$ is the class of Borel subsets of $\mathcal{Y}$ then the experiment $\mathcal{E}$ is more informative than the experiment $\mathcal{F}$ if and only if $\mathcal{E}$ is sufficient for $\mathcal{F}$.

Let T be a matrix of $n \times p$ and V be a positive definite matrix of $n \times n$.

By $N(X|T\beta, \sigma_0 V)$ will be denoted the experiment obtained by observing a normally distributed random vector X with the expectation $T\beta$ and the covariance $\sigma_0 V$, where β is an unknown vector from R^p and σ_0 is a given positive scalar.

Respective, by $N(X|T\beta, \sigma V;\ \sigma > 0)$ will be denoted the experiment obtained by observing a normally distributed random vector X with the expectation $T\beta$ and the covariance σV, where β and σ are unknown.

We need the following theorem, given by Hansen and Torgersen [4].

THEOREM 2.1. (i) *The experiment* $N(X|T\beta, \sigma_0 I_n)$ *is more informative than the experiment* $N(Y|U\beta, \sigma_0 I_m)$ *if and only if the matrix* $T'T - U'U$ *is non-negative definite.*

(ii) *The experiment* $N(X|T\beta, \sigma I_n;\ \sigma > 0)$ *is more informative than the experiment* $N(Y|U\beta, \sigma I_m;\ \sigma > 0)$ *if and only if the matrix* $T'T - U'U$ *is non-negative definite and* $n \geqq m + \operatorname{rank}(T'T - U'U)$.

Putting $n = m$ we obtain the following corollary.

COROLLARY 2.1. *The experiment* $N(X|T\beta, \sigma I_n;\ \sigma > 0)$ *is more informative than the experiment* $N(Y|U\beta, \sigma I_n;\ \sigma > 0)$ *if and only if* $T'T = U'U$.

Let n and k be positive integers such that $k \leqq n$. Any choice of non-negative integers $n_1, \cdots, n_k$ such that $\sum n_i = n$, is called an allocation of n observations is not more than k subclasses and is denoted by $A(n, k;\ n_1, \cdots, n_k)$.

Let $n_{i_1}, \cdots, n_{i_l}$ be positive elements of the sequence $n_1, \cdots, n_k$. It will be convenient to identify the allocations $A(n, k;\ n_1, \cdots, n_k)$ and $A(n, l;\ n_{i_1}, \cdots, n_{i_l})$.

The class of all allocations of n observations in not more than k subclasses, will be denoted by $L(n, k)$. According to the above convention we have

$$L(n, k) \subset L(n, k') \tag{2.1}$$

for $k \leqq k' \leqq n$.

An allocation $A(n, k; n_1, \cdots, n_k)$ may be considered in one of the following experiments (or models):

$$E^{\sigma_0, \rho_0}_{n_1, \cdots, n_k} = N(X \mid 1_n \mu, \sigma_0 \{I_n + \rho_0 \operatorname{diag}(1_{n_1} 1'_{n_1}, \cdots, 1_{n_k} 1'_{n_k})\}) ,$$

$$E^{\rho_0}_{n_1, \cdots, n_k} = N(X \mid 1_n \mu, \sigma \{I_n + \rho_0 \operatorname{diag}(1_{n_1} 1'_{n_1}, \cdots, 1_{n_k} 1'_{n_k})\} ; \sigma > 0) ,$$

or

$$E_{n_1, \cdots, n_k} = N(X \mid 1_n \mu, \sigma \{I_n + \rho \operatorname{diag}(1_{n_1} 1'_{n_1}, \cdots, 1_{n_k} 1'_{n_k})\} ; \rho > 0, \sigma > 0) ,$$

where 1_n denotes the n-dimensional column vector of ones, and $\operatorname{diag}(T_1, \cdots, T_k)$ denotes the block-diagonal matrix with diagonal elements $T_1, \cdots, T_k$.

We shall write:

$$A(n, k; n_1, \cdots, n_k) \overset{\sigma_0, \rho_0}{\geqq} A(n, k; m_1, \cdots, m_k)$$

if $E^{\sigma_0, \rho_0}_{n_1, \cdots, n_k}$ is more informative than $E^{\sigma_0, \rho_0}_{m_1, \cdots, m_k}$,

$$A(n, k; n_1, \cdots, n_k) \overset{\rho_0}{\geqq} A(n, k; m_1, \cdots, m_k)$$

if $E^{\rho_0}_{n_1, \cdots, n_k}$ is more informative than $E^{\rho_0}_{m_1, \cdots, m_k}$, and

$$A(n, k; n_1, \cdots, n_k) \geqq A(n, k; m_1, \cdots, m_k)$$

if $E_{n_1, \cdots, n_k}$ is more informative than $E_{m_1, \cdots, m_k}$.

DEFINITION 2.3. An allocation $A(n, k; n_1, \cdots, n_k)$ is said to be:

(a) locally optimal at $\sigma = \sigma_0$, $\rho = \rho_0$ if $A(n, k; n_1, \cdots, n_k) \overset{\sigma_0, \rho_0}{\geqq} A(n, k; m_1, \cdots, m_k)$ for all non-negative integers $m_1, \cdots, m_k$ such that $\sum m_i = n$,

(b) locally optimal at $\rho = \rho_0$ if $A(n, k; n_1, \cdots, n_k) \overset{\rho_0}{\geqq} A(n, k; m_1, \cdots, m_k)$ for all non-negative integers $m_1, \cdots, m_k$ such that $\sum m_i = n$,

(c) totally optimal if $A(n, k; n_1, \cdots, n_k) \geqq A(n, k; m_1, \cdots, m_k)$ for all non-negative integers $m_1, \cdots, m_k$ such that $\sum m_i = n$.

3. Comparison of allocations at $\sigma=\sigma_0$, $\rho=\rho_0$

Applying Theorem 2.1 (i) to the experiments $E^{\sigma_0,\rho_0}_{n_1,\cdots,n_k}$ and $E^{\sigma_0,\rho_0}_{m_1,\cdots,m_k}$ we obtain the following lemma.

LEMMA 3.1. *$A(n, k; n_1,\cdots, n_k) \overset{\sigma_0,\rho_0}{\geqq} A(n, k; m_1,\cdots, m_k)$ if and only if*

$$\sum_{i=1}^{k} \frac{n_i}{1+n_i\rho_0} \geqq \sum_{i=1}^{k} \frac{m_i}{1+m_i\rho_0} .$$

PROOF. Let X and Y be the vectors of the observations in the experiments $E^{\sigma_0,\rho_0}_{n_1,\cdots,n_k}$ and $E^{\sigma_0,\rho_0}_{m_1,\cdots,m_k}$, respectively, and let

$$X'=\sigma_0^{-1/2}\{I_n+\rho_0 \operatorname{diag}(1_{n_1}1'_{n_1},\cdots, 1_{n_k}1'_{n_k})\}^{-1/2}X ,$$

$$Y'=\sigma_0^{-1/2}\{I_n+\rho_0 \operatorname{diag}(1_{m_1}1'_{m_1},\cdots, 1_{m_k}1'_{m_k})\}^{-1/2}Y .$$

As the transformations $X\to X'$ and $Y\to Y'$ are non-singular, X is more informative than Y, if and only if, X' is more informative than Y'.

Thus we only need to consider the experiments $N(X'|T\mu, \sigma_0 I_n)$ and $N(Y'|U\mu, \sigma_0 I_n)$, where μ is an unknown scalar defined by the equality $\mu 1_n = \mathrm{E}\, X$,

$$T=T_{\sigma_0,\rho_0}=\sigma_0^{-1/2}\{I_n+\rho_0 \operatorname{diag}(1_{n_1}1'_{n_1},\cdots, 1_{n_k}1'_{n_k})\}^{-1/2}1_n ,$$

$$U=U_{\sigma_0,\rho_0}=\sigma_0^{-1/2}\{I_n+\rho_0 \operatorname{diag}(1_{m_1}1'_{m_1},\cdots, 1_{m_k}1'_{m_k})\}^{-1/2}1_n .$$

Now, by Theorem 2.1 (i) we obtain the desired result.

THEOREM 3.1. *An allocation $A(n, k; n_1,\cdots, n_k)$ is locally optimal at $\sigma=\sigma_0$, $\rho=\rho_0$ if and only if*

$$\max_{1\leqq i,j\leqq k} |n_i-n_j| \leqq 1 , \tag{3.1}$$

provided that $\rho_0>0$.

PROOF. Suppose that (3.1) is not satisfied. Then there exist numbers j, $l\in\{1,\cdots,k\}$ such that $n_j-n_l>1$.

Let

$$m_i=\begin{cases} n_i-1 & \text{if } i=j \\ n_i+1 & \text{if } i=l \\ n_i & \text{otherwise} . \end{cases} \tag{3.2}$$

We observe that $\sum_{i=1}^{k} \frac{m_i}{1+m_i\rho_0} - \sum_{i=1}^{k} \frac{n_i}{1+n_i\rho_0} = \frac{n_j-1}{1+(n_j-1)\rho_0} + \frac{n_l+1}{1+(n_l+1)\rho_0}$

$-\frac{n_j}{1+n_j\rho_0}-\frac{n_l}{1+n_l\rho_0}>0$. Thus, by Lemma 3.1, the allocation $A(n, k; n_1, \cdots, n_k)$ is not locally optimal at $\sigma=\sigma_0$, $\rho=\rho_0$.

Now let (3.1) be satisfied and let $m_1, \cdots, m_k$ be non-negative integers such that $\sum m_i=n$. Starting from the allocation $A(n, k; m_1, \cdots, m_k)$ we reach, after a finite numbers of "improving steps" of the form (3.2), to an allocation $A(n, k; r_1, \cdots, r_k)$ such that $\sum \frac{r_i}{1+r_i\rho_0} \geqq \sum \frac{m_i}{1+m_i\rho_0}$ and $\max_{1\leqq i,j\leqq k} |r_i-r_j|\leqq 1$.

Moreover, by invariability of $\sum \frac{n_i}{1+n_i\rho_0}$ with respect to any permutation of numbers $n_1, \cdots, n_k$ we obtain $\sum \frac{r_i}{1+r_i\rho_0}=\sum \frac{n_i}{1+n_i\rho_0}$. Thus, by Lemma 3.1, the allocation $A(n, k; n_1, \cdots, n_k)$ is locally optimal at $\sigma=\sigma_0$, $\rho=\rho_0$.

Putting $k=n$ we obtain the following corollary.

COROLLARY 3.1. *The allocation* $A(n, n; 1, \cdots, 1)$ *is locally optimal at* $\sigma=\sigma_0$, $\rho=\rho_0$ *for every positive* σ_0 *and* ρ_0.

Looking at (2.1) we observe that Corollary 3.1 is an another formulation of Theorem 3.3 in DeGroot [3].

We note that for $k=2$ and for every pair $A(n, 2; n_1, n_2)$ and $A(n, 2; m_1, m_2)$, holds at least one of the relations

$$A(n, 2; n_1, n_2) \overset{\sigma_0, \rho_0}{\geqq} A(n, 2; m_1, m_2)$$

or

$$A(n, 2; m_1, m_2) \overset{\sigma_0, \rho_0}{\geqq} A(n, 2; n_1, n_2) .$$

This property does not preserve for $k=3$. Counter-example: $A(10, 3; 1, 4, 5)$ and $A(10, 3; 2, 2, 6)$.

4. Comparison of allocations under σ unknown

An analogue of Lemma 3.1 is the following

LEMMA 4.1. $A(n, k; n_1, \cdots, n_k) \overset{\rho_0}{\geqq} A(n, k; m_1, \cdots, m_k)$ *if and only if*

$$\sum \frac{n_i}{1+n_i\rho_0}=\sum \frac{m_i}{1+m_i\rho_0} . \tag{4.1}$$

PROOF. See the Proof of Lemma 3.1, using Theorem 2.1 (ii), or Corollary 2.1, instead of Theorem 2.1 (i).

LEMMA 4.2. *There is no allocation being locally optimal at $\rho=\rho_0$, provided that $\rho_0>0$ and $k>1$.*

PROOF. Suppose, by contradiction, that an allocation $A(n, k; n_1, \cdots, n_k)$ is locally optimal at $\rho=\rho_0$. Then the condition (3.1) is satisfied.

On the other hand, by Lemma 4.1, the condition (4.1) holds for every integers $m_1, \cdots, m_k$ such that $\sum m_i=n$. Putting $m_1=n$ and $m_i=0$ for $i>1$ we observe that the conditions (3.1) and (4.1) are inconsistent.

This completes the proof.

Now we consider the case σ and ρ unknown. It follows from Definition 2.2 that if $A(n, k; n_1, \cdots, n_k)\geqq A(n, k; m_1, \cdots, m_k)$ then $A(n, k; n_1, \cdots, n_k)\overset{\rho_0}{\geqq} A(n, k; m_1, \cdots, m_k)$ for every $\rho_0>0$. From this and from Lemma 4.1 we obtain the following

COROLLARY 4.1. *$A(n, k; n_1, \cdots, n_k)\geqq A(n, k; m_1, \cdots, m_k)$ if and only if the sequence $m_1, \cdots, m_k$ is a permutation of numbers $n_1, \cdots, n_k$.*

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