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SENSITIVITY OF BAYES DECISIONS FOR THE SUCCESS PROBABILITY FOR SAMPLES FROM A NONBINOMIAL POPULATION

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Summary

A family of generalised negative binomial distributions is employed to investigate inference robustness of the Bayes estimator of the unknown parameter of the binomial distribution. A zone of sensitivity for the test of significance is constructed to forewarn the pro-Jeffreys Bayesians against indiscriminate choice of the probability in favour of the null hypothesis. A few selected tables are presented to illustrate the effect of relaxation of the 'binomiality' assumption.

1. Introduction

Inference robustness to non-normality of the Bayesian procedures has drawn attention of the researchers during the last two decades. Attempts have also been made to investigate the effect of choice of the prior distribution and also that of the loss function on the inference concerning the unknown parameters of the parent distribution.

Jain and Consul [3] introduced a family of generalised negative binomial distributions (GNBD) by compounding the negative binomial distribution with another parameter. It was shown that GNBD provides a better fit to some of the well known historical data than the negative binomial, Poisson, and the binomial distributions. This study suggests that, like that for normality assumption, the behaviour of the Bayes decision for the success probability as the binomial parameter to 'non-binomiality' of the parent population needs investigation.

2. Posterior distribution

Let $X_1, X_2, \cdots, X_k$ be a random sample of size k from some fixed member of the GNBD (see, Jain and Consul [3]) defined as follows:

Key words: Inference robustness, beta prior, quadratic loss, generalised negative binomial distribution, zone of sensitivity, non-binomiality.

For $\alpha \in (0, 1)$ and $|\alpha\beta| < 1$

$$(1) \qquad f(x;\alpha,n)=\begin{cases} \dfrac{n\Gamma(n+\beta x)}{x!\Gamma(n+\beta x-x+1)}\alpha^{x}(1-\alpha)^{n+\beta x-x}, & \\ & n>0;\ x=0,1,2,\cdots \\ 0; & \text{for } x\leqq m \text{ if } n+\beta m<0. \end{cases}$$

It is easy to verify that the binomial and the negative binomial distributions belong to this family of distributions for $\beta=0$ and $\beta=1$, respectively. The parameter β takes into account the variations in mean and variance in such a way that the both are positively correlated with the value of β. Here variance increases or decreases faster than the mean. In this paper we consider the parameter β as a measure of 'non-binomiality'.

For a realisation $\boldsymbol{x}=(x_1, x_2, \cdots, x_k)$ of the random sample for some fixed values of the parameters n and β of GNBD given in (1)

$$(2) \qquad f_k(\alpha\,|\,\boldsymbol{x})=\alpha^{s}(1-\alpha)^{kn+(\beta-1)s}\prod_{i=1}^{k}\frac{n\Gamma(n+\beta x_i)}{x_i!\Gamma(n+\beta x_i-x_i+1)}, \qquad s=\sum_{i=1}^{k}x_i,$$

is the likelihood function for the unknown parameter α. Let us assume that the prior distribution ξ of the success probability is beta

$$(3) \qquad \xi(\alpha)=\alpha^{u-1}(1-\alpha)^{v-1}/B(u, v),$$

with known parameters $u>0$, $v>0$.

The marginal distribution of $\boldsymbol{X}$ with respect to the beta prior (3) is

$$(4) \qquad f_{\xi}(\boldsymbol{x})=\frac{B(u', v')}{B(u, v)}\prod_{i=1}^{k}\frac{n\Gamma(n+\beta x_i)}{x_i!\Gamma(n+\beta x_i-x_i+1)},$$

and, therefore, the posterior distribution of A with respect to the prior ξ is

$$(5) \qquad \xi(\alpha\,|\,\boldsymbol{x})=\alpha^{u'-1}(1-\alpha)^{v'-1}/B(u', v'),$$

a beta with parameters $u'=u+s$ and $v'=v+nk+(\beta-1)s$. It is interesting to observe that beta distribution serves as a conjugate prior for the GNBD as well. As expected, for $\beta=0$, the posterior distribution (5) reduces to the one for binomial parent and as $\beta\to 1$ it tends to that for negative binomial case. Further, for $s=0$, the posterior distribution is seen to be independent of the 'non-binomiality' parameter β. Thus the Bayes decision for α will be insensitive to any amount of relaxation of the binomiality assumption when no success occurs in the observed sample. This will also be true for the negative binomial parent.

3. Bayes estimator and risk

Under quadratic loss function, $L(d, \alpha)=(d-\alpha)^2$, the Bayes estimator for the success probability α of the GNBD is given by

$$\delta_{\xi}^{*}(\boldsymbol{x})=\int_0^1 \alpha\xi(\alpha\,|\,\boldsymbol{x})d\alpha=u'/(u'+v') \ . \tag{6}$$

Clearly, this estimator tends to underestimate the true value of α even when sample size increases. The non-binomiality also adds to underestimation of α. In the binomial situation the number of successes s varies between 0 and nk and corresponding Bayes estimator depends on the sample observations only through the observed values of s. Numerical estimates of this effect for selected values of s and β are obtained in the Table I. The effect of non-binomiality increases as the number of successes in the observed sample increases. For example, as $\beta\rightarrow 1$ the posterior mean decreases by 0.005 for $s=1$, whereas, for $s=20$ it decreases by 0.4167. If posterior mode is used as the Bayes estimator of α, given by $(u'-1)/(u'+v'-2)$, Table I shows that it is

Table I. Bayes estimates for α (mean and mode of posterior distribution) and posterior variance when $u=v=n=2$, $k=10$.

s	$\beta\rightarrow$ 0.0	0.1	0.2	0.4	0.7	1.0
			Posterior Mean			
1	0.1250	0.1245	0.1240	0.1230	0.1215	0.1200
2	0.1667	0.1653	0.1639	0.1613	0.1575	0.1538
5	0.2917	0.2857	0.2800	0.2692	0.2545	0.2414
10	0.5000	0.4800	0.4615	0.4286	0.3871	0.3529
15	0.7500	0.6667	0.6296	0.5667	0.4928	0.4359
20	0.9167	0.8462	0.7857	0.6875	0.5789	0.5000
			Posterior Mode			
1	0.0909	0.0905	0.0900	0.0893	0.0881	0.0870
2	0.1364	0.1359	0.1339	0.1316	0.1282	0.1250
5	0.2727	0.2667	0.2609	0.2550	0.2353	0.2222
10	0.5000	0.4783	0.4583	0.4231	0.3793	0.3438
15	0.7727	0.6809	0.6400	0.5714	0.4923	0.4324
20	0.9545	0.8750	0.8077	0.7000	0.5833	0.5000
			Posterior Variance			
1	0.0044	0.0043	0.0043	0.0042	0.0042	0.0041
2	0.0056	0.0055	0.0054	0.0052	0.0050	0.0048
5	0.0083	0.0080	0.0078	0.0073	0.0067	0.0061
10	0.0100	0.0096	0.0092	0.0084	0.0074	0.0065
15	0.0083	0.0084	0.0083	0.0079	0.0070	0.0062
20	0.0031	0.0048	0.0058	0.0065	0.0063	0.0056

less affected by the non-binomiality than the posterior mean for small values of s and more by larger values of s.

It is well known that the extensive form of analysis, under quadratic loss function, yields Bayes risk as the posterior variance. The Bayes risk associated with the Bayes estimator (6) when sample $\boldsymbol{x}$ has been realised is

$$(7) \qquad \rho_{\xi}^{*}(\boldsymbol{x})=\int_{0}^{1}(\alpha-\delta_{\xi}^{*}(x))^{2}\xi(\alpha\,|\,\boldsymbol{x})d\alpha=u'v'/(u'+v'+1)(u'+v')^{2}\,.$$

As in the case of posterior mean, it is also affected by the presence of non-binomiality in the parent population and the effect increases with increase in k and β. Table I shows an interesting feature of this effect. It is seen to decrease for values of s upto 10, whereas, for $s>10$ it first increases by a small amount but then decreases steadily as β tends to 1.

4. Test of significance

Consider the problem of testing a null hypothesis H_0 : $\alpha=\alpha_0$ against the alternative H_1 : $\alpha\neq\alpha_0$ as a binary decision problem in which decision d_i amounts to acceptance of the hypothesis H_i $(i=0, 1)$. Let $L_i(\alpha)$ denote the loss incurred in taking decision d_i when $A=\alpha$. Further, let the loss function be quadratic and specified by

$$(8) \qquad \begin{aligned} &L_0(\alpha)=a(\alpha-\alpha_0)^2\geqq 0\,, \qquad \text{for } \alpha\in(0,1) \\ &L_1(\alpha)=\begin{cases} b\,, & \text{for } \alpha=\alpha_0\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \end{aligned}$$

with constants $a>0$ and $b>0$.

Following Jeffreys [4], we take the prior distribution of the success probability A as a mixed type, comprising discrete probability mass p at $\alpha=\alpha_0$ and a continuous distribution of total probability $(1-p)$ with density $\xi(\alpha)$ such that $\int_{\Omega_0}\xi(\alpha)d\alpha=1-p$, $\Omega_0=\{\alpha\colon\ \alpha\in(0,1)-\{\alpha_0\}\}$. In the absence of any knowledge about the 'true' prior, assume that $\xi(\alpha)$ for $\alpha\neq\alpha_0$ is a beta with parameters $u>0$ and $v>0$.

The sample data $\boldsymbol{x}$ refines the probability that $\alpha=\alpha_0$ and the density of A over the set Ω_0, in order to produce the mixed type posterior distribution of A. The Bayes risk in taking decision d_0 (accepting H_0) is

$$(9) \qquad \begin{aligned} \rho(d_0)&=\mathrm{E}\,[L_0(A)\,|\,\boldsymbol{X}=\boldsymbol{x}]=a(1-p)\int_0^1(\alpha-\alpha_0)^2\xi(\alpha\,|\,x)d\alpha \\ &=a(1-p)[\rho_{\xi}^{*}(\boldsymbol{x})+\{\delta_{\xi}^{*}(\boldsymbol{x})-\alpha_0\}^2]\,, \end{aligned}$$

and that in taking decision d_1 (accepting H_1) is

$$\rho(d_1)=b \text{ Prob.} [A=\alpha_0 | \boldsymbol{X}=\boldsymbol{x}]=bp/[p+(1-p)c] , \tag{10}$$

where $f_k(\boldsymbol{x}|\alpha)$ is the likelihood function obtained in (2) and

$$c=f_\xi(\boldsymbol{x})/f_k(\boldsymbol{x}|\alpha_0)=B(u', v')[B(u, v)\alpha_0^s(1-\alpha_0)^{nk+(\beta-1)s}]^{-1} . \tag{11}$$

Now the Bayes decision function (BDF) $D(\boldsymbol{x})$ for the binary decision problem takes decision d_0 if $\rho(d_0)<\rho(d_1)$, otherwise decision d_1.

The author (see, Bansal [1], [2]) developed a method to construct a 'zone of sensitivity' (ZS) for investigating the effect of non-normality on binary decision problem concerning unknown mean of a normal population. The same technique may easily be employed to obtain the ZS to investigate the effect of non-binomiality on the test of significance for α.

The risk curve in $p\rho$-plane for decision d_0, as in the normal theory, is a line segment with slope

$$c_2=-a[\rho_\xi^*(\boldsymbol{x})+\{\delta_\xi^*(\boldsymbol{x})-\alpha_0\}^2] , \tag{12}$$

and right end point (1, 0). For decision d_1, the risk curve is a segment of a rectangular hyperbola lying between the origin (0, 0) and the point $(1, b)$. These two segments are seen to intersect at

$$p^*=[(2cc_2-b-c_2)+\sqrt{(b+c_2)^2-4bcc_2}]/2c_2(c-1) . \tag{13}$$

The investigator may now modify the BDF in term of the critical value p^* of p to

$$D(\boldsymbol{x})=\begin{cases} d_0 , & \text{if } p<p^* \\ d_1 , & \text{otherwise} . \end{cases} \tag{14}$$

If p_0^* denotes the value of p^* when the sample is assumed to have come from the binomial population, write

$$p_1=\min(p_0^*, p^*) \qquad \text{and} \qquad p_2=\max(p_0^*, p^*) .$$

The shift in the point of intersection p^* due to non-binomiality of the population may be measured by the length of the interval (p_1, p_2). This interval is called the zone of sensitivity due to non-binomiality of the parent population. (see Fig.)

Jain and Consul [3] have given a maximum likelihood estimator for β. For a given sample and a chosen prior distribution the decision maker may use the maximum likelihood estimate of β to obtain the ZS. He may then choose the probability p in favour of the null hypothesis H_0 in the complement set $R=\{p: p\in(0, p_1)\cup(p_2, 1)\}$. This set R

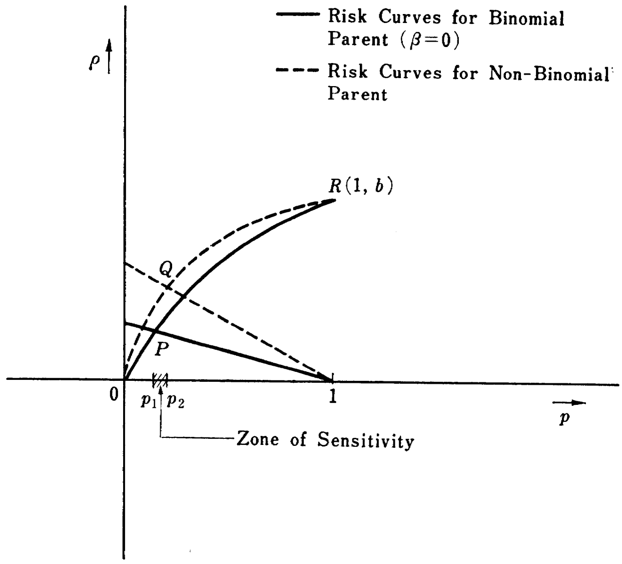

Fig. Construction of zone of sensitivity.

was called the 'region of robustness' by the author (see, Bansal [1]).

In order to illustrate the effect of non-binomiality on the zone of sensitivity for the test of significance, when u, v, and n are fixed in advance, we tabulate the critical value p^* of p for $k=5$, $s=1,\cdots,10$ and selected values of β. For $\beta=0$, p^* values are seen to be symmetric about the middle value of the range of s, which is $s=5$. For fixed β it is seen to decrease as s tends to 4, whereas, for fixed value of s the p^* values tend to increase with increase in the non-binomiality. The unequal counter-balancing effect disturbs uniformity in the pattern of the effect. Table II gives, for $\beta=0.4$, $k=5$, $u=v=n=s=2$, the zone

Table II. Point of intersection p^* of the two risk curves in the $p\rho$-plane when $u=v=n=2$, $k=5$.

			$\beta\rightarrow$			
s	0.0	0.1	0.2	0.4	0.7	1.0
1	0.3241	0.3327	0.3414	0.3590	0.3856	0.4126
2	0.1075	0.1159	0.1248	0.1443	0.1776	0.2159
3	0.0334	0.0368	0.0413	0.0520	0.0734	0.1027
4	0.0128	0.0142	0.0159	0.0206	0.0321	0.0491
5	0.0086	0.0084*	0.0086	0.0100	0.0151	0.0251
6	0.0128	0.0098	0.0082	0.0069*	0.0085	0.0139
7	0.0334	0.0205	0.0135	0.0075	0.0059*	0.0085
8	0.1075	0.0563	0.0311	0.0118	0.0053	0.0058
9	0.3241	0.1674	0.0826	0.0228	0.0061	0.0044
10	0.6575	0.4126	0.2159	0.0491	0.0085	0.0039

Table III. Point of intersection p^* of the two risk curves in the $p\rho$-plane for $s=u=v=n=2$.

k	$\beta\rightarrow$ 0.0	0.1	0.2	0.4	0.7	1.0
1	0.0659	0.0535	0.0444	0.0321	0.0226	0.0190
2	0.0190	0.0184	0.0182*	0.0185	0.0203	0.0239
3	0.0239	0.0254	0.0272	0.0313	0.0197	0.0492
4	0.0492	0.0533	0.0577	0.0676	0.0855	0.1075
5	0.1075	0.1159	0.1248	0.1442	0.1776	0.2159
6	0.2159	0.2297	0.2440	0.2739	0.3215	0.3717
7	0.3717	0.3888	0.4060	0.4405	0.4921	0.5425
8	0.5425	0.5589	0.5750	0.6064	0.6510	0.6924
9	0.6924	0.7054	0.7181	0.7422	0.7754	0.8051
10	0.8051	0.8143	0.8231	0.8397	0.8620	0.8816

of sensitivity as the interval (0.1075, 0.1443) and the region of robustness $R=\{p;\ p\in(0, 0.1075)\cup(0.1443, 1.0)\}$.

The decision maker may also like to get an idea of the effect of non-binomiality for increase in the sample size when the number of successes s in the observed sample is kept fixed. For $k=1$, p^* is seen to decrease as β tends to 1. However, for $k\geqq 4$, p^* steadily increases with increase in the value of β. In particular, for example, for $k=10$, p^* value for binomial parent ($\beta=0$) is 0.8051. This value increases to 0.8851 when the binomiality assumption is violated to the extent that the actual parent is negative binomial ($\beta=1.0$).

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