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DIFFERENTIAL OPERATORS ASSOCIATED WITH ZONAL POLYNOMIALS. I

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Summary

Associated with each zonal polynomial, $C_\kappa(S)$, of a symmetric matrix S, we define a differential operator ∂_κ, having the basic property that $\partial_\kappa C_\lambda(S)=\delta_{\kappa\lambda}$, δ being Kronecker's delta, whenever κ and λ are partitions of the non-negative integer k. Using these operators, we solve the problems of determining the coefficients in the expansion of (i) the product of two zonal polynomials as a series of zonal polynomials, and (ii) the zonal polynomial of the direct sum, $S\oplus T$, of two symmetric matrices S and T, in terms of the zonal polynomials of S and T. We also consider the problem of expanding an arbitrary homogeneous symmetric polynomial, $P(S)$ in a series of zonal polynomials. Further, these operators are used to derive identities expressing the doubly generalised binomial coefficients $\binom{\lambda}{P}$, $P(S)$ being a monomial in the power sums of the latent roots of S, in terms of the coefficients of the zonal polynomials, and from these, various results are obtained.

1. Introduction

The basic idea forming the foundation of this communication is to be found in the paper of Foulkes [6] who, taking advantage of the orthogonality properties of the characters of the symmetric group, associated with each Schur-function $\chi_\kappa(S)$ of a symmetric matrix S, a certain differential operator D_κ having the basic property that $D_\kappa\chi_\lambda(S)=\delta_{\kappa\lambda}$, δ being Kronecker's delta, whenever κ and λ are partitions of the non-negative integer k. Foulkes then and later (cf. [5]) used these operators in a variety of situations.

In similar fashion, with an eye on the orthogonality relations

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(James [9], [10]) for the zonal polynomial coefficients, we define a differential operator ∂_κ corresponding to each zonal polynomial $C_\kappa(S)$, which has the basic property that $\partial_\kappa C_\lambda(S)=\delta_{\kappa\lambda}$, whenever κ and λ are partitions of the same integer. With relative ease, solutions are obtained to the problems (Hayakawa [8], Khatri and Pillai [11], Davis [4]) of expanding (i) $C_\lambda(S)C_\tau(S)$ as a linear combination of zonal polynomials (Section 3), and (ii) $C_\lambda(S\oplus T)$, S and T being symmetric matrices, in terms of the zonal polynomials of S and T (Section 4); in both cases, the coefficients of the expansion are expressed in terms of the coefficients of the zonal polynomials.

Section 5 treats the problem of expanding an arbitrary homogeneous symmetric polynomial, $P(S)$, as a linear combination of zonal polynomials. When $P(S)$ is restricted to be a monomial in the power sums of the latent roots of S, it turns out that the generalised binomial coefficients (Constantine [3], Bingham [1]) can be expressed in terms of the zonal polynomial coefficients. This is used to show that a result of Bingham [1] is equivalent to a generalisation of one of James' [10] orthogonality relations.

2. Notation and preliminaries

1. Throughout, partitions will be denoted as follows:
(a) $\kappa\vdash k$ signifies that κ is a partition of the non-negative integer k. (This notation seems to be universal among algebraists; in any case, it has obvious advantages over the notation "$\kappa\in\mathcal{P}_k$".);
(b) $\kappa=(k_1, k_2,\cdots, k_m)$ signifies that the *parts* of κ are $k_1\geqq k_2\geqq\cdots\geqq k_m$;
(c) $\nu=\langle 1^{\nu_1}2^{\nu_2}\cdots k^{\nu_k}\rangle$ signifies that exactly ν_i parts of ν are equal to i.

We shall also denote the *length* of a partition κ, i.e. the number of non-zero parts of κ by $\#(\kappa)$.

If $\nu=\langle 1^{\nu_1}2^{\nu_2}\cdots k^{\nu_k}\rangle\vdash k$, $\rho=\langle 1^{\rho_1}2^{\rho_2}\cdots l^{\rho_l}\rangle\vdash l$, we define $|\nu|=\nu_1+\nu_2+\cdots+\nu_k$, $\nu!=\nu_1!\,\nu_2!\cdots\nu_k!$, and $\nu+\rho=\langle 1^{\nu_1+\rho_1}2^{\nu_2+\rho_2}\cdots\rangle\vdash k+l$.

2. Let $\kappa=(k_1, k_2,\cdots, k_m)\vdash k$, and S be an $m\times m$ symmetric matrix, with $s_1, s_2,\cdots, s_k,\cdots,$ being the power sums of the latent roots of S. Let

$$C_\kappa(S)=(\chi_{[2\kappa]}(1)2^k(k!)/(2k)!)Z_\kappa(S) \tag{2.1}$$

be the zonal polynomial (Constantine [2], James [10]) corresponding to κ where

$$Z_\kappa(S)=\sum_{\nu\vdash k} z_{\kappa\nu}s_1^{\nu_1}s_2^{\nu_2}\cdots s_k^{\nu_k} \tag{2.2}$$

is the zonal polynomial of James [9], and $\chi_{[2\kappa]}(1)$ is the degree of the representation $[2\kappa]$ of the symmetric group on $2k$ symbols. If the coefficients $c_{\kappa\nu}$ are defined analogously for $C_\kappa(S)$, James [9], [10] gives orthogonality relations for the $z_{\kappa\nu}$, $c_{\kappa\nu}$ which we state as

$$\sum_\nu z_{\kappa\nu} c_{\lambda\nu} / z_{(k)\nu} = \delta_{\kappa\lambda} ; \tag{2.3}$$

$$\sum_\kappa z_{\kappa\nu} c_{\kappa\rho} = \delta_{\nu\rho} z_{(k)\rho} . \tag{2.4}$$

It is well-known (cf. James [10], Gupta and Richards [7]) that

$$z_{(k)\nu} = 2^k(k!)/\nu! 2^{\nu_1} 4^{\nu_2} \cdots (2k)^{\nu_k} , \qquad \nu = \langle 1^{\nu_1} 2^{\nu_2} \cdots \rangle \vdash k . \tag{2.5}$$

DEFINITION. Let $\kappa \vdash k$. The differential operator associated with $C_\kappa(S)$ is

$$\partial_\kappa = \sum_\nu (z_{\kappa\nu}/\nu! z_{(k)\nu}) \frac{\partial^{|\nu|}}{\partial s_1^{\nu_1} \partial s_2^{\nu_2} \cdots \partial s_k^{\nu_k}} . \tag{2.6}$$

Using (2.3), it is easy to see that $\partial_\kappa C_\lambda(S) = \delta_{\kappa\lambda}$, $\lambda \vdash k$, $\kappa \vdash k$.

3. Expansion of the product of zonal polynomials

Let $\kappa \vdash k$, $\lambda \vdash l$, $k+l=f$. Hayakawa [8] (cf. Khatri and Pillai [11]) defines coefficients $b_{\kappa,\lambda}^\phi$, $\phi \vdash f$, by the expansion

$$C_\kappa(S) C_\lambda(S) = \sum_\phi b_{\kappa,\lambda}^\phi C_\phi(S) . \tag{3.1}$$

We then have

THEOREM 3.1. *If the coefficients* $b_{\kappa,\lambda}^\phi$ *are as in* (3.1), *then*

$$b_{\kappa,\lambda}^\phi = \sum_{\mu \vdash f} (z_{\phi\mu}/z_{(f)\mu}) \sum_{\substack{\nu \vdash k \\ \nu+\rho=\mu}} \sum_{\rho \vdash l} c_{\kappa\nu} c_{\lambda\rho} . \tag{3.2}$$

PROOF. Applying the differential operator ∂_ϕ to both sides of (3.1), we obtain

$$\begin{aligned} b_{\kappa,\lambda}^\phi &= \partial_\phi \{C_\kappa(S) C_\lambda(S)\} \\ &= \sum_{\mu \vdash f} (z_{\phi\mu}/z_{(f)\mu}\mu!) \frac{\partial^{|\mu|}}{\partial s_1^{\mu_1} \cdots \partial s_f^{\mu_f}} \left(\sum_{\nu \vdash k} c_{\kappa\nu} s_1^{\nu_1} \cdots s_k^{\nu_k} \right) \left(\sum_{\rho \vdash l} c_{\lambda\rho} s_1^{\rho_1} \cdots s_l^{\rho_l} \right) . \end{aligned} \tag{3.3}$$

But when the operator $\partial^{|\mu|}/\partial s_1^{\mu_1} \cdots \partial s_f^{\mu_f}$ is applied to the monomial $s_1^{\nu_1+\rho_1} \cdot s_2^{\nu_2+\rho_2} \cdots$, the result is zero if $\nu+\rho \neq \mu$, and $\mu!$ if $\nu+\rho=\mu$. From this, (3.2) follows.

A number of identities for the coefficients $b_{\kappa,\lambda}^\phi$ can be obtained from Theorem 3.1. We express them as

COROLLARY 3.1. *Let* $\tau = \langle 1^{\tau_1} 2^{\tau_2} \cdots f^{\tau_f} \rangle \vdash f$. *Then,* (3.4) $$\sum_{\phi} b^{\phi}_{\kappa,\lambda} c_{\phi\tau} = \sum_{\nu+\rho=\tau} c_{\kappa\nu} c_{\lambda\rho} \ .$$

PROOF. One way to prove (3.4) is to use (3.2) and the orthogonality relation (2.4) and proceed in a straightforward manner. A second proof can in fact be derived by applying the differential operator $\partial^{|\tau|}/\partial s_1^{\tau_1} \cdots \times \partial s_f^{\tau_f}$ to both sides of (3.1).

The identities given by Corollary 3.1 serve as useful checks in computation of the $b^{\phi}_{\kappa,\lambda}$.

4. Expansion of the zonal polynomial of $S \oplus T$

Let S and T be symmetric matrices of orders m and n respectively, and let $\phi \vdash f$. Hayakawa [8] defines the coefficients $a^{\phi}_{\kappa,\lambda}$ by the expansion

(4.1) $$C_{\phi}(S \oplus T) = \sum_{k=0}^{f} \sum_{\kappa \vdash k} \sum_{\lambda \vdash f-k} a^{\phi}_{\kappa,\lambda} C_{\kappa}(S) C_{\lambda}(T) \ .$$

THEOREM 4.1. *With the coefficients $a^{\phi}_{\kappa,\lambda}$ defined via* (4.1), *we have*

(4.2) $$a^{\phi}_{\kappa,\lambda} = \sum_{\mu} \sum_{\nu} \sum_{\rho} z_{\kappa\nu} z_{\lambda\rho} c_{\phi\mu} \mu! / z_{(k)\nu} z_{(f-k)\rho} \nu! \rho! \ , \quad \mu = \nu + \rho$$

$\kappa \vdash k$, $\lambda \vdash f-k$.
In particular, $a^{\phi}_{\kappa,\lambda} = \delta_{\phi\kappa}$ (*respectively,* $\delta_{\phi\lambda}$) *if* $\kappa \vdash f$ (*respectively,* $\lambda \vdash f$).

PROOF. Temporarily denoting the differential operators associated with $C_{\kappa}(S)$ and $C_{\lambda}(T)$ by $\partial_{\kappa}(S)$ and $\partial_{\lambda}(T)$ respectively, we apply both to (4.1) to obtain

(4.3) $$a^{\phi}_{\kappa,\lambda} = \partial_{\kappa}(S) \partial_{\lambda}(T) C_{\phi}(S \oplus T)$$

(4.4) $$= \left(\sum_{\nu} (z_{\kappa\nu}/z_{(k)\nu} \nu!) \frac{\partial^{|\nu|}}{\partial s_1^{\nu_1} \partial s_2^{\nu_2} \cdots} \right) \left(\sum_{\rho} (z_{\lambda\rho}/z_{(f-k)\rho} \rho!) \frac{\partial^{|\rho|}}{\partial t_1^{\rho_1} \partial t_2^{\rho_2} \cdots} \right) \cdot \left(\sum_{\mu} c_{\phi\mu} (s_1+t_1)^{\mu_1} (s_2+t_2)^{\mu_2} \cdots \right) ,$$

where $t_1, t_2, \cdots$ are the power sums of the latent roots of T. Using the same reasoning as in the proof of Theorem 3.1, it follows that (4.4) reduces to (4.2). When $\kappa \vdash f$ in particular, the orthogonality relation (2.3) shows that $a^{\phi}_{\kappa,\lambda} = \delta_{\phi\kappa}$, and similarly when $\lambda \vdash f$.

COROLLARY 4.1. *Let* $\nu \vdash k$, $\rho \vdash f-k$, $\mu \vdash f$. *Then,*

(4.5) $$\sum_{\kappa \vdash k} \sum_{\lambda \vdash f-k} a^{\phi}_{\kappa,\lambda} c_{\kappa\nu} c_{\lambda\rho} = \delta_{\mu,\rho+\nu} c_{\phi\mu} \mu! / \nu! \rho! \ .$$

The proof is similar to that of Corollary 3.1 and is omitted.

COROLLARY 4.2. *Define (cf. Khatri and Pillai [11]) the coefficients* a_κ^ϕ *by*

$$C_\phi(S_1)=\sum_{k=0}^{f} t^{f-k} \sum_{\kappa\vdash k} a_\kappa^\phi C_\kappa(S) , \tag{4.6}$$

where $S_1=\text{diag}(S, t)$. *Then*

$$a_\kappa^\phi=\sum_{\nu} \sum_{\mu=\nu+\langle 1^{f-k}\rangle} z_{\kappa\nu} c_{\phi\mu} \mu!/z_{(k)\nu} \nu!(f-k)! . \tag{4.7}$$

Again, identities similar to Corollaries 3.1 and 4.1 are easily derived.

5. Expansions of homogeneous symmetric polynomials in terms of zonal polynomials

For the rest of the paper, $\lambda=(l_1, l_2,\cdots, l_m)\vdash l$ where $l\geqq k$. Suppose now that

$$P(S)=\sum_{\nu} P_\nu s_1^{\nu_1} s_2^{\nu_2}\cdots s_k^{\nu_k} \tag{5.1}$$

is an arbitrary homogeneous symmetric polynomial of degree k in the elements of S. Bingham [1] defines the doubly generalised binomial coefficient $\binom{\lambda}{P}$, via the expansion

$$P(S)\exp(\text{tr } S)/k!=\sum_{l=k}^{\infty} \sum_{\lambda\vdash l} \binom{\lambda}{P} C_\lambda(S)/l! . \tag{5.2}$$

When $P(S)$ is a zonal polynomial, $\binom{\lambda}{P}$ reduces to Constantine's [3] generalised binomial coefficient. We now have

THEOREM 5.1. *Let* $\nu^*=\nu+\langle 1^{l-k}\rangle$. *Then,*

$$\binom{\lambda}{P}=\binom{l}{k} \sum_{\nu} P_\nu z_{\lambda\nu^*}/z_{(l)\nu^*} . \tag{5.3}$$

Further, if $l_i<k_i$ *for any* $i=1, 2,\cdots, m$, *(and in particular, if* $\#(\lambda)<\#(k)$*), then*

$$\sum_{\nu} c_{\kappa\nu} z_{\lambda\nu^*}/z_{(l)\nu^*}=0 . \tag{5.4}$$

PROOF. To prove (5.3), one only has to substitute (5.1) into (5.2), and then apply the differential operator associated with $C_\lambda(S)$ to (5.2), evaluating the result at $S=0$. Next, (5.4) follows by putting $P(S)=C_\kappa(S)$ in (5.3) and appealing to Bingham ([1], Lemma 2).

When $k=l$, then (5.4) reduces to (2.3). Again, for some special partitions, we can use (5.3) to obtain explicit results. Thus, when $k=l$, we find that $\binom{\lambda}{\kappa}=\delta_{\lambda\kappa}$, a result previously obtained by Bingham ([1], eq. (3.1)). For arbitrary k and l, it is easy—but somewhat tedious—to show, using the explicit result for $C_{(1^k)}(S)$ given by Gupta and Richards ([7], eq. (2.7)), that

$$\binom{(1^l)}{(1^k)}=\binom{l}{k}, \tag{5.5}$$

and, using (2.3) and (2.5) that

$$\binom{(l)}{\kappa}=\delta_{(k)\kappa}\binom{l}{k}, \qquad \kappa\vdash k, \tag{5.6}$$

a special case of (5.6) having been previously conjectured by Pillai and Jouris [13]. (cf. Muirhead [12]).

Finally, we remark that in the case when $P(S)$ is a monomial in $s_1,\cdots,s_k$, (5.3) can be used in conjunction with various results of Bingham [1] to obtain explicit results for various $z_{\lambda\nu}$'s.

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REFERENCES

[1] Bingham, C. (1974). An identity involving partitional generalized binomial coefficients, *J. Multivariate Anal.*, **4**, 210-223.

[2] Constantine, A. G. (1963). Some non-central distribution problems in multivariate analysis, *Ann. Math. Statist.*, **34**, 1270-1285.

[3] Constantine, A. G. (1966). The distribution of Hotelling's generalised T_0^2, *Ann. Math. Statist.*, **37**, 215-225.

[4] Davis, A. W. (1979). Invariant polynomials with two matrix arguments extending the zonal polynomials: applications to multivariate distribution theory, *Ann. Inst. Statist. Math.*, **31**, A, 465-485.

[5] Foata, D., Morris, A. O. and Perfect, H. (1979). Herbert Owen Foulkes, Obituary, *Bull. London Math. Soc.*, **11**, 208-214.

[6] Foulkes H. O. (1949). Differential operators associated with S-functions, *J. London Math. Soc.*, **24**, 136-143.

[7] Gupta, R. D. and Richards, D. (1979). Calculation of zonal polynomials of 3×3 positive definite symmetric matrices, *Ann. Inst. Statist. Math.*, **31**, A, 207-213.

[8] Hayakawa, T. (1967). On the distribution of the maximum latent root of a positive definite symmetric random matrix, *Ann. Inst. Statist. Math.*, **19**, 1-17.

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[9] James, A. T. (1961). The distribution of non-central means with known covariance, *Ann. Math. Statist.*, **32**, 874-882.

[10] James, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples, *Ann. Math. Statist.*, **35**, 475-501.

[11] Khatri, C. G. and Pillai, K. C. S. (1968). On the non-central distributions of two test criteria in multivariate analysis of variance, *Ann. Math. Statist.*, **39**, 215-226.

[12] Muirhead, R. J. (1974). On the calculation of generalized binomial coefficients, *J. Multivariate Anal.*, **4**, 341-346.

[13] Pillai, K. C. S. and Jouris, G. M. (1969). On the moments of elementary symmetric functions of the roots of two matrices, *Ann. Inst. Statist. Math.*, **21**, 309-320.