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LINEAR COMBINATION OF CONCOMITANTS OF ORDER STATISTICS WITH APPLICATION TO TESTING AND ESTIMATION

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Summary

Let $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$ be i.i.d. as (X, Y). The Y-variate paired with the rth ordered X-variate X_{rn} is denoted by Y_{rn} and terms the concomitant of the rth order statistic. Statistics of the form $T_n = n^{-1} \sum_{i=1}^{n} J(t_{ni}) Y_{in}$ are considered. The asymptotic normality of T_n is established. The asymptotic results are used to test univariate and bivariate normality, to test independence and linearity of X and Y, and to estimate regression coefficient based on complete and censored samples.

1. Introduction

Let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be i.i.d. as some bivariate random variable (X, Y). Let X_{in} be the ith ordered X-variate and Y_{in} be the Y-variate paired with X_{in}. Following David [3], [4] the Y_{in} will be called the concomitants of order statistics. Bhattacharya [1] called them induced order statistics. Statistics of the form $T_n = n^{-1} \sum_{i=1}^{n} J(t_{ni}) Y_{in}$, where J is some weight function defined on $(0, 1)$, are considered in this paper.

In Section 2, the asymptotic normality for T_n is established under fairly general conditions. This asymptotic result is established by applying the asymptotic results for functions of order statistics given in Shorack [10] and Wellner [12]. Bhattacharya [1], [2] established weak convergence results for the partial sum process of the Y_{in} based on stronger conditions than those given here.

In Section 3, three applications are considered. In Section 3.1,

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statistics of the form T_n are used to construct large sample tests for univariate and bivariate normality. The univariate normal tests are analogous to those proposed in Shapiro and Wilk [8] and LaBrecque [6]. For testing bivariate normality, tests based on alternative model of nonlinearity in the regression function are proposed. In Section 3.2, various T_n are used to construct tests for independence of X and Y which will also give us information on the nature of the dependence of X and Y. In Section 3.3, T_n are used to obtain estimates for the regression coefficient based on both complete and censored observations. In the case of bivariate normality and complete data, the proposed estimator is asymptotically as efficient as the maximum likelihood estimator. The proposed estimators will be useful when the data are censored, because the maximum likelihood estimators are usually complicated.

2. Asymptotic normality

2.1. *Notation*

For convenience, the following notation will also be adopted throughout this paper.

$F(x)$=c.d.f. of the X-variate.

$F^{-1}(u)=\inf\{x \mid F(x)\geqq u\}$.

$\Phi(x)$=c.d.f. of a standard normal random variable.

$g(x)=\mathrm{E}(Y|X=x)$.

$\tau(x)=\mathrm{Var}(Y|X=x)$.

2.2. *Main theorem*

The derivation of the asymptotic normality for T_n is an application of the general results of Shorack [10] and Wellner [12]. We shall first prove a lemma which will be needed in the proof of the main theorem.

LEMMA 1. *Let $(X_1, Y_1), (X_2, Y_2), \cdots$, be a sequence of random vectors and for each $n\geqq 1$ $(X_1, Y_1), \cdots, (X_n, Y_n)$ possesses a joint distribution. Let $\boldsymbol{Z}_n=((X_1, Y_1), \cdots, (X_n, Y_n))$ and $\boldsymbol{X}_n=(X_1, \cdots, X_n)$, and let $W_n(\boldsymbol{Z}_n)$ and $S_n(\boldsymbol{X}_n)$ be respectively measurable vector-valued functions of $\boldsymbol{Z}_n$ and $\boldsymbol{X}_n$. Suppose S_n converges in distributin to F_s, and the conditional distribution of W_n given $\boldsymbol{X}_n$ converges weakly to a distribution F_w which does not depend on the X_i's. Then $(W_n, S_n)\xrightarrow{d}F_wF_s$.*

PROOF. Let W and S be random vectors with distributions F_w and F_s respectively. Write

$$|\mathrm{E}\{\exp[i(t_1'W_n+t_2'S_n)]\}-\mathrm{E}\{\exp(it_1'W)\}\,\mathrm{E}\{\exp(it_2'S)\}|$$
$$\leqq|\mathrm{E}\{\mathrm{E}[\exp(it_1'W_n+it_2'S_n)\,|\,\boldsymbol{X}_n]\}-\mathrm{E}[\exp(it_1'W)]\,\mathrm{E}[\exp(it_2'S_n)]|$$

$$+|\mathrm{E}\,[\exp(it_1'W)]\,\mathrm{E}\,[\exp(it_2'S_n)]-\mathrm{E}\,[\exp(it_1'W)]\,\mathrm{E}\,[\exp(it_2'S)]|$$
$$\leqq \mathrm{E}\,\{|\mathrm{E}\,[\exp(it_1'W_n)\,|\,X_n]-\mathrm{E}\,[\exp(it_1'W)]|\}$$
$$+|\mathrm{E}\,[\exp(it_2'S_n)]-\mathrm{E}\,[\exp(it_2'S)]|\,.$$

By assumptions, the last two expressions tend to zero as $n\to\infty$.

We shall consider $n^{1/2}(T_n-\mu_n)$ where

$$\mu_n=\int_0^1 g[F^{-1}(t)]J_n(t)dt\,,$$

$$J_n(t)=\begin{cases} J(t_{n1}) & t=0 \\ J(t_{ni}) & (i-1)/n<t\leqq i/n \end{cases}$$, and J is the weight function used in T_n. For fixed α_1, α_2, M and $\delta>0$ define

$$D(t,\alpha_1,\alpha_2)=Mt^{-1/2+\alpha_1+\delta}(1-t)^{-1/2+\alpha_2+\delta} \qquad \text{for } 0<t<1\,,$$

$$B(t,\alpha_1,\alpha_2)=Mt^{-\alpha_1}(1-t)^{-\alpha_2} \qquad \text{for } 0<t<1\,.$$

Let J denote a fixed measurable function on (0, 1). Define $q(t)=[t(1-t)]^{1-\delta/2}$ on (0, 1).

ASSUMPTION 1. *J is continuous except at a finite number of points at which $g[F^{-1}]$ is continuous.*

ASSUMPTION 2. $\max_{1\leqq i\leqq n}|t_{ni}-i/n|\to 0$ as $n\to\infty$ and where for some $a>0$ $a[(i/n)\wedge(1-i/n)]\leqq t_{ni}\leqq 1-a[(i/n)\wedge(1-i/n)]$ for $i=1,\cdots,n$.

ASSUMPTION 3. $g[F^{-1}]$ and $\tau[F^{-1}]$ are left continuous on (0, 1) and of bounded variation on $(\theta, 1-\theta)$ for every $\theta>0$.

ASSUMPTION 4. For some fixed b_1, b_2, d_1, d_2, $|g[F^{-1}(t)]|\leqq D(t, b_1, b_2)$ and $|\tau[F^{-1}(t)]|\leqq D(t, d_1-1/2, d_2-1/2)$.

ASSUMPTION 5. $|J(t)|\leqq B(t, c_1, c_2)$ on (0, 1) where $c_i=\min\{b_i, d_i/2, 1/2\}$, $i=1, 2$.

ASSUMPTION 6. $\int_0^1 B(t, c_1, c_2)q(t)d|\tau[F^{-1}(t)]|<\infty$.

ASSUMPTION 7. $\sigma_s^2=\int_0^1\int_0^1 (s\wedge t-st)J(s)J(t)dg[F^{-1}(s)]dg[F^{-1}(t)]<\infty$ and $\sigma_w^2=\int_0^1 J^2(t)\tau[F^{-1}(t)]dt<\infty$.

ASSUMPTION 8. J' exists and is continuous on (0, 1) with $|J'|\leqq B(t, 1+c_1, 1+c_2)$.

THEOREM 1. *If Assumptions* 1 *to* 7 *hold, then* $n^{1/2}(T_n-\mu_n)\xrightarrow{d}N(0,$

$\sigma_w^2+\sigma_s^2)$ *with* μ_n *finite.*

PROOF. Let $S_n=n^{-1}\sum_{i=1}^{n} J(t_{ni})g(X_{in})$. Then by Theorem 1 and Example 1 of Shorack [10], we have $n^{1/2}(S_n-\mu_n)\xrightarrow{d}N(0,\sigma_s^2)$. Next consider $W_n=n^{-1}\sum_{i=1}^{n} J(t_{ni})[Y_{in}-g(X_{in})]$. Bhattacharya [1] showed that $Y_{in},\cdots,Y_{nn}$ are conditionally independent given $X_{1n},\cdots,X_{nn}$ with respectively conditional c.d.f. $G_{x_{1n}}(y),\cdots,G_{x_{nn}}(y)$ where $G_x(y)=G(Y\leqq y\,|\,X=x)$. Let $(Z_{1n},\cdots,Z_{nn})$ be a set of random variables distributed according to the conditional distribution of $(Y_{1n}-g(X_{1n}),\cdots,Y_{nn}-g(X_{nn}))$ given $X_{1n},\cdots,X_{nn}$. We shall show that $W_{nx}=n^{-1}\sum_{i=1}^{n} J(t_{ni})Z_{in}$ upon suitable standardization converges in distribution to $N(0,1)$. Let $n^{-1}C_n^2=n^{-2}\sum_{i=1}^{n} J^2(t_{ni})\tau(X_{in})$. By Lindeberg's normal convergence criterion $n^{1/2}W_{nx}/C_n\xrightarrow{d}N(0,1)$ iff for every $\varepsilon>0$ $n^{-1}C_n^{-2}\sum_{i=1}^{n} J^2(t_{ni})H_{a_{ni}}(X_{in})\to 0$ as $n\to\infty$ where $H_K(x)=\int_A [y-g(x)]^2 dG_x(y)$ with $A=\{|y-g(x)|\geqq K\}$ and $a_{ni}=\varepsilon J^{-1}(t_{ni})n^{1/2}C_n$. By the strong law of large number for a function of order statistics given in Wellner [12] for almost all realization of the sequence $X_1,X_2,\cdots,C_n^2\to\sigma_w^2$ and for any $K>0$, we have $n^{-1}\sum_{i=1}^{n} J^2(t_{ni})H_K(X_{in})\to\int_0^1 J^2(t)H_K[F^{-1}(t)]dt$ as $n\to\infty$. Since $a_{ni}\to\infty$ as $n\to\infty$, Lindeberg's criterion holds for almost all realizations of the sequence $X_1,X_2,\cdots$, and thus $n^{1/2}W_{nx}/C_n\xrightarrow{d}N(0,1)$. By Lemma 1, we have $[n^{1/2}W_n/C_n,\, n^{1/2}(S_n-\mu_n)]\xrightarrow{d}N(0,1)N(0,\sigma_s^2)$. Since $n^{1/2}(T_n-\mu_n)=C_n(n^{1/2}W_n/C_n)+n^{1/2}(S_n-\mu_n)$, $n^{1/2}(T_n-\mu_n)\xrightarrow{d}N(0,\sigma_w^2+\sigma_s^2)$.

With additional Assumption 8, we have the following useful result.

THEOREM 2. *Let* $\mu=\int_0^1 J(t)g[F^{-1}(t)]dt$ *if Assumptions* 1 *to* 8 *hold; then* $N^{1/2}(T_n-\mu)\xrightarrow{d}N(0,\sigma_w^2+\sigma_s^2)$ *with* μ *finite.*

Remark 1. The above results can be obtained with weaker conditions (e.g., more general scores) similar to those given in Shorack's [10] paper. However, the above theorems are probably sufficient for most applications.

Remark 2. Often in application Assumption 4 is replaced by moment conditions on $g(x)$ and $\tau(x)$:

ASSUMPTION 4*. For some r, $s>0$, $\int_0^1 |g[F^{-1}(t)]|^r dt<\infty$ and $\int_0^1 |\tau[F^{-1}\cdot(t)]|^s dt<\infty$.

Then Assumptions 5 and 8 are replaced respectively by:

ASSUMPTION 5*. $|J(t)| \leqq B(t, c, c)$ for $0<t<1$ with $c=\min\{1/2-1/r-\delta, 1/2-1/(2s)-\delta/2, 1/2\}$.

ASSUMPTION 8*. J' exists and is continuous on $(0, 1)$ with $|J'(t)| \leqq B(t, 1+c, 1+c)$ on $(0, 1)$.

Remark 3. The distributions of X and Y need not be continuous.

3. Applications

3.1. *Testing normality*

Consider $T_{nk}=n^{-1}\sum_{i=1}^{n} H_k[\Phi^{-1}(i/n-1/(2n))]Y_{in}$, $k=1, 2, \cdots$, where the H_k are Hermite polynomials (Sansone, [7], p. 303). The first few are $H_0=1$, $H_1=x$, $H_2=x^2-1$, $H_3=x^3-3x, \cdots$. Since the T_{nk} only involve the ranks of the X_i's, we may assume that the X_i's are i.i.d. $N(0, 1)$. Let

$$\mu_k=\int_{-\infty}^{\infty} H_k(x)g(x)d\Phi(x)$$

$$\sigma_k^2=\int_{-\infty}^{\infty} H_k^2(x)\tau(x)d\Phi(x)+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\Phi(x\wedge y)-\Phi(x)\Phi(y)]H_k(x)H_k(y) \cdot dg(x)dg(y) .$$

If appropriate conditions are satisfied, then $n^{1/2}(T_{nk}-\mu_k)\xrightarrow{d} N(0, \sigma_k^2)$. In fact $n^{1/2}[(T_{n1}, \cdots, T_{nq})-(\mu_1, \cdots, \mu_q)]\xrightarrow{d} N_q(0, \Sigma)$ where $\Sigma=\|\sigma_{ij}\|$ with $\sigma_{ij}=\int_{-\infty}^{\infty} H_i(x)H_j(x)\tau(x)d\Phi(x)+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\Phi(x\wedge y)-\Phi(x)\Phi(y)]H_i(x)H_j(x)dg(x)dg(y)$.

Suppose we wish to test the null hypothesis that the $(X_i Y_i)$ come from a bivariate normal distribution with correlation ρ, $\mathrm{E}(Y)=\mu_y$ and $\mathrm{Var}(Y)=\sigma_y^2$. Under the normal hypothesis, using the fact that $\mathrm{E}[H_j\cdot(X)H_k(X)]=k!\delta_{jk}$ and $H'_{k+1}(x)=(k+1)H_k(x)$, we can show that $\mu_1=\rho\sigma_y$, $\mu_k=0$ for $k\geqq 2$ and $\sigma_{jk}=k!\sigma_y^2(1-\rho^2k/(k+1))\delta_{jk}$. Let $\hat{\sigma}_k^2=k!\hat{\sigma}_y^2(1-\hat{\rho}^2k/(k+1)$ where $\hat{\sigma}_y^2$ and $\hat{\rho}$ are some consistent estimators for σ_y^2 and ρ, e.g., the sample variance and sample correlation. If $|T_{nk}|\geqq n^{-1/2}\hat{\sigma}_k\Phi^{-1}(1-\alpha/2))$ $(k\geqq 2)$, a departure from linearity in the regression function with significant kth order polynomial trend is indicated. The asymptotic power of the test is apporoximately $1-\Phi[\Phi^{-1}(1-\alpha/2)\hat{\sigma}_k/\sigma_k-n^{1/2}\mu_k/\sigma_k]+\Phi[-\Phi^{-1}\cdot(1-\alpha/2)\hat{\sigma}_k/\sigma_k-n^{1/2}\mu_k/\sigma_k]$ which depends on $|\mu_k/\sigma_k|$. If we want to detect the over all nonlinearity in $g(x)$, we might want to use the test criterion $n\sum_{k=2}^{q} T_{nk}^2/\hat{\sigma}_k^2>c$ where c is the percentage point of a chi-square distribution with $(q-1)$ degrees of freedom. Under the alternative this chi-square test statistic is approximately a weighted sum of $(q-1)$ noncentral chi-square random variables with one degree of freedom. The weights and the non-centrality parameters depend on the μ_k and σ_{ij}. It is not clear how one should choose q but clearly q cannot be too large relative to the sample size n. It should be mentioned that the tests proposed here can also be used as tests for linearity of the regression function.

The special case $p=1$ in the preceding development can be used to test univariate normality. In this case the Y_{in} are the order statistics of the sample under consideration. Under normality, the asymptotic means and variances of T_{nk} are respectively

$$\mu_k = \begin{cases} \sigma_y & k=1 \\ 0 & k=2, 3, \cdots, \end{cases}$$

$$\sigma_k^2 = \sigma_y^2 (k+1)!(k+1)^{-2} .$$

Let $\hat{\sigma}_k^2 = \hat{\sigma}_y^2 (k+1)!(k+1)^{-2}$ where $\hat{\sigma}_y^2$ is the sample variance. Following the ideas of LaBrecque [6], T_{nk} $(k \geqq 2)$ and the chi-square test statistics defined in the bivariate normal case can be used to test against the alternatives of various nonlinearity in normal probability plots. For example, a large value of $|T_{n2}|$ indicates asymmetry of the distribution, and a large value of $|T_{n3}|$ indicates deviation from the normal kurtosis. Since past studies have shown that Shapiro-Wilk test and its extensions (Shapiro, Wilk and Chen [9]; and LaBrecque [6]) perform well, one expects that for large samples the tests proposed here will perform well.

3.2. *Testing independence*

Here we shall assume that F is continuous, but the distribution of Y can be discrete. Since T_n only involves the rank of the X_i's, we may assume that $F=N(0, 1)$. If X and Y are independent, then the asymptotic mean and variance of T_{nk} $(k \geqq 1)$ defined in Section 3.1 are respectively $\mu_k = 0$ and $\sigma_k^2 = \sigma_y^2 k!$. Let $\hat{\sigma}_k^2 = \hat{\sigma}_y^2 k!$ where $\hat{\sigma}_y^2$ is some consistent estimator for σ_y^2. If $|T_{nk}| > n^{-1/2} \hat{\sigma}_k \Phi^{-1}(1-\alpha/2)$, a kth order polynomial trend in $g(x)$ is indicated. As in Section 3.1 for detecting general functional relationship between X and Y, a chi-square test using $n \sum_{k=1}^{q} T_{nk}^2 / \hat{\sigma}_k^2$ is recommended. The asymptotic efficiency of these two types of tests as before will depend on μ_k/σ_k for the former and on μ_k and σ_{ij} for the latter. Other nonparametric tests of independence, e.g., the Spearman's rank correlation test, do not by themselves indicate the nature of the dependence. The tests proposed here give information about the specific nature of the dependence which might be useful in exploratory research. If the distribution F is known, then $\Phi^{-1}(i/n - 1/(2n))$ should be replaced by $F^{-1}(i/n - 1/(2n))$ in T_{nk} and orthogonal polynomials with respect to the measure induced by F should be used rather than the Hermite polynomials.

3.3. *Estimation of regression coefficient*

Assume that $Y_i=\alpha+\beta X_i+Z_i$ $(i=1,\cdots,n)$ where the Z_i's are i.i.d. random variables independent of the X_i's. Let $\sigma_y^2=\mathrm{Var}(Y)$, $\sigma_x^2=\mathrm{Var}(X)$ and ρ be the correlation between X and Y.

We shall first study the bivariate normal case with censored data —a situation which is quite common in practice. For example, the X_{in} are the entrance scores and the Y_{in} $(i=r+1,\cdots,n)$ later scores of the successful candidates. Suppose we have $(X_{rn}, Y_{rn}),\cdots,(X_{sn}, Y_{sn})$ with $1\leqq r<s\leqq n$. Let

$$\boldsymbol{Y}=(Y_{rn},\cdots,Y_{sn})'$$

$$\boldsymbol{\alpha}=(\alpha_{rn},\cdots,\alpha_{sn})' , \qquad \alpha_{in}=\mathrm{E}(X_{in})/\sigma_x$$

$$C=\text{covariance matrix of } (X_{rn},\cdots,X_{sn})'\sigma_x^{-1}$$

$$J=(1,\cdots,1)'$$

$$\Omega=[\rho^2C+(1-\rho^2)I]^{-1} .$$

Using Gauss-Markov least-squares theorem, we can show that the best linear unbiased estimator for $\rho\sigma_y$ is

$$\widehat{\rho\sigma}_y=\frac{[(J'\Omega J)\boldsymbol{\alpha}-(J'\Omega\boldsymbol{\alpha})J']\Omega\boldsymbol{Y}}{[(J'\Omega J)(\boldsymbol{\alpha}'\Omega\boldsymbol{\alpha})-(\boldsymbol{\alpha}'\Omega J)^2]} .$$

For large n, the elements of the C matrix are generally small (of order n^{-1}). By neglecting the term ρ^2C in Ω, for large n, $\widehat{\rho\sigma}_y$ is approximately equal to $(\rho\sigma_y)^*=\left[\sum_{i=r}^{s}(\alpha_{in}-\bar{\alpha})^2\right]^{-1}\sum_{i=r}^{s}(\alpha_{in}-\bar{\alpha})Y_{in}$ where $\bar{\alpha}=(s-r)^{-1}\sum_{i=r}^{s}\alpha_{in}$. Watterson [11] has shown that this type of simple estimator is generally low in variance. Since $Y_{in}=\alpha+\beta X_{in}+Z_{in}$, we may directly apply Shorack's [10] result to establish the asymptotic normality of $(\rho\sigma_y)^*$. Suppose $a<b$ and $r/n\to\Phi(a)$ and $s/n\to\Phi(b)$ as $n\to\infty$. Then $n^{1/2}[(\rho\sigma_y)^*-\rho\sigma_y]\xrightarrow{d}N(0,\tau_1)$ where

$$\tau_1=[\Phi(b)-\Phi(a)]^{-1}V_{ab}^{-1}\sigma_y^2(1-\rho^2)+[4V_{ab}^2]^{-1}[\Phi(b)-\Phi(a)]^{-1}\rho^2\sigma_y^2D_{ab}$$

and

$$D_{ab}=[\Phi(b)-\Phi(a)]^{-1}\int_a^b(x-\mu_{ab})^4d\Phi(x)-V_{ab}^2 ,$$

$$\mu_{ab}=\int_a^b xd\Phi(x) , \qquad V_{ab}=[\Phi(b)-\Phi(a)]^{-1}\int_a^b(x-\mu_{ab})^2d\Phi(x) .$$

A simplified best linear unbiased estimator for σ_x by taking $C=I$ (Gupta [5] is $\sigma_n^* = \left[\sum_{i=r}^{s} (\alpha_{in}-\alpha)^2\right]^{-1} \sum_{i=r}^{s} (\alpha_{in}-\bar{\alpha})X_{in}$. If $r/n \to \Phi(a)$ and $s/n \to \Phi(b)$ as $n\to\infty$, then $n^{1/2}(\sigma_x^* - \sigma_x) \xrightarrow{d} N(0, \tau_2)$ where $\tau_2 = [4V_{ab}^2]^{-1}[\Phi(b)-\Phi(a)]^{-1}\sigma_x^2 D_{ab}$. Hence a consistent estimator for β is $\beta^* = (\rho\sigma_y)^*/\sigma_x^*$. Using the Cramer-Wold Device, we can show that $n^{1/2}[((\rho\sigma_y)^* - \rho\sigma_y), (\sigma_x^* - \sigma_x)] \xrightarrow{d} N(\mathbf{0}, \Gamma)$ where $\Gamma = \|\gamma_{ij}\|$ with $\gamma_{11}=\tau_1$, $\gamma_{22}=\tau_2$ and $\gamma_{12}=[4V_{ab}^2]^{-1}[\Phi(b)-\Phi(a)]^{-1}\rho\sigma_y\sigma_x D_{ab}$. Hence $\beta^* = (\rho\sigma_y)^*/\sigma_x^*$ is asymptotically normal with mean β and variance $n^{-1}[V_{ab}\sigma_x^2(\Phi(b)-\Phi(a))]^{-1}\sigma_y^2(1-\rho^2)$. For a complete sample, we have $\beta^* = n^{-1}\sum_{i=1}^{n} \alpha_{in} Y_{in} / n^{-1} \sum_{i=1}^{n} \alpha_{in} X_{in}$ with asymptotic variance $n^{-1}\sigma_x^{-2}\sigma_y^2(1-\rho^2)$. Hence β^* is asymptotically as efficient as the maximum likelihood estimator for β. A large sample $100(1-\alpha)\%$ confidence interval for β is

$$\beta^* \pm \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\{[nV_{ab}[\Phi(b)-\Phi(a)]\}^{-1/2}(\sigma_x^*)^{-1}[\hat{\sigma}_y^2 - (\rho\sigma_y)^*]^{1/2}$$

where $\hat{\sigma}_y^2$ is some consistent estimator for σ_y^2. For n sufficiently large (e.g., $n>50$) we may replace α_{in} by $\Phi(i/n-1/(2n))$ in these estimators.

In the case of censored data, the proposed computationally simple estimators will be of use because the maximum likelihood estimators are rather complicated and not always practical. Moreover, if the X_i's are suspected to contain outliers, a small tail portions of the X_{in}'s can be trimmed and the β^* computed from the trimmed data will be robust against outlying X-observations.

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