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ASYMPTOTIC BEHAVIOR OF FUNCTIONALS OF EMPIRICAL DISTRIBUTION FUNCTIONS FOR THE TWO-SAMPLE PROBLEM

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Summary

Mises functional is extended for the two-sample problem. It is shown that the extended Mises functional also has the asymptotic property given by von Mises (1947, *Ann. Math. Statist.*, 18, 309–348) and by Filippova (1962, *Theory Prob. Appl.*, 7, 24–57) in the one-sample case. Asymptotic behavior of U-statistic in the two-sample case, the statistic of Cramér-von Mises type for testing homogeneity and so forth are investigated as important examples of the theory.

1. Introduction

R. von Mises [7] derived the asymptotic distribution of a certain class of functionals of empirical distribution function. Filippova [3] investigated the asymptotic distribution for a wider class of functionals of empirical distribution function. By using this theory we can give the asymptotic distribution of many statistics (for example see Filippova [3] and Aki [1]).

The purpose of the present paper is to extend this theory for the two-sample problem. In Section 2 we extend the definition of Mises functional for the two-sample problem. We obtain the analogous result to the one derived by Filippova [3] in the one-sample case. In Section 3 three examples are investigated. The first one is the U-statistic in the two-sample case. Though the asymptotic normality of the statistic was proved by Lehmann [5], another proof of the property is given here as an application of Theorem 2.2 of Section 2. The second one is the statistic of Cramér-von Mises type for testing homogeneity. The asymptotic distribution of this statistic is equal to the asymptotic distribution of the one-sample statistic. Rosenblatt [6] and Fisz [4] derived this property. We, however, give another derivation as an application of Theorem 2.3 of Section 2. The last example is an extension of the second one in a sense.

2. Results

Let $X_1, X_2, \cdots, X_n, \cdots$ and $Y_1, Y_2, \cdots, Y_m, \cdots$ be two independent sequences of random variables, where X_i's $(i=1, 2, \cdots)$ are identically distributed with a continuous distribution function (d.f.) F and Y_j's $(j=1, 2, \cdots)$ are identically distributed with a continuous d.f. G. F_n and G_m denote the empirical distribution functions of the variables $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_m$ respectively. Throughout the paper we assume that there exists $\lambda \geqq 0$ such that $n/m \to \lambda$ when $n \to \infty$ and $m \to \infty$. Let V be the totality of real functions of bounded variation. Suppose a real valued functional T is defined on a subset σ_T of the direct product $V \times V$.

DEFINITION 2.1. The functional T is called ν times differentiable at the point (J, K) with respect to the set τ $(\subset \sigma_T)$ which is assumed to be star-shaped at the point (J, K) if the following conditions are satisfied:

(1) For any $t \in [0, 1]$, $p=1, 2, \cdots, \nu$ and any element $(J', K') \in \tau$,

$$\frac{d^p}{dt^p} T[((1-t)J+tJ', (1-t)K+tK')]$$

exists.

(2) There exist functions $T_j^{(p)}[(J, K): x_1, \cdots, x_p]$, $p=1, \cdots, \nu$; $j=1, \cdots, 2^p$, which depend on (J, K) and p real parameters $(x_1, \cdots, x_p)$, such that for any element $(J', K') \in \tau$ the relation

$$\frac{d^p}{dt^p} T[((1-t)J+tJ', (1-t)K+tK')]\bigg|_{t=0}$$
$$=\sum_j \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} T_j^{(p)}[(J, K): x_1, \cdots, x_p] \prod_{i=1}^{p} dM_{ji}(x_i)$$

holds, where $M_{ji}=[J'(x_i)-J(x_i)]$ or $[K'(x_i)-K(x_i)]$ depending on j and i.

DEFINITION 2.2. The functional T is called a Mises functional of order ν at the point (F, G) (F and G are distribution functions) if the following conditions are satisfied:

(1) There exists a star-shaped set $\tau_T \subset \sigma_T$ at the point (F, G) such that

$$\lim \mathrm{P}[(F_n, G_m) \in \tau_T]=1 \qquad \text{when } n \to \infty,\ m \to \infty \text{ and } n/m \to \lambda .$$

(2) The functional T is ν times differentiable at the point (F, G) with respect to the set τ_T.

(3) $\lim \mathrm{P}\left[n^{(p/2)-\delta} \sup_t \left|\frac{d^p}{dt^p} T[(F_n^{(t)}, G_m^{(t)})]\right| > \varepsilon\right]=0$, for any $\varepsilon>0$, $\delta>0$ and $p=1,\cdots,\nu$, when $n\to\infty$, $m\to\infty$ and $n/m\to\lambda$, where $F_n^{(t)}=(1-t)\cdot F+tF_n$, $G_m^{(t)}=(1-t)G+tG_m$.

THEOREM 2.1. *Let T be a Mises functional of order $(\nu+1)$ at the point (F, G). Let moreover $T_j^{(p)}[(F, G)\colon x_1,\cdots,x_p]=0$ identically in $x_1, \cdots, x_p$ for $j=1, 2,\cdots, 2^p$; $p=1, 2,\cdots,\nu-1$, then*

$$n^{\nu/2}[T[(F_n, G_m)]-T[(F, G)]] \\ -(n^{\nu/2}/\nu!)\sum_{j=1}^{2^\nu}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} T_j^{(\nu)}[(F, G)\colon x_1,\cdots,x_\nu]\prod_{i=1}^{\nu} dH_{ji}(x_i)$$

converges to zero in probability as $n\to\infty$, $m\to\infty$ and $n/m\to\lambda$, where $H_{ji}(x_i)=[F_n(x_i)-F(x_i)]$ or $[G_m(x_i)-G(x_i)]$ depending on j and i.

PROOF. Consider the Taylor expansion of $T[(F_n^{(t)}, G_m^{(t)})]$ at $t=0$.

$$T[(F_n^{(t)},G_m^{(t)})]=T[(F, G)]+(t/1!)\frac{d}{dt}T[(F_n^{(t)}, G_m^{(t)})]\Big|_{t=0}+\cdots \\ +(t^\nu/\nu!)\frac{d^\nu}{dt^\nu}T[(F_n^{(t)}, G_m^{(t)})]\Big|_{t=0} \\ +(t^{\nu+1}/(\nu+1)!)\frac{d^{\nu+1}}{dt^{\nu+1}}T[(F_n^{(t)}, G_m^{(t)})]\Big|_{t=\theta}. \tag{2.1}$$

From the assumption of Theorem 2.1, we can see that

$$\frac{d^p}{dt^p}T[(F_n^{(t)}, G_m^{(t)})]\Big|_{t=0}=0\,, \qquad p=1, 2,\cdots,\nu-1\,.$$

Let $t=1$ in (2.1), then we have

$$T[(F_n, G_m)]-T[(F, G)] \\ =(1/\nu!)\frac{d^\nu}{dt^\nu}T[(F_n^{(t)}, G_m^{(t)})]\Big|_{t=0}+(1/(\nu+1)!)\frac{d^{\nu+1}}{dt^{\nu+1}}T[(F_n^{(t)}, G_m^{(t)})]\Big|_{t=\theta'} \\ =(1/\nu!)\sum_{j=1}^{2^\nu}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} T_j^{(\nu)}[(F, G)\colon x_1,\cdots,x_\nu]\prod_{i=1}^{\nu} dH_{ji}(x_i) \\ +(1/(\nu+1)!)\frac{d^{\nu+1}}{dt^{\nu+1}}T[(F_n^{(t)}, G_m^{(t)})]\Big|_{t=\theta'}.$$

By (3) of Definition 2.2,

$$(n^{\nu/2}/(\nu+1)!)\frac{d^{\nu+1}}{dt^{\nu+1}}T[(F_n^{(t)}, G_m^{(t)})]\Big|_{t=\theta'}$$

converges to zero in probability as $n\to\infty$, $m\to\infty$ and $n/m\to\lambda$. Therefore we have

$$n^{\nu/2}[T[(F_n, G_m)]-T[(F, G)]]$$
$$-(n^{\nu/2}/\nu!)\sum_{j=1}^{2^\nu}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} T_j^{(\nu)}[(F, G): x_1, \cdots, x_\nu]\prod_{i=1}^{\nu} dH_{ji}(x_i)$$

converges to zero in probability as $n\to\infty$, $m\to\infty$ and $n/m\to\lambda$.

From Theorem 2.1 and Theorem 2.3 which will be given below, the asymptotic distribution of $n^{\nu/2}[T[(F_n, G_m)]-T[(F, G)]]$ is the distribution represented by a sum of multiple stochastic integrals of independent Brownian bridges. In particular the asymptotic distribution is normal when $\nu=1$ as we shall show in the following Theorem 2.2. Of course we can regard Theorem 2.2 as a corollary of Theorem 2.3. But, because we can give an elementary proof when $\nu=1$, we will state Theorem 2.2 first.

THEOREM 2.2. *Let T be a Mises functional of order* 2 *at the point* (F, G). *Then we have*

$$n^{1/2}[T[(F_n, G_m)]-T[(F, G)]]\to N(0, \sigma^2) \qquad \textit{in distribution}$$

as $n\to\infty$, $m\to\infty$ and $n/m\to\lambda$, where

$$\sigma^2=\int_{-\infty}^{\infty}[T_1^{(1)}[(F, G): x]]^2dF(x)-\left[\int_{-\infty}^{\infty}T_1^{(1)}[(F, G): x]dF(x)\right]^2$$
$$+\lambda\left[\int_{-\infty}^{\infty}[T_2^{(1)}[(F, G): x]]^2dG(x)-\left[\int_{-\infty}^{\infty}T_2^{(1)}[(F, G): x]dG(x)\right]^2\right].$$

PROOF. By Theorem 2.1 it is sufficient to investigate the asymptotic distribution of

$$(2.2)\qquad n^{1/2}\int_{-\infty}^{\infty}T_1^{(1)}[(F, G): x]d[F_n(x)-F(x)]$$
$$+n^{1/2}\int_{-\infty}^{\infty}T_2^{(1)}[(F, G): x]d[G_m(x)-G(x)].$$

From Theorem 4 of Filippova [3],

$$n^{1/2}\int_{-\infty}^{\infty}T_1^{(1)}[(F, G): x]d[F_n(x)-F(x)]\to\int_0^1 T_1^{(1)}[(F, G): F^{-1}(x)]d\beta(x),$$

$$n^{1/2}\int_{-\infty}^{\infty}T_2^{(1)}[(F, G): x]d[G_m(x)-G(x)]\to\int_0^1\sqrt{\lambda}\,T_2^{(1)}[(F, G): G^{-1}(x)]d\beta(x),$$

in distribution as $n\to\infty$, $m\to\infty$ and $n/m\to\lambda$ hold, where $\{\beta(t);\ 0\leqq t\leqq 1\}$ is a Brownian bridge,

$$\int_0^1 T_1^{(1)}[(F, G): F^{-1}(x)]d\beta(x)=N(0, \sigma_1^2) \qquad \text{(in distribution)},$$

$$\int_0^1 \sqrt{\lambda}\, T_2^{(1)}[(F, G): G^{-1}(x)]d\beta(x) = N(0, \lambda\sigma_2^2) \text{ (in distribution)},$$

$$\sigma_1^2 = \int_0^1 [T_1^{(1)}[(F, G): F^{-1}(x)]]^2 dx - \left[\int_0^1 T_1^{(1)}[(F, G): F^{-1}(x)]dx\right]^2$$

and

$$\sigma_2^2 = \int_0^1 [T_2^{(1)}[(F, G): G^{-1}(x)]]^2 dx - \left[\int_0^1 T_2^{(1)}[(F, G): G^{-1}(x)]dx\right]^2.$$

Moreover each terms of (2.2) are independent. Therefore we get the result of Theorem 2.2.

DEFINITION 2.3 (Filippova). We denote by $\bar{\bar{L}}_2^{(\nu)}(F)$ ($F(x)$ is a distribution function) the set of functions $f(x_1, \cdots, x_\nu)$ such that

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f^2(x_{i_1}, \cdots, x_{i_\nu}) \prod_{l=1}^{r} dF(x_{j_l}) < \infty$$

$r = 1, \cdots, \nu$, $j_l = 1, \cdots, \nu$, $l = 1, \cdots, r$, where $j_{l_1} \neq j_{l_2}$ for $l_1 \neq l_2$ and each of the indices i_k, $k = 1, \cdots, \nu$, takes on one of the values j_l, $l = 1, \cdots, r$. In the space $\bar{\bar{L}}_2^{(\nu)}(F)$ we introduce the norm

$$\|f\|_{\bar{\bar{L}}_2^{(\nu)}(F)} = \sum \left\{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f^2(x_{i_1}, \cdots, x_{i_\nu}) \prod_{l=1}^{r} dF(x_{j_l})\right\}^{1/2},$$

where the sum is taken over all $r \leqq \nu$, and the r different among the ν arguments must be permuted in all possible ways. The space $\bar{\bar{L}}_2^{(\nu)}(F)$ is a Banach space. If $F(x)$ is the distribution function of uniform distribution on $[0, 1]$, $\bar{\bar{L}}_2^{(\nu)}(F)$ will be denoted by $\bar{\bar{L}}_2(D^\nu)$.

Suppose that $\{A_n\}$, $n = 1, 2, \cdots$, $\{B_m\}$, $m = 1, 2, \cdots$, A and B are random elements defined on a single probability space and their range is some measurable space. Let moreover A_n and B_n be independent and A and B be independent. If A_n converges to A in distribution and B_n converges to B in distribution, then (A_n, B_n) converges to (A, B) in distribution. Therefore if β_1 and β_2 are independent Brownian bridges, then $(n^{1/2}(F_n(F^{-1}(u)) - u), m^{1/2}(G_m(G^{-1}(u)) - u))$ converges to $(\beta_1(u), \beta_2(u))$ in distribution.

Note that we can define multiple stochastic integral of two independent Brownian bridges β_1 and β_2, similarly as Filippova [3]. A function which is given in D^ν and constant on the parallelopipeds $t_{i_k} < x_k \leqq t_{i_{k+1}}$, $k = 1, \cdots, \nu$, $(0 = t_0 < t_1 < \cdots < t_{r-1} < t_r = 1$ is some finite partition of the segment $[0, 1])$ shall be called a step-function. Let $h(u_1, \cdots, u_\nu)$ be a step-function. We define multiple stochastic integral of two independent Brownian bridges for the step-function $h(u_1, \cdots, u_\nu)$ by

$$B(h, \beta_1, \beta_2)=\sum_{i_1=1}^{r}\cdots\sum_{i_\nu=1}^{r} h_{i_1,\dots,i_\nu}\prod_{l=1}^{\nu}[\beta^*(t_{i_l})-\beta^*(t_{i_l-1})] ,$$

where $h_{i_1,\dots,i_\nu}$ is the value which the function $h(u_1,\cdots,u_\nu)$ takes on for $t_{i_k-1}<u_k\leqq t_{i_k}$, $k=1,\cdots,\nu$, and $\beta^*=\beta_1$ or β_2 depending on l and i_l. If $\beta_l^*=\beta_1$ for all l and i_l, then $B(h, \beta_1, \beta_2)=B(h, \beta_1)$ which is the multiple stochastic integral defined by Filippova [3]. Let $S(D^\nu)$ be the totality of step-functions. We let $(\Omega, \mathcal{B}, P)$ be the probability space on which β_1 and β_2 are defined. Consider a linear operator L defined by

$$\begin{aligned} L:\ & S(D^\nu)\longrightarrow L_2(\Omega, \mathcal{B}, P) \\ & \ \ \cup\!\!\!\mid \\ & \ \ h \ \longmapsto B(h, \beta_1, \beta_2) . \end{aligned}$$

Similarly as Section 2 of Filippova [3], it can be established that L is a bounded linear operator. Because $S(D^\nu)$ is dense in $\bar{\bar{L}}_2(D^\nu)$, $B(h, \beta_1, \beta_2)$ is defined naturally for all $h\in\bar{\bar{L}}_2(D^\nu)$. Suppose $D[0, 1]$ is the space of functions on $[0, 1]$ and $\mathcal{D}$ is the σ-field on $[0, 1]$ defined by Billingsley [2] (Chapter 3). If we define

$$\begin{aligned} \psi_h:\ & D[0, 1]\times D[0, 1]\longrightarrow R \\ & \qquad\quad \cup\!\!\!\mid \\ & \qquad (x, y) \longmapsto \psi_h(x, y)=B(h, x, y) , \end{aligned}$$

$\psi_h(x, y)$ is continuous except for a null set measured by the product measure of P_{β_1} and P_{β_2}, where P_{β_i} is the probability measure on $(D[0, 1], \mathcal{D})$ induced by β_i $(i=1, 2)$. It is proved by the following inequality.

$$\begin{aligned} (2.3)\qquad & |\psi_h(x, y)-\psi_h(x', y')| \\ & \leqq|\psi_h(x, y)-\psi_{h_n}(x, y)|+|\psi_{h_n}(x, y)-\psi_{h_n}(x', y')| \\ & \quad +|\psi_{h_n}(x', y')-\psi_h(x', y')| , \end{aligned}$$

where $x, y, x', y'\in D[0, 1]$ and h_n is a step-function. Since ψ_h is a bounded linear operator with respect to h, the first and the third terms of the right-hand side of (2.3) can be sufficiently small uniformly in the arguments except for a null set by taking h_n near h. If the distance between (x, y) and (x', y') is small, the second term is small by the definition of $\psi_{h_n}(x, y)$. Therefore by using Theorem 5.1 of Billingsley [2], we obtain the following Theorem 2.3.

THEOREM 2.3. *If $h_j(u_1,\cdots,u_\nu)\in\bar{\bar{L}}_2(D^\nu)$, $j=1,\cdots,2^\nu$, the asymptotic distribution of*

$$n^{\nu/2}\sum_{j=1}^{2^\nu}\int_0^1\cdots\int_0^1 h_j(u_1,\cdots,u_\nu)\prod_{i=1}^{\nu}dH_{ji}^*(u_i)$$

is the distribution of

$$\sum_{j=1}^{2^\nu} \int_0^1 \cdots \int_0^1 c_j h_j(u_1, \cdots, u_\nu) \prod_{i=1}^{\nu} d\beta_j^*(u_i) ,$$

when $n \to \infty$, $m \to \infty$ *and* $n/m \to \lambda$, *where* $H_{ji}^*(u_i)$ *is either* $[F_n(F^{-1}(u_i)) - u_i]$ *or* $[G_m(G^{-1}(u_i)) - u_i]$ *depending on* j *and* i, $\beta_j^*(u_i)$ *is either* $\beta_1(u_i)$ *or* $\beta_2(u_i)$ *depending on* j *and* i, *and where* $c_j = \lambda^{n_j/2}$, n_j *is the number of* $[G_m(G^{-1} \cdot (u_i)) - u_i]$ *in the set* $\{H_{ji}^*(u_i);\ i = 1, \cdots, \nu\}$.

Remark 2.1. If T is a Mises functional, from Theorem 2.1 its asymptotic distribution is equal to the asymptotic distribution of

$$(2.4) \qquad (n^{\nu/2}/\nu!) \sum_{j=1}^{2^\nu} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} T_j^{(\nu)}[(F, G):\ x_1, \cdots, x_\nu] \prod_{i=1}^{\nu} dH_{ji}(x_i)$$

for some ν, where $H_{ji}(x_i) = [F_n(x_i) - F(x_i)]$ or $[G_m(x_i) - G(x_i)]$. Define

$$\bar{H}_{ji}(x) = \begin{cases} F(x) & \text{if } H_{ji}(x) = F_n(x) - F(x) \\ G(x) & \text{if } H_{ji}(x) = G_m(x) - G(x) . \end{cases}$$

Then the distribution of (2.4) is equal to the distribution of

$$(n^{\nu/2}/\nu!) \sum_{j=1}^{2^\nu} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} T_j^{(\nu)}[(F, G):\ \bar{H}_{j1}^{-1}(u_1), \cdots, \bar{H}_{j\nu}^{-1}(u_\nu)] \prod_{i=1}^{\nu} dH_{ji}^*(u_i) .$$

3. Examples

Example 1 (U-statistic for the two-sample problem). Let $h(x_1, \cdots, x_a, y_1, \cdots, y_b)$ be symmetric in the x's alone and in the y's alone. For simplicity we assume that $h(x_1, \cdots, x_a, y_1, \cdots, y_b) = 0$ if for some i, j $(i \neq j)$ $x_i = x_j$ or $y_i = y_j$. Define

$$(3.1) \quad U_{nm} = \binom{n}{a}^{-1} \binom{m}{b}^{-1} \sum h(X_{i_1}, \cdots, X_{i_a}, Y_{j_1}, \cdots, Y_{j_b}) ,$$

where the summation is extended over all subscripts $1 \leqq i_1 < \cdots < i_a \leqq n;\ 1 \leqq j_1 < \cdots < j_b \leqq m$.

If we assume that

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h^2(x_1, \cdots, x_a, y_1, \cdots, y_b) dF(x_1) \cdots dF(x_a) dG(y_1) \cdots dG(y_b) < \infty ,$$

asymptotic normality of U_{nm} holds as follows. Note that

$$\begin{aligned} U_{nm} (&= T[(F_n, G_m)]) \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(a!\, b! \binom{n}{a} \binom{m}{b}\right)^{-1} n^a m^b h(x_1, \cdots, x_a, y_1 \cdots, y_b) \\ &\quad \times dF_n(x_1) \cdots dF_n(x_a) dG_m(y_1) \cdots dG_m(y_b) . \end{aligned}$$

$T[(F_n, G_m)]$ is a Mises functional of order 2 at (F, G), and it holds

$$\frac{d}{dt}T[(F_n^{(t)}, G_m^{(t)})]\Big|_{t=0}$$
$$=\left(a!b!\binom{n}{a}\binom{m}{b}\right)^{-1} n^a m^b\Big[\int_{-\infty}^{\infty}\Big\{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} ah(x, x_1,\cdots, x_{a-1},$$
$$y_1,\cdots, y_b)dF(x_1)\cdots dF(x_{a-1})dG(y_1)\cdots dG(y_b)\Big\}$$
$$\times d[F_n(x)-F(x)]+\int_{-\infty}^{\infty}\Big\{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} bh(x_1,\cdots, x_a, x,$$
$$y_1,\cdots, y_{b-1})dF(x_1)\cdots dF(x_a)dG(y_1)\cdots dG(y_{b-1})\Big\}$$
$$\times d[G_m(x)-G(x)]\Big] .$$

If we apply Theorem 2.2 with

$$T_1^{(1)}[(F, G): x]=\left(a!b!\binom{n}{a}\binom{m}{b}\right)^{-1} n^a m^b a\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h(x, x_1,\cdots, x_{a-1},$$
$$y_1,\cdots, y_b)dF(x_1)\cdots dF(x_{a-1})dG(y_1)\cdots dG(y_b) ,$$
$$T_2^{(1)}[(F, G): x]=\left(a!b!\binom{n}{a}\binom{m}{b}\right)^{-1} n^a m^b b\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h(x_1,\cdots, x_a, x,$$
$$y_1,\cdots, y_{b-1})dF(x_1)\cdots dF(x_a)dG(y_1)\cdots dG(y_{b-1}) ,$$

we can see the asymptotic normality of U_{nm}, that is,

$$n^{1/2}\Big[U_{nm}-\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h(x_1,\cdots, x_a, y_1,\cdots, y_b)$$
$$\times dF(x_1)\cdots dF(x_a)dG(y_1)\cdots dG(y_b)\Big]$$

converges to $N(0, \sigma^2)$ in distribution as $n\to\infty$, $m\to\infty$ and $n/m\to\lambda$, where

$$\sigma^2=a^2\,\mathrm{Var}\left[\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h(X_1, x_1,\cdots, x_{a-1}, y_1,\cdots, y_b)dF(x_1)\cdots dG(y_b)\right]$$
$$+\lambda b^2\,\mathrm{Var}\left[\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h(x_1,\cdots, x_a, Y_1 y_1,\cdots, y_{b-1})dF(x_1)\cdots dG(y_{b-1})\right] .$$

Example 2 (The statistic of Cramér-von Mises type for testing homogeneity). We can mention the following statistic (3.2) as a test statistic for the two-sample problem.

$$\text{(3.2)} \qquad \frac{nm}{n+m}\int_{-\infty}^{\infty}[F_n(x)-G_m(x)]^2 d\left[\frac{n}{n+m}F_n(x)+\frac{m}{n+m}G_m(x)\right] .$$

The next theorem is well known.

THEOREM 3.1 (Rosenblatt [6] and Fisz [4]). *The statistic* (3.2) *has the same asymptotic distribution when* $n\to\infty$, $m\to\infty$ *and* $n/m\to\lambda$, *as the one-sample von Mises statistic under the assumption* "$F=G$".

We will prove Theorem 3.1 by our method. We denote the formula (3.2) by $(nm/(n+m))T[(F_n, G_m)]$. Then the following (3.3), (3.4), (3.5) and (3.6) hold.

$$\begin{aligned}(3.3)\quad &\frac{d}{dt}T[(F_n^{(t)}, G_m^{(t)})]\\ &=\int_{-\infty}^{\infty} 2[F_n^{(t)}(x)-G_m^{(t)}(x)][(F_n(x)-F(x))-(G_m(x)-G(x))]\\ &\quad\times d\left[\frac{n}{n+m}F_n^{(t)}(x)+\frac{m}{n+m}G_m^{(t)}(x)\right]\\ &\quad+\int_{-\infty}^{\infty}[F_n^{(t)}(x)-G_m^{(t)}(x)]^2 d\left[\frac{n}{n+m}(F_n(x)-F(x))\right.\\ &\quad\left.+\frac{m}{n+m}(G_m(x)-G(x))\right],\end{aligned}$$

$$(3.4)\quad \frac{d}{dt}T[(F_n^{(t)}, G_m^{(t)})]\bigg|_{t=0}=0\,, \qquad \text{under the assumption "}F=G\text{"},$$

$$\begin{aligned}(3.5)\quad &\frac{d^2}{dt^2}T[(F_n^{(t)}, G_m^{(t)})]\\ &=\int_{-\infty}^{\infty} 2[(F_n(x)-F(x))-(G_m(x)-G(x))]^2\\ &\quad\times d\left[\frac{n}{n+m}F_n^{(t)}(x)+\frac{m}{n+m}G_m^{(t)}(x)\right]\\ &\quad+\int_{-\infty}^{\infty}4[F_n^{(t)}(x)-G_m^{(t)}(x)][(F_n(x)-F(x))-(G_m(x)-G(x))]\\ &\quad\times d\left[\frac{n}{n+m}(F_n(x)-F(x))+\frac{m}{n+m}(G_m(x)-G(x))\right],\end{aligned}$$

$$\begin{aligned}(3.6)\quad &\frac{d^2}{dt^2}T[(F_n^{(t)}, G_m^{(t)})]\bigg|_{t=0}\\ &=\int_{-\infty}^{\infty} 2[(F_n(x)-F(x))-(G_m(x)-G(x))]^2 dF(x)\\ &=\int_{-\infty}^{\infty} 2[F_n(x)-F(x)]^2 dF(x)\\ &\quad-\int_{-\infty}^{\infty}4[F_n(x)-F(x)][G_m(x)-G(x)]dF(x)\\ &\quad+\int_{-\infty}^{\infty}2[G_m(x)-G(x)]^2 dF(x)\end{aligned}$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2g(y, z)d[F_n(y)-F(y)]d[F_n(z)-F(z)]$$
$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 4g(y, z)d[F_n(y)-F(y)]d[G_m(z)-G(z)]$$
$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2g(y, z)d[G_m(y)-G(y)]d[G_m(z)-G(z)] ,$$

where

$$g(y, z)=\int_{-\infty}^{\infty} \chi_{(-\infty,x]}(y)\chi_{(-\infty,x]}(z)dF(x)$$
$$=1-F(\max\{y, z\}) , \qquad \text{under the assumption "}F=G\text{"} .$$

By using Theorems 2.1 and 2.3, we can see under the assumption "$F=G$",

$$\tag{3.7} \frac{nm}{n+m}T[(F_n, G_m)]$$
$$\longrightarrow \int_0^1\int_0^1 h(u, v)(\lambda/(1+\lambda))d\beta_1(u)d\beta_1(v)$$
$$-\int_0^1\int_0^1 h(u, v)(2\lambda^{1/2}/(1+\lambda))d\beta_1(u)d\beta_2(v)$$
$$+\int_0^1\int_0^1 h(u, v)(1/(1+\lambda))d\beta_2(u)d\beta_2(v)$$

in distribution, where $h(u, v)=g(F^{-1}(u), F^{-1}(v))$, β_1 and β_2 are independent Brownian bridges, when $n\to\infty$, $m\to\infty$ and $n/m\to\lambda$. Note that the right-hand side of (3.7) is equal to

$$\tag{3.8} \int_0^1\int_0^1 h(u, v)d[(\lambda/(1+\lambda))^{1/2}\beta_1(u)-(1/(1+\lambda))^{1/2}\beta_2(u)]$$
$$\times d[(\lambda/(1+\lambda))^{1/2}\beta_1(v)-(1/(1+\lambda))^{1/2}\beta_2(v)] .$$

Moreover $(1/(1+\lambda))^{1/2}\beta_1-(1/(1+\lambda))^{1/2}\beta_2$ is a Brownian bridge since β_1 and β_2 are independent. The distribution of (3.8) is equal to the asymptotic distribution of the goodness of fit test statistic of Cramér-von Mises type (see Filippova [3]).

Example 3. We shall consider the following two statistics as an extension of Example 2.

$$\tag{3.9} n^2\int_{-\infty}^{\infty} [F_n(x)-F(x)]^4 dF(x) ,$$

$$\tag{3.10} \frac{n^2m^2}{(n+m)^2}\int_{-\infty}^{\infty} [F_n(x)-G_m(x)]^4 d\left[\frac{n}{n+m}F_n(x)+\frac{m}{n+m}G_m(x)\right] .$$

We want to use (3.9) as a statistic for testing goodness of fit and to use (3.10) as a statistic for testing homogeneity. Unfortunately, we can not use them as test statistics for the present, because we do not know the distribution function of the asymptotic distribution of (3.9) and (3.10), though it is true that the asymptotic distribution of (3.9) and (3.10) exists and does not depend on F under the null hypothesis. Anyway we deal with this as only a mathematical example since this is a very interesting example for Theorem 2.3. We let

$$T[(F_n, G_m)]=\int_{-\infty}^{\infty} [F_n(x)-G_m(x)]^4 d\left[\frac{n}{n+m}F_n(x)+\frac{m}{n+m}G_m(x)\right] .$$

Similarly as Example 2, we get under the assumption "$F=G$"

$$\frac{d}{dt}T[(F_n^{(t)}, G_m^{(t)})]\Big|_{t=0}=0 , \tag{3.11}$$

$$\frac{d^2}{dt^2}T[(F_n^{(t)}, G_m^{(t)})]\Big|_{t=0}=0 , \tag{3.12}$$

$$\frac{d^3}{dt^3}T[(F_n^{(t)}, G_m^{(t)})]\Big|_{t=0}=0 \qquad \text{and} \tag{3.13}$$

$$\begin{aligned}
&\frac{d^4}{dt^4}T[(F_n^{(t)}, G_m^{(t)})]\Big|_{t=0} \\
&=4!\int_{-\infty}^{\infty} [(F_n(x)-F(x))-(G_m(x)-G(x))]^4 dF(x) \\
&=4!\Big[\int_{-\infty}^{\infty} (F_n(x)-F(x))^4 dF(x) \\
&\quad -4\int_{-\infty}^{\infty} (F_n(x)-F(x))^3(G_m(x)-G(x))dF(x) \\
&\quad +6\int_{-\infty}^{\infty} (F_n(x)-F(x))^2(G_m(x)-G(x))^2 dF(x) \\
&\quad -4\int_{-\infty}^{\infty} (F_n(x)-F(x))(G_m(x)-G(x))^3 dF(x) \\
&\quad +\int_{-\infty}^{\infty} (G_m(x)-G(x))^4 dF(x)\Big] \\
&=4!\Big[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y_1, y_2, y_3, y_4)d[F_n(y_1)-F(y_1)] \\
&\quad \times d[F_n(y_2)-F(y_2)]d[F_n(y_3)-F(y_3)]d[F_n(y_4)-F(y_4)] \\
&\quad -4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y_1, y_2, y_3, y_4)d[F_n(y_1)-F(y_1)] \\
&\quad \times d[F_n(y_2)-F(y_2)]d[F_n(y_3)-F(y_3)]d[G_m(y_4)-G(y_4)] \\
&\quad +6\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y_1, y_2, y_3, y_4)d[F_n(y_1)-F(y_1)] \\
&\quad \times d[F_n(y_2)-F(y_2)]d[G_m(y_3)-G(y_3)]d[G_m(y_4)-G(y_4)]
\end{aligned} \tag{3.14}$$

$$-4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y_1, y_2, y_3, y_4)d[F_n(y_1)-F(y_1)]$$
$$\times d[G_m(y_2)-G(y_2)]d[G_m(y_3)-G(y_3)]d[G_m(y_4)-G(y_4)]$$
$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y_1, y_2, y_3, y_4)d[G_m(y_1)-G(y_1)]$$
$$\times d[G_m(y_2)-G(y_2)]d[G_m(y_3)-G(y_3)]d[G_m(y_4)-G(y_4)]\Big] ,$$

where

$$g(y_1, y_2, y_3, y_4)=\int_{-\infty}^{\infty} \chi_{(-\infty,x]}(y_1)\chi_{(-\infty,x]}(y_2)\chi_{(-\infty,x]}(y_3)\chi_{(-\infty,x]}(y_4)dF(x)$$
$$=1-F(\max\{y_1, y_2, y_3, y_4\}) .$$

By using Theorems 2.1 and 2.3, we get under the assumption "$F=G$".

$$(3.15)\quad \frac{n^2m^2}{(n+m)^2}T[(F_n, G_m)]$$
$$\longrightarrow\int_0^1\int_0^1\int_0^1\int_0^1 h(u_1, u_2, u_3, u_4)(\lambda^2/(1+\lambda)^2)d\beta_1(u_1)d\beta_1(u_2)d\beta_1(u_3)d\beta_1(u_4)$$
$$-4\int_0^1\int_0^1\int_0^1\int_0^1 h(u_1, u_2, u_3, u_4)(\lambda^{3/2}/(1+\lambda)^2)d\beta_1(u_1)d\beta_1(u_2)d\beta_1(u_3)d\beta_2(u_4)$$
$$+6\int_0^1\int_0^1\int_0^1\int_0^1 h(u_1, u_2, u_3, u_4)(\lambda/(1+\lambda)^2)d\beta_1(u_1)d\beta_1(u_2)d\beta_2(u_3)d\beta_2(u_4)$$
$$-4\int_0^1\int_0^1\int_0^1\int_0^1 h(u_1, u_2, u_3, u_4)(\lambda^{1/2}/(1+\lambda)^2)d\beta_1(u_1)d\beta_2(u_2)d\beta_2(u_3)d\beta_2(u_4)$$
$$+\int_0^1\int_0^1\int_0^1\int_0^1 h(u_1, u_2, u_3, u_4)(1/(1+\lambda)^2)d\beta_2(u_1)d\beta_2(u_2)d\beta_2(u_3)d\beta_2(u_4)$$

in distribution, where

$$h(u_1, u_2, u_3, u_4)=g(F^{-1}(u_1), F^{-1}(u_2), F^{-1}(u_3), F^{-1}(u_4))$$
$$=1-\max\{u_1, u_2, u_3, u_4\} ,$$

β_1 and β_2 are independent Brownian bridges, when $n\to\infty$, $m\to\infty$ and $n/m\to\lambda$. Note that the right-hand side of (3.15) is equal to

$$(3.16)\quad \int_0^1\int_0^1\int_0^1\int_0^1 h(u_1, u_2, u_3, u_4)d[(\lambda/(1+\lambda))^{1/2}\beta_1(u_1)-(1/(1+\lambda))^{1/2}\beta_2(u_1)]$$
$$\times d[(\lambda/(1+\lambda))^{1/2}\beta_1(u_2)-(1/(1+\lambda))^{1/2}\beta_2(u_2)]$$
$$\times d[(\lambda/(1+\lambda))^{1/2}\beta_1(u_3)-(1/(1+\lambda))^{1/2}\beta_2(u_3)]$$
$$\times d[(\lambda/(1+\lambda))^{1/2}\beta_1(u_4)-(1/(1+\lambda))^{1/2}\beta_2(u_4)] .$$

It is easily seen by Filippova's results [3] that the distribution of (3.16) is equal to the asymptotic distribution of (3.9) under the null hypothesis.

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