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MIXING NORMAL APPROXIMATIONS OF VECTORS OF SUMS AND MAXIMUM SUMS

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Summary

The conditioned central limit theorem for the vector of maximum partial sums based on independent identically distributed random vectors is investigated and the rate of convergence is discussed. The conditioning is that of Rényi (1958, *Acta Math. Acad. Sci. Hungar.*, **9**, 215–228). Analogous results for the vector of partial sums are obtained.

1. Introduction

Let $\boldsymbol{X}_1,\cdots,\boldsymbol{X}_n,\cdots$ be a sequence of p-dimensional independent identically distributed (iid) random vectors defined on the same probability space $(\Omega, \mathcal{A}, P)$ with distribution functions (df) $F(x)$ and such that

$$\text{(1.1)}\qquad \mathrm{E}\,\boldsymbol{X}_1=\boldsymbol{\mu} \quad\text{and}\quad \mathrm{E}\,(\boldsymbol{X}_1-\boldsymbol{\mu})(\boldsymbol{X}_1-\boldsymbol{\mu})'=\boldsymbol{\Sigma}=(\sigma_{ij})\,, \qquad \text{say}\,.$$

Let ρ_{ij} denote the correlation coefficient between X_{i1} and X_{j1} for all $i\neq j$, $i, j=1,\cdots,p$, i.e., ρ_{ij} is the correlation coefficient between the ith and the jth coordinates of $\boldsymbol{X}_1=(X_{11},\cdots,X_{p1})'$. Further, for $i=1,\cdots,p$ let $S_{in}=\sum_{j=1}^{n} X_{ij}$ and set $S_{in}^*=\max_{1\leq j\leq n} S_{ij}$. Write

$$\text{(1.2)}\qquad \boldsymbol{S}_n=(S_{1n},\cdots,S_{pn})' \quad\text{and}\quad \boldsymbol{S}_n^*=(S_{1n}^*,\cdots,S_{pn}^*)'\,.$$

When $p=1$, Rényi [5] established that for any $B\in\mathcal{A}$ with $\mathrm{P}(B)>0$, $\mathrm{P}[S_n-n\mu\leqq\sigma x\sqrt{n}\,|B]\to\Phi(x)$ as $n\to\infty$. Next, let

$$\text{(1.3)}\qquad \Delta_n(B)=\sup_x |\mathrm{P}[S_n-n\mu\leqq\sigma x\sqrt{n}\,|B]-\Phi(x)|$$

where $\Phi(\cdot)$ denotes the df of the standard normal variate. More recently, Landers and Rogge [4] gave a counter example demonstrating

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that there does not exist a sequence of real numbers $\{\delta_n\}$ such that $\delta_n \to \infty$ and $\delta_n \Delta_n(B)$ remain bounded for each $B \in \mathcal{A}$. It is not surprising that the rate of convergence should depend on B. Landers and Rogge [4] gave an interesting inequality for rates of convergence of $\Delta_n(B)$ that depends on B and used it to give an elegant proof of convergence of (1.3) to zero.

On the other hand, the rate of convergence in the multidimensional central limit theorem is a relatively new area of investigation. Sazonov [6] and Bikyalis [3] showed that if $\mathrm{E}|X_{i1}|^{2+\delta} < \infty$, $0 < \delta \leqq 1$, $i = 1, \cdots, p$ and if $\boldsymbol{X}_1$ has a nonsingular covariance matrix, then

$$(1.4) \qquad \Delta_n = \sup_{\boldsymbol{x}} |\mathrm{P}[S_{1n} - n\mu_1 \leqq \sigma_1 x_1 \sqrt{n}, \cdots, S_{pn} - n\mu_p \leqq \sigma_p x_p \sqrt{n}] - \Phi_R(\boldsymbol{x})| = O(n^{-\delta/2}),$$

where $R = (\rho_{ij})$ is the correlation matrix of $\boldsymbol{X}_1$. Assuming that $\mu_i > 0$, $i = 1, \cdots, p$, Ahmad [1] showed that under the same conditions of Sazonov [6] and Bikyalis [3], and for any $0 < \delta \leqq 1$,

$$(1.5) \qquad \Delta_n^* = \sup_{\boldsymbol{x}} |\mathrm{P}[S_{1n}^* - n\mu_1 \leqq \sigma_1 x_1 \sqrt{n}, \cdots, S_{pn}^* - n\mu_p \leqq \sigma_p x_p \sqrt{n}] - \Phi_R(\boldsymbol{x})| = O(n^{-\delta/2}).$$

For any $B \in \mathcal{A}$ such that $\mathrm{P}(B) > 0$, define

$$(1.6) \qquad \Delta_n(B) = \sup_{\boldsymbol{x}} |\mathrm{P}[\{S_{1n} - n\mu_1 \leqq \sigma_1 x_1 \sqrt{n}, \cdots, S_{pn} - n\mu_p \leqq \sigma_p x_p \sqrt{n}\} | B] - \Phi_R(\boldsymbol{x})|,$$

and

$$\Delta_n^*(B) = \sup_{\boldsymbol{x}} |\mathrm{P}[\{S_{1n}^* - n\mu_1 \leqq \sigma_1 x_1 \sqrt{n}, \cdots, S_{pn}^* - n\mu_p \leqq \sigma_p x_p \sqrt{n}\} | B] - \Phi_R(\boldsymbol{x})|.$$

The purpose of present note is to invetigate the conditions under which $\Delta_n(B)$ and $\Delta_n^*(B)$ converge to 0 as $n \to \infty$ for any $B \in \mathcal{A}$ and discuss the rate of convergence.

For ease of presentation we shall only deal with the case $p = 2$ and let $\Phi(x_1, x_2)$ denote the bivariate standard normal df (i.e. bivariate normal df with $\mathrm{E}\,X_i = 0$, $\mathrm{Var}\,X_i = 1$, and $\mathrm{cov}(X_1, X_2) = \rho$, the correlation coefficient) and $\Phi_{(1)}(\phi_{(1)})$ denote the df (probability density function (pdf)) of the univariate standard normal.

2. Conditioned CLT for partial sums

Throughout this section we shall assume that $p = 2$. Let $\mathcal{F}_k$ denote the σ-field generated by $(X_{11}, X_{12})', \cdots, (X_{1k}, X_{2k})'$ $k = 1, 2, \cdots$. Write

$$(2.1)\qquad \Delta_n(x_1, x_2)=|\mathrm{P}\,[S_{1n}-n\mu_1\leqq\sigma_1x_1\sqrt{n}\,,\ S_{2n}-n\mu_2 \leqq\sigma_2x_2\sqrt{n}\,]-\Phi(x_1, x_2)|\ ,$$

$$(2.2)\qquad \Delta_n(x_1, x_2\,|\,B)=|\mathrm{P}\,[S_{1n}-n\mu_1\leqq\sigma_1x_1\sqrt{n}\,,\ S_{2n}-n\mu_2 \leqq\sigma_2x_2\sqrt{n}\,|\,B]-\Phi(x_1, x_2)|\ ,$$

and

$$(2.3)\qquad \Delta_n(x_1, x_2\,|\,\mathcal{F}_k)=|\mathrm{P}\,[S_{1n}-n\mu_1\leqq\sigma_1x_1\sqrt{n}\,,\ S_{2n}-n\mu_2 \leqq\sigma_2x_2\sqrt{n}\,|\,\mathcal{F}_k]-\Phi(x_1, x_2)|\ .$$

We start with the following inequality which is a bivariate extension of Theorem 1 of Landers and Rogge [4] whose method of proof utilizes a conditioning device that enables us to establish an appropriate bound.

INEQUALITY 2.1. For all $n>k$ and all (x_1, x_2),

$$(2.4)\qquad \Delta_n(x_1, x_2\,|\,\mathcal{F}_k)\leqq\sup_{x_1,x_2}\Delta_{n-k}(x_1, x_2)+C_1|S_{1k}-k\mu_1|/(n-k)^{1/2} +C_2|S_{2k}-k\mu_2|/(n-k)^{1/2}+C_3(k/(n-k))\ ,$$

where C_1, C_2 and C_3 are positive constants dependent on σ_1, σ_2 and ρ but independent of n.

PROOF. Write $S_{i,(n-k)}=\sum_{j=k+1}^{n}X_{ij}$, $i=1, 2$. Since S_{ik} and $S_{i,(n-k)}$ are independent and $S_{i,(n-k)}$ and $S_{i(n-k)}=\sum_{j=1}^{n-k}X_{ij}$ have the same distribution we have

$$(2.5)\qquad \Delta_n(x_1, x_2\,|\,\mathcal{F}_k)\leqq\sup_{x_1,x_2}\Delta_{n-k}(x_1, x_2) +\left|\Phi\left(\frac{\sigma_1x_1\sqrt{n}-(S_{1k}-k\mu_1)}{\sigma_1\sqrt{n-k}},\ \frac{\sigma_2x_2\sqrt{n}-(S_{2k}-k\mu_2)}{\sigma_2\sqrt{n-k}}\right)-\Phi(x_1, x_2)\right|\ .$$

Thus we need to show that the second term of the upper bound of (2.5) is less than or equal to the sum of the last three terms of the right member of (2.4). Note that

$$\left|\Phi\left(\frac{\sigma_1x_1\sqrt{n}-(S_{1k}-k\mu_1)}{\sigma_1\sqrt{n-k}},\ \frac{\sigma_2x_2\sqrt{n}-(S_{2k}-k\mu_2)}{\sigma_2\sqrt{n-k}}\right)-\Phi(x_1, x_2)\right| \leqq I_{1n}+I_{2n}+I_{3n}+I_{4n}\ ,$$

where

$$I_{1n}=\left|\Phi\left(\frac{\sigma_1x_1\sqrt{n}-(S_{1k}-k\mu_1)}{\sigma_1\sqrt{n-k}},\ \frac{\sigma_2x_2\sqrt{n}-(S_{2k}-k\mu_2)}{\sigma_2\sqrt{n-k}}\right)\right.$$
$$\left.-\Phi\left(x_1\sqrt{\frac{n}{n-k}},\ \frac{\sigma_2x_2\sqrt{n}-(S_{2k}-k\mu_2)}{\sigma_2\sqrt{n-k}}\right)\right|,$$

$$I_{2n}=\left|\Phi\left(x_1\sqrt{\frac{n}{n-k}},\ \frac{\sigma_2x_2\sqrt{n}-(S_{2k}-k\mu_2)}{\sigma_2\sqrt{n-k}}\right)\right.$$
$$\left.-\Phi\left(x_1\sqrt{\frac{n}{n-k}},\ x_2\sqrt{\frac{n}{n-k}}\right)\right|,$$

$$I_{3n}=\left|\Phi\left(x_1\sqrt{\frac{n}{n-k}},\ x_2\sqrt{\frac{n}{n-k}}\right)-\Phi\left(x_1,\ x_2\sqrt{\frac{n}{n-k}}\right)\right|,\qquad\text{and}$$

$$I_{4n}=\left|\Phi\left(x_1,\ x_2\sqrt{\frac{n}{n-k}}\right)-\Phi(x_1,\ x_2)\right|.$$

Now, it is not difficult to show that

$$(2.6)\qquad I_{1n}\leqq\int_{-\infty}^{(\sigma_2x_2\sqrt{n}-(S_{2k}-k\mu_2))/\sigma_2\sqrt{n-k}}\left|\Phi\left(x_1\sqrt{\frac{n}{n-k}}-\frac{(S_{1k}-k\mu_1)}{\sigma_1\sqrt{n-k}},\ w\right)\right.$$
$$\left.-\Phi\left(x_1\sqrt{\frac{n}{n-k}},\ w\right)\right|\phi_{(1)}(w)dw$$
$$=\int_{-\infty}^{(\sigma_2x_2\sqrt{n}-(S_{2k}-k\mu_2))/\sigma_2\sqrt{n-k}}\left|\Phi_{(1)}\left(\left(x_1\sqrt{\frac{n}{n-k}}-\frac{S_{1k}-k\mu_1}{\sigma_1\sqrt{n-k}}\right.\right.\right.$$
$$\left.\left.\left.-\rho w\right)\Big/\sqrt{1-\rho^2}\right)-\Phi_{(1)}\left(\left(x_1\sqrt{\frac{n}{n-k}}-\rho w\right)\Big/\sqrt{1-\rho^2}\right)\right|\phi_{(1)}(w)dw$$
$$\leqq C_1\frac{|S_{1k}-k\mu_1|}{\sqrt{n-k}},$$

since $|\Phi_{(1)}(x+\varepsilon)-\Phi_{(1)}(x)|\leqq C|\varepsilon|$ (see Theorem 1 of Landers and Rogge [4]). Similarly we can show that $I_{2n}\leqq C_2(|S_{2k}-k\mu_2|/\sqrt{n-k})$, where C_1 and C_2 depend on σ_1, σ_2 and ρ but are independent of n. Next,

$$(2.7)\qquad I_{3n}\leqq\int_{-\infty}^{x_2\sqrt{n/(n-k)}}\left|\Phi_{(1)}\left(\frac{x_1\sqrt{n/(n-k)}-\rho w}{\sqrt{1-\rho^2}}\right)-\Phi_{(1)}\left(\frac{x_1-\rho w}{\sqrt{1-\rho^2}}\right)\right|\phi_{(1)}(w)dw$$
$$\leqq C_3\left(\sqrt{\frac{n}{n-k}}-1\right)\leqq C_3^*\left(\frac{k}{n-k}\right),$$

since $(\sqrt{n/(n-k)}-1)\leqq k/(n-k)$ and $|\Phi_{(1)}(\varepsilon x)-\Phi_{(1)}(x)|\leqq C|\varepsilon-1||x|e^{-x^2/2}\leqq C|\varepsilon-1|$; similarly we can show that $I_{4n}\leqq C_4(k/(n-k))$, where C_3 and C_4 are positive constants that may depend on σ_1 and σ_2. The desired conclusion now follows.

Using Inequality 2.1 we can now establish rates of convergence for

$\Delta_n(x_1, x_2|B)$ for $B \in F_k$ such that $\mathrm{P}(B)>0$. Let $\Delta_n(B)=\sup_{x_1,x_2} \Delta_n(x_1, x_2|B)$.

THEOREM 2.1. *If $\Delta_n=O(n^{-\delta/2})$, $0<\delta\leqq 1$, and if for each r such that $2\leqq r\leqq 2+\delta$, $\mathrm{E}|X_{i1}|^r<\infty$, $i=1, 2$, then there exists a constant C_r which depends on σ_1, σ_2 and ρ such that for all $B\in\mathcal{F}_k$ with $\mathrm{P}(B)>0$,*

$$\Delta_n(B)\leqq C_r(\mathrm{P}(B))^{-1/r}\left(\frac{k}{n}\right)^{\delta/2}. \tag{2.8}$$

PROOF. Using Hölder's inequality,

$$\begin{aligned}\sup_{x_1,x_2}|\mathrm{P}[S_{1n}-n\mu_1\leqq\sigma_1x_1\sqrt{n}, S_{2n}-n\mu_2\leqq\sigma_2x_2\sqrt{n}, B]-\Phi(x_1, x_2)\mathrm{P}(B)|\\ \leqq[\mathrm{P}(B)]^{(r-1)/r}\|\Delta_n(\mathcal{F}_k)\|_r,\end{aligned} \tag{2.9}$$

where $\Delta_n(\mathcal{F}_k)=\sup \Delta_n(x_1, x_2|\mathcal{F}_k)$. But using Inequality 2.1 and Minkowski's inequality we see that

$$\begin{aligned}\|\Delta_n(\mathcal{F}_k)\|_r\leqq\Delta_{n-k}+\frac{C_1}{\sqrt{n-k}}\mathrm{E}^{1/r}|S_{1k}-k\mu_1|^r\\ +\frac{C_2}{\sqrt{n-k}}\mathrm{E}^{1/r}|S_{2k}-k\mu_2|^r+C_3\left(\frac{k}{n-k}\right),\end{aligned} \tag{2.10}$$

where C_1, C_2, and C_3 are positive constants dependent on σ_1, σ_2 and ρ. But since $\sup_k (1/\sqrt{k})\mathrm{E}^{1/r}|S_{ik}-k\mu_i|^r<\infty$, $i=1, 2$, the desired conclusion follows by taking without loss of generality $k\leqq(n/2)$.

Another consequence of Inequality 2.1 is a multivariate extension of the Theorem of Rényi [5]. Note also that the proof follows that of Landers and Rogge [4], Corollary 3.

THEOREM 2.2. *Let $(X_{11}, X_{21}),\cdots,(X_{1n}, X_{2n})$ be defined on the probability space $(\Omega, \mathcal{A}, P)$ and let $B\in\mathcal{A}$ such that $\mathrm{P}(B)>0$. Then*

$$\mathrm{P}[S_{1n}-n\mu_1\leqq\sigma_1x_1\sqrt{n}, S_{2n}-n\mu_2\leqq\sigma_2x_2\sqrt{n}\,|B]\to\Phi(x_1, x_2), \quad \text{as } n\to\infty. \tag{2.11}$$

PROOF. Let $\mathcal{F}_\infty=\sigma((X_{11}, X_{21}),\cdots,(X_{1n}, X_{2n}),\cdots)$. Then there exists $\mathcal{F}_n$-measurable functions ζ_n, $0\leqq\zeta_n\leqq 1$ and $\mathrm{E}|\mathrm{P}(B|\mathcal{F}_\infty)-\zeta_n|\to 0$ as $n\to\infty$. Thus

$$\begin{aligned}&|\mathrm{P}[S_{1n}-n\mu_1\leqq\sigma_1x_1\sqrt{n}, S_{2n}-n\mu_2\leqq\sigma_2x_2\sqrt{n}, B]-\Phi(x_1, x_2)\mathrm{P}(B)|\\ &\quad\leqq|\mathrm{P}[S_{1n}-n\mu_1\leqq\sigma_1x_1\sqrt{n}, S_{2n}-n\mu_2\leqq\sigma_2x_2\sqrt{n}, \mathrm{P}(B|\mathcal{F}_\infty)]\\ &\qquad-\mathrm{P}[S_{1n}-n\mu_1\leqq\sigma_1x_1\sqrt{n}, S_{2n}-n\mu_2\leqq\sigma_2x_2\sqrt{n}, \zeta_k]|\\ &\qquad+|\mathrm{P}[S_{1n}-n\mu_1\leqq\sigma_1x_1\sqrt{n}, S_{2n}-n\mu_2\leqq\sigma_2x_2\sqrt{n}, \zeta_k]\\ &\qquad\quad-\Phi(x_1, x_2)\mathrm{E}\,\zeta_k|+\Phi(x_1, x_2)|\mathrm{E}\,\zeta_k-\mathrm{P}(B)|\end{aligned}$$

$$\leqq \frac{\varepsilon}{2} + \mathrm{E}\,\{\zeta_k | \mathrm{P}\,[S_{1n} - n\mu_1 \leqq \sigma_1 x_1 \sqrt{n}\,,\, S_{2n} - n\mu_2 \leqq \sigma_2 x_2 \sqrt{n}\,|\,\mathcal{F}_k]$$
$$-\Phi(x_1, x_2)|\}$$
$$\leqq \frac{\varepsilon}{2} \| |\mathrm{P}\,[S_{1n} - n\mu_1 \leqq \sigma_1 x_1 \sqrt{n}\,,\, S_{2n} - n\mu_2 \leqq \sigma_2 x_2 \sqrt{n}\,|\,\mathcal{F}_k]$$
$$-\Phi(x_1, x_2)| \|_2$$
$$\leqq \frac{\varepsilon}{2} + \Delta_{n-k} + C_1\Big(\frac{k}{n-k}\Big)^{1/2} + C_2\Big(\frac{k}{n-k}\Big) \leqq \varepsilon\,,$$

for sufficiently large n, using the multidimensional Central Limit Theorem and Inequality 2.1 with Minkowski's inequality.

Remark 2.1. We remark here that if $\mathrm{E}|X_{i1}|^{2+\delta} < \infty$, $i=1, 2$, $0 < \delta \leqq 1$, then $\Delta_n = O(n^{-\delta/2})$, $0 < \delta \leqq 1$. This result for $\delta = 1$ is given by Sazonov [6] and for $0 < \delta < 1$ is due to Bikyalis [3]. Thus Theorem 2.1 is valid if $\mathrm{E}|X_{i1}|^{2+\delta} < \infty$, $i=1, 2$. The extension to any $p > 2$ is immediate.

3. Conditioned CLT for maximum sums

In this section we assume that $\mathrm{E}\,X_{i1} = \mu_i > 0$, $i=1, 2$. Define $\Delta_n^*(x_1, x_2)$, $\Delta_n^*(x_1, x_2 | B)$, and $\Delta_n^*(x_1, x_2 | \mathcal{F}_k)$ as in (2.1)–(2.3) with S_{in} replaced by $S_{in}^* = \max_{1 \leqq j \leqq n} S_{ij}$, $i=1, 2$. Note that

$$S_{in}^* = \max\,\{S_{ik}^*,\, S_{ik} + S_{i,(n-k)}^*\}\,, \qquad i=1, 2\,, \tag{3.1}$$

where $S_{i,(n-k)}^* = \max_{k+1 \leqq j \leqq n} S_{i,(j-k)}$ with $S_{i,(n-k)} = \sum_{j=k+1}^{n} X_{ij}$, $i=1, 2$. Further, let $y_{in} = x_i \sigma_i \sqrt{n} + n\mu_i$, $i=1, 2$. Thus

$$\begin{aligned} (3.2)\quad & |\mathrm{P}\,[S_{1n}^* \leqq y_{1n},\, S_{2n}^* \leqq y_{2n} | \mathcal{F}_k] - \Phi(x_1, x_2)| \\ &= |\mathrm{P}\,[S_{1,(n-k)}^* \leqq y_{1n} - S_{1k},\, S_{2,(n-k)}^* \leqq y_{2n} - S_{2k},\, S_{1k}^* \leqq y_{1n},\, S_{2k}^* \leqq y_{2n} | \mathcal{F}_k] \\ &\quad - \Phi(x_1, x_2)| \\ &\leqq |\mathrm{P}\,[S_{1,(n-k)}^* \leqq y_{1n} - S_{1k},\, S_{2,(n-k)}^* \leqq y_{2n} - S_{2k} | \mathcal{F}_k] - \Phi(x_1, x_2)|\, I_{A_{n,k}} \\ &\quad + (1 - I_{A_{n,k}})\Phi(x_1, x_2) = I_{1n} + I_{2n}\,, \qquad \text{say}\,, \end{aligned}$$

where $A_{n,k} = \{S_{1k}^* \leqq y_{1n},\, S_{2k}^* \leqq y_{2n}\}$ and I_A denotes the indicator function of a set A. But since $(X_{11}, X_{21}), \cdots, (X_{1n}, {}_{2n})$ are iid then it is easy to see that

$$\begin{aligned} (3.3)\quad I_{1n} &\leqq \Delta_{n-k}^* + \left|\Phi\Big(\frac{y_{1n} - S_{1k} - (n-k)\mu_1}{\sigma_1 \sqrt{n-k}},\, \frac{y_{2n} - S_{2k} - (n-k)\mu_2}{\sigma_2 \sqrt{n-k}}\Big) - \Phi(x_1, x_2)\right| \\ &= \Delta_{n-k}^* + \left|\Phi\Big(x_1 \sqrt{\frac{n}{n-k}} - \frac{S_{1k} - k\mu_1}{\sigma_1 \sqrt{n-k}},\, x_2 \sqrt{\frac{n}{n-k}} - \frac{S_{2k} - k\mu_2}{\sigma_2 \sqrt{n-k}}\Big)\right. \end{aligned}$$

$$-\Phi(x_1, x_2)\Big|$$
$$\leqq \Delta^*_{n-k} + C_1\frac{|S_{1k}-k\mu_1|}{\sqrt{n-k}} + C_2\frac{|S_{2k}-k\mu_2|}{\sqrt{n-k}} + C_3\left(\frac{k}{n-k}\right),$$

as proved in Inequality 2.1. Hence we arrive at the following bivariate extension of Theorem 3.1 of Ahmad [2].

INEQUALITY 3.1. For any $n>k$ and all (x_1, x_2)

$$\Delta^*_n(x_1, x_2|\mathcal{F}_k) \leqq \Delta^*_{n-k} + C_1\frac{|S_{1k}-k\mu_1|}{\sqrt{n-k}} + C_2\frac{|S_{2k}-k\mu_2|}{\sqrt{n-k}} + C_3\left(\frac{k}{n-k}\right) + \mathrm{P}\,(A^c_{n,k}|\mathcal{F}_k)\,, \tag{3.4}$$

where C_1, C_2 and C_3 are positive constants dependent on σ_1, σ_2 and ρ but are independent of n, A^c denotes the complement of a set A and $A_{n,k}=\{S^*_{1k}\leqq y_{1n}, S^*_{2k}\leqq y_{2n}\}$ where $y_{in}=\sigma_i x_i\sqrt{n}+n\mu_i$, $i=1, 2$.

Again we shall use Inequality 3.1 to obtain a rate of convergence in the conditioned CLT for maximum partial sums. This is done in the following theorem.

THEOREM 3.1. *If* $\Delta^*_n=O(n^{-\delta/2})$, $0<\delta\leqq 1$, *and if for any* r, $2\leqq r\leqq 2+\delta$, $\mathrm{E}|X_{i1}|^r<\infty$, $i=1, 2$, *then there exists a positive constant* C_r *which depends on* σ_1, σ_2 *and* ρ *such that for all* $B\in\mathcal{F}_k$ *with* $\mathrm{P}\,(B)>0$,

$$\Delta^*_n(B)\leqq C_r(\mathrm{P}\,(B))^{-1/r}\left(\frac{k}{n}\right)^{\delta/2}, \qquad 0<\delta\leqq 1\,. \tag{3.5}$$

PROOF. Note that by using conditional expectation and Hölder's inequality we have for any (x_1, x_2) and $n>k$,

$$\begin{aligned}&|\mathrm{P}\,[S^*_{1n}\leqq\sigma_1x_1\sqrt{n}+n\mu_1, S^*_{2n}\leqq\sigma_2x_2\sqrt{n}+n\mu_2, B]-\Phi(x_1, x_2)\,\mathrm{P}\,(B)|\\ &\leqq \mathrm{E}\,\{|\mathrm{P}\,[S^*_{1n}\leqq y_{1n}, S^*_{2n}\leqq y_{2n}|\mathcal{F}_k]-\Phi(x_1, x_2)|\,\mathrm{P}\,(B)\}\\ &\leqq [\mathrm{P}\,(B)]^{(r-1)/r}\|\Delta^*(x_1, x_2|\mathcal{F}_k)\|_r\,.\end{aligned} \tag{3.6}$$

Now, applying Inequality 3.1 to $\Delta^*(x_1, x_2|\mathcal{F}_k)$ and using Minkowski's inequality we see that the last upper bound of (3.6) is less than or equal to

$$\begin{aligned}&[\mathrm{P}\,(B)]^{(r-1)/r}\Big\{\Delta^*_{n-k}+\frac{C_1}{\sqrt{n-k}}\,\mathrm{E}^{1/r}|S_{1k}-k\mu_1|^r+\frac{C_2}{\sqrt{n-k}}\,\mathrm{E}^{1/r}|S_{2k}-k\mu_2|\\ &\quad+C_3\left(\frac{k}{n-k}\right)+\mathrm{E}^{1/r}\,[\mathrm{P}\,(A^c_{n,k}|\mathcal{F}_k)]\Phi(x_1, x_2)\Big\}\\ &\leqq [\mathrm{P}\,(B)]^{(r-1)/r}\Big\{O((n-k)^{-\delta/2})+C_4\left(\frac{k}{n-k}\right)^{1/2}+C_5\left(\frac{k}{n-k}\right)\end{aligned}$$

$$+\mathrm{P}(A^c_{n,k})\Phi(x_1, x_2)\Big\}$$

$$\leqq [\mathrm{P}(B)]^{(r-1)/r}\Big\{O((n-k)^{-\delta/2})+C_6\Big(\frac{k}{n-k}\Big)^{1/2}$$

$$+\sum_{i=1}^{2}\{\mathrm{P}[S^*_{ik}\geqq y_{in}]\}^{1/(2+\delta)}\Phi_{(1)}(x_i)\Big\} ;$$

since $\mathrm{E}^{1/r}[\mathrm{P}(A^c_{n,k}|\mathcal{F}_k)]\leqq \mathrm{E}[\mathrm{P}(A^c_{n,k}|\mathcal{F}_k)]=\mathrm{P}(A^c_{n,k})$. Let us examine the last terms in the above upper bound. Let

$$J_{in}=\{\mathrm{P}[S^*_{ik}\geqq y_{in}]\}^{1/(2+\delta)}\Phi_{(1)}(x_i) , \qquad i=1, 2 .$$

We distinguish between three cases

(i) If $x_i\geqq 0$, then

$$J_{in}=\{\mathrm{P}[S^*_{ik}\geqq x_i\sigma_i\sqrt{n}+n\mu_i]\}^{1/(2+\delta)}\Phi_{(1)}(x_i)\leqq \{\mathrm{P}[S^*_{ik}\geqq n\mu_i]\}^{1/(2+\delta)}\Phi_{(1)}(x_i)$$
$$\leqq C_1\Big(\frac{k^{1+\delta/2}}{n^{2+\delta}}\Big)^{1/(2+\delta)}\leqq C_2\Big(\frac{k}{n}\Big) , \qquad i=1, 2 .$$

(ii) If $-\mu_i\sqrt{n}/(2\sigma_i)\leqq x_i\leqq 0$, then

$$J_{in}\leqq \Big\{\mathrm{P}\Big[S^*_{ik}\geqq \frac{n\mu_i}{2}\Big]\Big\}^{1/(2+\delta)}\Phi_{(1)}(x_i)\leqq C_3\Big(\frac{k}{n}\Big) ,$$

as in case (i), $i=1, 2$.

(iii) If $x_i\leqq -\mu_i\sqrt{n}/(2\sigma_i)$, then

$$J_{in}\leqq \Phi_{(1)}(x_i)\leqq \Phi_{(1)}\Big(-\frac{\mu_i\sqrt{n}}{2\sigma_i}\Big)=O(n^{-\delta/2}) , \qquad i=1, 2 .$$

Hence in all cases we have that $J_{in}=O((k/n)^{\delta/2})$. The desired conclusion follows.

Next, we give a conditioned central limit theorem of maximum partial sums. The proof of the next theorem is modeled after that of Theorem 2.2 and hence is omitted.

THEOREM 3.2. *Under the conditions of Theorem* 2.2,

$$(3.7) \quad \mathrm{P}[S^*_{1n}\leqq \sigma_1 x_1\sqrt{n}+\mu_1 n,\ S^*_{2n}\leqq \sigma_2 x_2\sqrt{n}+\mu_2 n\,|\,B]\to\Phi(x_1, x_2) \quad \textit{as } n\to\infty .$$

Remark 3.1. Ahmad [1], Theorem 1, has shown that if $\Delta_n(n^{-\delta/2})$, $0<\delta\leqq 1$, then $\Delta^*_n=O(n^{-\delta/2})$, this, in conjunction with Remark 2.1, leads to showing that if $\mathrm{E}|X_{i1}|^{2+\delta}<\infty$, $i=1, 2$, $0<\delta\leqq 1$, then Theorem 3.1 holds.

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