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# ON THE STABILITY OF CHARACTERIZATIONS OF THE EXPONENTIAL DISTRIBUTION*

RYOICHI SHIMIZU

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## Summary

It is shown that if the distribution of $\min\{X_1/a_1, X_2/a_2, \cdots, X_N/a_N\}$ is close to that of $X_1$, then the distribution is close to the exponential distribution.

## 1. Introduction

In a previous paper Shimizu [2] proved that the minimum, $Z$, of the independent variables $X_1/a_1, X_2/a_2, \cdots, X_N/a_N$, where the $X$'s are identically distributed positive random variables and the $a$'s are positive constants such that $a_1+a_2+\cdots+a_N=1$ and such that $\log a_j/\log a_k$ is an irrational number for some $j$ and $k$, has the same distribution as $X_1$ if and only if the distribution is exponential. In fact the theorem was stated in a more general form and the result was substantially extended by Shimizu-Davies [4] in which it was proved, in particular, that the above mentioned characterization theorem holds true if the $a$'s are random variables independent of the $X$'s.

The purpose of the present article is to prove a stability theorem for these characterizations of the exponential distribution. In the next section we shall show that if the ratio of $\Pr\{Z>x\}$ to $\Pr\{X_1>x\}$ is close to 1 then $\Pr\{X_1>x\}$ is close to $e^{-\lambda x}$. An extension of this is given in Section 3.

## 2. A stability theorem

Let $m_1, m_2, \cdots, m_N$ $(N\geqq 2)$ be positive integers and let $X_{j,k}$, $j=1, 2, \cdots, m_k$; $k=1, 2, \cdots, N$ be i.i.d. positive random variables with common distribution $F$. Let $a_1, a_2, \cdots, a_N$ be positive random variables in-

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dependent of the $X$'s and let $(\Theta, \mathcal{A}, P)$ be the probability space on which the $a$'s are defined. Thus, $a_k$ is an $\mathcal{A}$-measurable real function on $\Theta$: $a_k=a_k(\theta)$, $k=1, 2, \cdots, N$. We assume that the following conditions are satisfied.

$$(1) \qquad a\equiv\max_k \text{ess.}\sup_\theta a_k(\theta)<1$$

$$(2) \qquad \Pr\{a_1(\theta)+a_2(\theta)+\cdots+a_N(\theta)=1\}=1 \quad \text{and}$$

$$(3) \qquad \Pr\{A_j(\theta)/A_k(\theta) \text{ is irrational for some } j \text{ and } k\}>0$$

where $A_k(\theta)\equiv-\log a_k(\theta)/m_k$, $k=1, 2, \cdots, N$. For $k=1, 2, \cdots, N$, put $Z_k \equiv \min_{1\leqq j\leqq m_k} \{X_{j,k}\}$. We compare the distribution of the random variable

$$Z\equiv \min_{1\leqq k\leqq N} \{m_k Z_k/a_k(\theta)\}$$

with the underlying distribution $F$. For $x^0>0$, let $R(x)$ be the real function defined by

$$(4) \qquad \Pr\{Z>x\}=(1-R(x))\Pr\{X_{1,1}>x\}\,, \qquad 0\leqq x<x^0\,.$$

If $x^0=\infty$ and $R(x)\equiv0$, then the condition (4) means that $Z$ has the same distribution as $X_{1,1}$, which is possible if and only if $F$ is exponential: $F(x)=1-e^{-\lambda x}$, $x>0$. For the proof of this characterization theorem for the exponential distribution the reader is referred to Shimizu [2] and Shimizu-Davies [4]. We shall prove the following stability theorem.

THEOREM 1. *Let* $\varepsilon$ *and* $\delta$ *be non-negative numbers less than* 1/2 *and let* $R(x)$ *be such that*

$$R(x)=\begin{cases} \varepsilon\zeta x^2\,, & \text{if } x\leqq1 \\ \delta\zeta\,, & \text{if } x>1 \end{cases}$$

*where and in what follows* $\zeta$ *denotes a quantity bounded in absolute value by* 1. *If* (4) *is satisfied for some* $x^0>0$, *then* $\lambda=\lim_{x\downarrow0} F(x)/x$ *exists finitely and we have*

$$\bar F(x)\equiv\Pr\{X_{1,1}>x\} =\begin{cases} e^{-\lambda x}+\varepsilon_1\zeta x^2\,, & \text{if } 0<x\leqq1\,, \\ e^{-\lambda x}+(\varepsilon_1+\delta_1)\zeta\cdot x\cdot\exp\{-(\lambda-(\varepsilon_1+\delta_1))x\}\,, & \text{if } 1<x<x^0\,, \end{cases}$$

*where* $\varepsilon_1=2\varepsilon(1-a)^{-1}$ *and* $\delta_1=2\delta(1-a)^{-1}$.

It is easy to see that if $X$'s are i.i.d. random variables such that $\lim_{x\downarrow0}\Pr\{X_j\leqq x\}/x$ exists and positive, then the distribution of $n\cdot\min_{1\leqq j\leqq n}\{X_j\}$

satisfies the condition of Theorem 1 provided $n$ is sufficiently large.

PROOF. We assume that $x^0>1$. We can rewrite the condition (4) in the form

$$(5)\qquad (1-R(x))\bar{F}(x)=\int_\Theta \prod_{k=1}^{N}(\bar{F}(a_k(\theta)x/m_k))^{m_k}dP(\theta)\,, \qquad 0<x<x^0\,.$$

As $1-R(x)$ is bounded away from 0, (5) implies that $\bar{F}(x)$ is positive for all $0<x<x^0$, and the function

$$g(x)=-(\log \bar{F}(x))/x$$

is defined for $0<x<x^0$. By Hölder's and Jensen's inequalities we have for sufficiently small $x$

$$\begin{aligned}\text{R.H.S. of (5)}&=\int_\Theta \exp\left\{-x\sum_{k=1}^{N}a_k(\theta)g(a_k(\theta)x/m_k)\right\}dP(\theta)\\ &\leqq\left(\int_\Theta \exp\left\{-\sum_{k=1}^{N}a_k(\theta)g(a_k(\theta)x/m_k)\right\}dP(\theta)\right)^x\\ &\leqq\left(\int_\Theta \sum_{k=1}^{N}a_k(\theta)\exp\{-g(a_k(\theta)x/m_k)\}dP(\theta)\right)^x.\end{aligned}$$

Also we have for sufficiently small $x$

$$\text{L.H.S. of (5)}\geqq(1-\varepsilon x^2)e^{-xg(x)}\geqq((1-\varepsilon x)e^{-g(x)})^x\,.$$

It follows that

$$(6)\qquad (1-\varepsilon x)e^{-g(x)}\leqq\int_\Theta \sum_{k=1}^{N}a_k(\theta)\exp\{-g(a_k(\theta)x/m_k)\}dP(\theta)\,,$$

which holds for sufficiently small $x>0$. For $x>x_0\equiv-\log x^0$, put

$$H(x)=g(e^{-x}) \quad\text{and}\quad H_0(x)=e^{-H(x)}(1-\varepsilon(1-a)^{-1}e^{-x})\,,$$

and let $G(x)$ be the distribution function on the half interval $[0,\infty)$ defined by

$$G(x)=\int_\Theta \sum_{k=1}^{N}a_k(\theta)\mathcal{E}(x-A_k(\theta))dP(\theta)\,,$$

where $\mathcal{E}(x)$ denotes the distribution function degenerate at $x=0$. The inequality (6) implies

$$(7)\qquad H_0(x)\leqq\int_0^\infty H_0(x+y)dG(y)\,, \qquad \text{for sufficiently large } x\,.$$

By Lemma 2 of Shimizu-Davies [5] we can find a sequence $\{x_n\}_{n=1}^{\infty}$ of positive numbers and positive constants $K_1$, $K_2$, and $K_3$ such that

$$(8)\qquad K_1\leqq x_n-x_{n-1}\leqq K_2$$

and

$$H_0(x_n)\geqq K_3\ . \tag{9}$$

Note that this is a simple consequence of (7) when $\operatorname{ess.inf}_\theta a_k(\theta)$ is positive for all $k$. In fact in this case the distribution $G$ is concentrated on the finite interval $[A, B]$, where $0<A\equiv -\log a\leqq \min_k \operatorname{ess.inf}_\theta A_k(\theta)< \max_k \operatorname{ess.sup}_\theta A_k(\theta)\equiv B<\infty$, and the inequality (7) implies that for any $x$ there exists an $x'$ such that $A\leqq x'\leqq B$ and $H_0(x)\leqq H_0(x+x')$.

The inequality (9) is tantamount to saying that $H(x_n)$, $n=1, 2,\cdots$, are bounded from above by a constant $C$, say. Let $x$ be any real number between $x_n$ and $x_{n+1}$. Then,

$$H(x)=-e^x\log \bar{F}(e^{-x})\leqq e^{x-x_n}(-e^{x_n}\log \bar{F}(e^{-x_n}))\leqq e^{K_2}H(x_n)\leqq Ce^{K_2}\ .$$

As $H(x)$ is bounded in any finite interval we conclude that it is bounded for all $x\geqq 0$. Now, we can rewrite the equation (5) to obtain

$$(1-R(e^{-x}))e^{-e^{-x}H(x)}=\int_\Theta \exp\left\{-e^{-x}\sum_{k=1}^N a_k(\theta)H(x+A_k(\theta))\right\}dP(\theta)\ , \tag{10}$$

or

$$H(x)=-e^x\log\int_\Theta \exp\left\{-e^{-x}\sum_{k=1}^N a_k(\theta)H(x+A_k(\theta))\right\}dP(\theta)+S(x)\ , \tag{11}$$

which hold for all $x>x_0$ and where

$$S(x)\equiv e^x\log(1-R(e^{-x}))=\begin{cases} 2\varepsilon\zeta e^{-x}\ , & \text{if } x\geqq 0\ ,\\ 2\delta\zeta e^{x}\ , & \text{if } x_0<x<0\ .\end{cases} \tag{12}$$

For $x\geqq 0$ we have, as $H$ is bounded,

$$\begin{aligned}\text{R.H.S. of (10)}&=\int_\Theta\left(1-e^{-x}\sum_{k=1}^{N'} a_kH(x+A_k)+Be^{-2x}\right)dP\\ &=1-e^{-x}\int_0^\infty H(x+y)dG(y)+Be^{-2x}\ , \qquad \text{and}\end{aligned}$$

$$\begin{aligned}\text{L.H.S. of (10)}&=(1-Be^{-x})(1-e^{-x}H(x)+Be^{-2x})\\ &=1-e^{-x}H(x)+Be^{-2x}\ ,\end{aligned}$$

where and in what follows in this and in the next section the symbol $B$ denotes a quantity which is bounded by a constant independent of $x$ and $\theta$. It follows that $H$ satisfies the functional equation with an error term

$$H(x)=\int_0^\infty H(x+y)dG(y)+S_0(x)e^{-x}\ , \qquad x\geqq 0\ , \tag{13}$$

where $S_0(x)$ is a bounded function. By Theorem 3 of Shimizu [3], there exist positive constants $\lambda$ and $\gamma$ such that

$$H(x)=\lambda+\gamma\zeta e^{-x}\,, \qquad x\geqq 0\,. \tag{14}$$

In particular we have

$$\lambda=\lim_{x\to\infty} H(x)=-\lim_{x\to\infty} e^x \log \bar{F}(e^{-x})=-\lim_{x\downarrow 0} x^{-1}\log \bar{F}(x)=\lim_{x\downarrow 0} F(x)/x\,.$$

We can take $\gamma=\varepsilon_1(=2\varepsilon(1-a)^{-1})$ or

$$H(x)=\lambda+\varepsilon_1\zeta e^{-x}\,, \qquad x\geqq 0\,. \tag{15}$$

In fact we can prove, by mathematical induction, that

$$H(x)=\lambda+\gamma_j\zeta e^{-x}\,, \qquad j=0, 1, \cdots, \text{ and } x\geqq 0\,, \tag{16}$$

where

$$\gamma_j=\begin{cases} \gamma\zeta\,, & \text{if } j=0\,,\\ \gamma a^j\zeta+2\varepsilon(1+a+\cdots+a^{j-1})\zeta\,, & \text{if } j\geqq 1\,. \end{cases}$$

The equality (16) reduces to (14) if $j=0$. Therefore assume that (16) is true for some $j$. If $x\geqq 0$, then $x+A_k(\theta)\geqq 0$ for a.e. $\theta$ and $k=1, 2, \cdots, N$, and

$$H(x+A_k)=\lambda+\gamma_j\zeta e^{-(x+A_k)}=\lambda+\gamma_j a\zeta e^{-x}\,, \qquad k=1, 2, \cdots\,.$$

Substituting these in (11) we obtain

$$\begin{aligned} H(x)&=-e^x \log \int_\Theta \exp\{-e^{-x}(\lambda+\gamma_j a\zeta e^{-x})\}\,dP+2\varepsilon\zeta e^{-x}\\ &=\lambda+\gamma_j a\zeta e^{-x}+2\varepsilon\zeta e^{-x}=\lambda+\gamma_{j+1}\zeta e^{-x} \end{aligned}$$

as was to be proved. As $j$ is arbitrary (15) follows from (16).

Now, let us derive a similar expression for negative $x$. We shall prove, by mathematical induction, that

$$H(x)=\lambda+\varepsilon_1\zeta+\lambda_j\zeta\,, \qquad \text{if } x\geqq -jA,\ j=0, 1, 2, \cdots\,, \tag{17}$$

where $A=-\log a\leqq A_k(\theta)$, and

$$\lambda_j=\begin{cases} 0\,, & \text{if } j=0\,,\\ 2\delta(1+a+\cdots+a^{j-1})\,, & \text{if } j\geqq 1\,. \end{cases}$$

The assertion is clear when $j=0$. Suppose (17) is true for some $j$. Then $x\geqq -jA$ implies $H(x)=\lambda+\varepsilon_1\zeta+\lambda_j\zeta=\lambda+\varepsilon_1\zeta+\lambda_{j+1}\zeta$. We assume therefore $-jA\geqq x\geqq -(j+1)A$. Then we have $x+A_k(\theta)\geqq -jA$ for a.e. $\theta$ and $k=1, 2, \cdots, N$, and the induction hypothesis yields

$$H(x+A_k)=\lambda+\varepsilon_1\zeta+\lambda_j\zeta\,, \qquad \text{a.e. } \theta \text{ and } k=1, 2, \cdots, N.$$

Substituting these in (11) we obtain

$$\begin{aligned} H(x) &= -e^{-x}\log\int_\Theta \exp\{-e^{-x}(\lambda+\varepsilon_1\zeta+\lambda_j\zeta)\}\,dP+2\delta\zeta e^x \\ &= \lambda+\varepsilon_1\zeta+\lambda_j\zeta+2\delta a^j\zeta=\lambda+\varepsilon_1\zeta+\lambda_{j+1}\zeta\,, \end{aligned}$$

as was to be proved. As $j$ is arbitrary we obtain from (17)

$$H(x)=\lambda+(\varepsilon_1+\delta_1)\zeta\,, \qquad \text{if } x_0<x\leqq 0\,. \tag{18}$$

The desired result follows from (15) and (18). q.e.d.

## 3. Generalization

The result of the preceding section can be further generalized. We can prove the stability of the general characterization theorem of the exponential distribution as given by Shimizu-Davies [4], of which Theorem 1 is a special case.

Let $(\Theta, \mathcal{A}, P)$ be a probability space and let $\{V(\cdot\,;\theta)\,|\,\theta\in\Theta\}$ be a family of distribution functions on the open interval $(0, a)$, where $0<a<1$, with $\Theta$ as its parameter space. A family of distributions on the half interval $[-\log a, \infty)$ is defined by

$$U(x;\theta)=1-V(e^{-x-0};\theta)\,, \qquad \theta\in\Theta\,.$$

Let $G$ be the mixture of the $U$'s with respect to $P$:

$$G(x)=\int_\Theta U(x;\theta)\,dP(\theta)\,.$$

THEOREM 2. *Suppose the distribution $G$ has a finite mean and is not concentrated on a lattice $\rho, 2\rho, \cdots$ for any $\rho>0$. Then the conclusion of Theorem 1 is valid if $F$ satisfies the functional equation*

$$(1-R(x))\bar{F}(x)=\int_\Theta \exp\left\{\int_0^a u^{-1}\log\bar{F}(ux)\,V(du;\theta)\right\}dP(\theta)\,, \qquad 0\leqq x<x^0\,. \tag{19}$$

PROOF. Theorem 2 can be proved as in Theorem 1 without any essential change. In fact if we define the functions $g$, $H_0$, and $H$ as in the proof of Theorem 1, then we have the following inequality, instead of (6):

$$(1-\varepsilon x)e^{-g(x)}\leqq\int_\Theta dP(\theta)\int_0^a e^{-g(yx)}\,V(dy;\theta)=\int_0^\infty e^{-g(yx)}\,dG(y)\,.$$

The boundedness of $H$ follows from this. Corresponding to the equalities (10) and (11) we have

$$(1-R(e^{-x}))e^{-e^{-x}H(x)}=\int_{\Theta}\exp\left\{-e^{-x}\int_0^{\infty}H(x+y)U(dy;\theta)\right\}dP(\theta)\ , \tag{20}$$

and

$$H(x)=-e^{x}\log\int_{\Theta}\exp\left\{-e^{-x}\int_0^{\infty}H(x+y)U(dy;\theta)\right\}dP(\theta)+S(x)\ , \tag{21}$$

respectively. The equation (20) yields (13), which in turn implies (14). On substituting (14) in the left-hand side of (21) and remembering the fact that the $U$'s are concentrated on the interval $[-\log a, \infty)$, we obtain the desired result. q.e.d.

Note that the equality (19) reduces to (11) if we take

$$V(x;\theta)=\sum_{k=1}^{N}a_k(\theta)\mathcal{E}(x-a_k(\theta)/m_k)\ .$$

Another specialization of Theorem 2 gives the stability of the characterization theorem established by Davies [1].

COROLLARY. *Let $X_1, X_2, \cdots$ be a sequence of i.i.d. positive random variables with common distribution $F$, and let $N$ be an integer valued random variable independent of the $X$'s. We assume that $N\geqq 2$ with probability one and that the distribution of* $\log N$ *has a finite mean and is non-lattice. Then the conclusion of Theorem* 1 *with $a=1/2$ and $\bar{F}(x)=\Pr\{X_1>x\}$ follows from the condition*

$$(1-R(x))\Pr\{X_1>x\}=\Pr\{N\cdot\min\{X_1, X_2,\cdots, X_N\}>x\}\ ,\qquad 0<x<x^0\ .$$

PROOF. The condition can be written as

$$(1-R(x))\bar{F}(x)=\sum_{k=2}^{\infty}\Pr\{N=n\}\bar{F}^n(x/n)\ .$$

This is a special case of (19) in which $\Theta=\{2,3,\cdots\}$, $P$ is the probability measure corresponding to the random variable $N$ and $V(x;n)$ is the probability measure degenerate at $x=1/n$. q.e.d.

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