Ann. Inst. Statist. Math.
33 (1981), Part A, 297-313

ON THE CONSTRUCTION OF A CLASS OF INVARIANT POLYNOMIALS IN SEVERAL MATRICES, EXTENDING THE ZONAL POLYNOMIALS

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(Received July 14, 1980; revised Jan. 26, 1981)

Summary

The construction of a class of invariant polynomials in several matrices extending the zonal polynomials is discussed. The method adopted generalizes the original group-theoretic approach of James [9]. A table of three-matrix polynomials up to degree 5 is presented.

1. Introduction

Following the work of James [9], [11] and Constantine [5], many multivariate distributions have been represented in terms of the zonal polynomials $C_\kappa(X)$, where X is an $m\times m$ symmetric matrix, and κ is an ordered partition of a nonnegative integer k into not more than m parts. These polynomials arise from the group representation theory of the real linear group $Gl(m, R)$ of nonsingular $m\times m$ matrices. An extension of the zonal polynomials to invariant polynomials $C_\phi^{\kappa[r]}(X_{[r]})$ in r matrices, $X_{[r]}=(X_1,\cdots,X_r)$ has been given for $r=2$ by Davis [7] and for $r\geqq 3$ by Chikuse [4]. Here $\kappa[r]=(\kappa(1),\cdots,\kappa(r))$, where the $\kappa(i)$ are ordered partitions of nonnegative integers $k(i)$ into $\leqq m$ parts ($i=1,\cdots,r$) and ϕ is an ordered partition of $f=\sum_{i=1}^r k(i)$. The basic property of the polynomials is the following,

$$(1.1)\quad \int_{O(m)}\prod_{i=1}^r C_{\kappa(i)}(A_iH'X_iH)dH=\sum_{\phi\in\kappa(1)\cdot\kappa(2)\cdot\cdots\cdot\kappa(r)} C_\phi^{\kappa[r]}(A_{[r]})C_\phi^{\kappa[r]}(X_{[r]})/C_\phi(I_m)\ ,$$

where the A_i are $m\times m$ matrices, dH is the invariant Haar measure over the group $O(m)$ of $m\times m$ orthogonal matrices H, and I_m denotes the $m\times m$ unit matrix. The summation on the right-hand side of (1.1) extends over those partitions ϕ for which the irreducible representa-

AMS 1970 subject classifications: 62E15, 62H10.
Key words and phrases: Invariant polynomials, zonal polynomials, group representations, multivariate distributions, Young tableaux.

tions of $Gl(m, R)$ indexed by $[2\phi]$ occur in the decomposition of the Kronecker product $\bigotimes_{i=1}^{r}[2\kappa(i)]$. The $C_{\phi}^{\kappa[r]}(X_{[r]})$ are invariant under the *simultaneous* transformations

$$X_i \rightarrow H'X_iH\,, \qquad H \in O(m),\ i=1,\cdots,r\,. \tag{1.2}$$

The basic existence theory for the polynomials, together with some of their properties, is summarized in Section 2. Applications to multivariate distribution theory have been presented by Davis [6], Chikuse [2], [3], Phillips [13], and Richards and Gupta [14]. The problems of construction and convergence associated with expansions in these polynomials are, of course, considerably more serious even than in the case of the zonal polynomials (see discussion in Muirhead [12]). However, an application of the lower degree polynomials in two matrices has been given by Davis [8], relating to the effects of moderate multivariate nonnormality on the MANOVA tests.

The present paper is concerned with the construction of the $C_{\phi}^{\kappa[r]}$, using an extension of the original group-theoretic approach of James [9], [10] for the zonal polynomials. A tabulation of invariant polynomials up to degree 5 in the case $r=2$ has been given in Davis [6], and the corresponding polynomials for $r=3$ are presented here in Table 1.

2. Summary of existence theory

We first summarize some results concerning the existence of the invariant polynomials, following Davis [7] and Chikuse [4]. Let $X_1, \cdots, X_r$ denote $m\times m$ complex symmetric matrices, and let $P_{k[r]}(X_{[r]})$ denote the class of homogeneous polynomials of degree $k(1),\cdots,k(r)$ in the elements of $X_1,\cdots,X_r$ respectively,

$$P_{k[r]}(X_{[r]}) = \bigotimes_{i=1}^{r} P_{k(i)}(X_i)\,, \tag{2.1}$$

the Kronecker product of the classes of polynomials of degree $k(i)$ in the elements of X_i $(i=1,\cdots,r)$. The simultaneous congruence transformations by matrices $L \in Gl(m, R)$

$$X_i \rightarrow LX_iL'\,, \qquad i=1,\cdots,r \tag{2.2}$$

produce linear transformations in $P_{k[r]}$ regarded as a vector space over the basis consisting of all monomials of degree k_i in X_i $(i=1,\cdots,r)$. Their linear transformations constitute a *representation* of $Gl(m, R)$ in $P_{k[r]}$ which is the Kronecker product of the representations induced by (2.2) in the $P_{k(i)}(X_i)$ $(i=1,\cdots,r)$. This leads to a decomposition of $P_{k[r]}$

into a direct sum of subspaces which are invariant and irreducible under this representation,

$$P_{k[r]}(X_{[r]})=\bigoplus_{\kappa[r]}\bigoplus_{\Phi}\mathcal{V}_{\Phi}^{\kappa[r]}(X_{[r]}) \tag{2.3}$$

where the $\kappa[r]$ are all partitions of $k_1,\cdots,k_r$ into $\leqq m$ parts, and Φ runs through all partitions of $2f$ into $\leqq m$ parts for which the representation $[\Phi]$ occurs in the decomposition of $\bigotimes_{i=1}^{r}[2\kappa(i)]$.

When $\Phi=2\phi$, ϕ a partition of f into $\leqq m$ parts, $\mathcal{V}_{2\phi}^{\kappa[r]}$ contains a one-dimensional subspace which is invariant under the restriction of (2.2) to $H\in O(m)$. This subspace is generated by a suitably normalized polynomial $\Gamma_{\phi}^{\kappa[r]}(X_{[r]})$, which is invariant under (1.2). We note that $[2\phi]$ may occur with a multiplicity $n_{\phi}^{\kappa[r]}>1$ in the decomposition of $\bigotimes_{i=1}^{r}[2\kappa(i)]$. The direct sum of the corresponding equivalent irreducible subspaces

$$\mathcal{U}_{\phi}^{\kappa[r]}=\bigoplus_{\phi'\equiv\phi}\mathcal{V}_{2\phi'}^{\kappa[r]} \tag{2.4}$$

is then uniquely defined, but the individual $\mathcal{V}_{2\phi'}^{\kappa[r]}$, and hence the $\Gamma_{\phi'}^{\kappa[r]}$, are not.

In obtaining a resolution of this nonuniqueness which is sufficient for practical purposes, we first note that for a given $k[r]$, the set of all *distinct* products of traces

$$(\operatorname{tr} X_1^{a(1)}X_2^{a(2)}\cdots X_r^{a(r)}X_1^{b(1)}X_2^{b(2)}\cdots)^{p(1)} \cdot (\operatorname{tr} X_1^{g(1)}X_2^{g(2)}\cdots X_r^{g(r)}X_1^{h(1)}X_2^{h(2)}\cdots)^{p(2)}\cdots, \tag{2.5}$$

of total degree $k(i)$ in X_i $(i=1,\cdots,r)$, constitute a basis for the $\Gamma_{\phi}^{\kappa[r]}$. (Clearly, all monomials (2.5) are invariant under (1.2).) Now write

$$G=\sum_{i=1}^{r}\alpha_i A_i\otimes X_i\ ,$$

where the A_i are $m\times m$ matrices, and the α_i are arbitrary real numbers, and let

$$\det(I_{m^2}-G)^{-1/2}=\sum_{k(1),\cdots,k(r)=0}^{\infty}\alpha_1^{k(1)}\cdots\alpha_r^{k(r)}E_{k[r]}/k(1)!\cdots k(r)!\ .$$

It may be shown that

$$E_{k[r]}=\boldsymbol{\pi}'(A_{[r]})\Delta_{k[r]}\boldsymbol{\pi}(X_{[r]})\ , \tag{2.6}$$

where $\boldsymbol{\pi}(X_{[r]})$ is the vector of all monomials (2.5), and $\Delta_{k[r]}$ is a diagonal matrix. Then it is sufficient to (1) construct $n_{\phi}^{\kappa[r]}$ invariant polynomials $\tilde{\Gamma}_{\phi}^{\kappa[r]}(X_{[r]})$ in $\mathcal{U}_{\phi}^{\kappa[r]}$ whose coefficients are *orthonormal* with respect to the diagonal elements of $\Delta_{k[r]}$ as weights, and (2) multiply these by $z_\phi^{1/2}$ to obtain polynomials $C_\phi^{\kappa[r]}$ satisfying (1.1), where $z_\phi = C_\phi(I_m)/2^f(m/2)_\phi$ and $(m/2)_\phi$ is a generalized hypergeometric coefficient (Constantine [5]). The total number of $C_\phi^{\kappa[r]}$ for a given $k[r]$ is equal to the number of monomials (2.5).

Two further properties of the invariant polynomials which will be used in this paper are that

$$\prod_{i=1}^{r} C_{\kappa(i)}(X_i) = \sum_{\phi \in \kappa(1)\cdot\kappa(2)\cdot \cdots \cdot\kappa(r)} \theta_\phi^{\kappa[r]} C_\phi^{\kappa[r]}(X_{[r]}) , \tag{2.7}$$

where

$$\theta_\phi^{\kappa[r]} = C_\phi^{\kappa[r]}(I_m, \cdots, I_m)/C_\phi(I_m) , \tag{2.8}$$

and the multinomial expansion

$$C_\phi\left(\sum_{i=1}^{r} X_i\right) = \sum_{\kappa[r](\phi \in \kappa(1)\cdot\kappa(2)\cdot \cdots \cdot\kappa(r))} \binom{f}{k(1), \cdots, k(r)} \theta_\phi^{\kappa[r]} C_\phi^{\kappa[r]}(X_{[r]}) . \tag{2.9}$$

Equation (2.9) allows certain polynomials to be directly constructed from the zonal polynomials. Since the irreducible representation $[2f]$ occurs only in the decomposition of $\bigotimes_{i=1}^{r} [2k(i)]$, with multiplicity 1,

$$\binom{f}{k(1), \cdots, k(r)}^{-1} C_f^{\kappa[r]}(X_{[r]})$$

is given by the terms of degree $k(i)$ in X_i $(i=1, \cdots, r)$ in the expansion of $C_f\left(\sum_{i=1}^{r} X_i\right)$. The coefficients of the basis elements (2.5) in $1\times 3 \times \cdots \times (2f-1) C_f^{k[r]}$ are precisely the diagonal terms of $\Delta_{k[r]}$.

Example 1. Writing $(X) = \mathrm{tr}\,(X)$, the following results are obtained.

$$C_3(X) = \frac{1}{15}[(X)^3 + 6(X^2)(X) + 8(X^3)] ,$$

$$C_3^{2,1}(X, Y) = \frac{1}{15}[(X)^2(Y) + 4(XY)(X) + 2(X^2)(Y) + 8(X^2 Y)] ,$$

$$\Delta_{2,1} = \mathrm{diag}\,(1, 4, 2, 8) .$$

(The factor 5 in equation (4.9) of Davis [7] should be deleted.)

$$C_3^{1,1,1}(X, Y, Z) = \frac{1}{15}[(X)(Y)(Z) + 2\{(X)(YZ) + (Y)(ZX) + (Z)(XY)\} + 8(XYZ)] ,$$

$$\Delta_{1,1,1}=\text{diag}\,(1, 2, 2, 2, 8)\ .$$

Similarly,

$$\binom{f}{k(1),\cdots, k(r)}^{-1} C_{1f}^{1^{k(1)},\cdots,1^{k(r)}}(X_{[r]})$$

may be obtained from $C_{1f}\left(\sum_{i=1}^{r} X_i\right)$.

3. Representations of $Gl(m, R)$ in tensor space

The basic notion in constructing the invariant polynomials is that the component of an arbitrary polynomial invariant under (1.2) $\left(\text{say } \prod_{i=1}^{r} (\text{tr } X_i)^{k(i)}\right)$ in a subspace $\mathcal{V}_{2\phi}^{\kappa[r]}$ must itself be invariant, and hence, provided it is not identically zero, must be proportional to the corresponding $\Gamma_{\phi}^{\kappa[r]}(X_{[r]})$. The problem thus arises of constructing the idempotents which project upon the $\mathcal{V}_{2\phi}^{\kappa[r]}$, and the approach adopted here essentially follows that of James [9], [10].

We first note that any polynomial $p(X_{[r]}) \in P_{k[r]}$ may be written as

$$(3.1)\qquad p(X_{[r]})=\sum_{i(1)=1}^{m}\cdots\sum_{i(2f)=1}^{m} a(i(1),\cdots, i(2f))x_1(i(1), i(2))\cdots x_1(i(2k(1)-1), i(2k(1)))\cdots x_r(i(2f-1), i(2f))$$

where $X_l=(x_l(i, j))$, $l=1,\cdots, r$. The coefficients $a(i(1),\cdots, i(2f))$ are uniquely defined by the requirement that they should be invariant under the group $U_{2k[r]}$ of permutations u which (i) permute within each of the f pairs

$$(3.2)\qquad (1, 2), (3, 4),\cdots, (2f-1, 2f)$$

and (ii) if the pairs are consecutively grouped into r sets of sizes $k(1)$, $\cdots$, $k(r)$ in that order, permute the *pairs* independently within each set. $U_{2k[r]}$ thus has order $2^f \prod_{i=1}^{r} k(i)!$.

Regarding each polynomial as a point with coordinates $\{a(i(1),\cdots, i(2f))\}$ in the space $C^{m^{2f}}$ of complex-valued tensors $\{c(i(1),\cdots, i(2f))\}$, $i(j)=1,\cdots, m$, $j=1,\cdots, 2f$, it follows that $P_{k[r]}(X_{[r]})$ corresponds to a linear subspace of $C^{m^{2f}}$, defined by invariance under $U_{2k[r]}$. Further, letting $L=(l(i, j))$, the transformation (2.2) applied to (3.1) yields

$$\begin{aligned} p(X_{[r]})&\rightarrow p(L'X_1L,\cdots, L'X_rL)\\ &=\sum_{(i)} a^*(i(1),\cdots, i(2f))x_1(i(1), i(2))\cdots x_r(i(2f-1), i(2f))\ , \end{aligned}$$

where $\sum_{(i)}$ denotes the summation in (3.1) and

$$a^*(i(1),\cdots,i(2f))=\sum_{(j)} l(i(1),j(1))\cdots l(i(2f),j(2f)) \cdot a(j(1),\cdots,j(2f))\ . \tag{3.3}$$

Equation (3.3) is a component of the representation of $Gl(m, R)$ in $\mathcal{C}^{m^{2f}}$ defined by the transformation

$$c^*(i(1),\cdots,i(2f))=\sum_{(j)} l(i(1),j(1)\cdots l(i(2f),j(2f)) \cdot c(j(1),\cdots,j(2f))\ , \tag{3.4}$$

i.e. the $2f$th Kronecker power representation

$$L\rightarrow[L]^{2f}=L\otimes L\otimes\cdots\otimes L\ (2f\ \text{terms})\ ,\qquad L\in Gl(m,R)\ . \tag{3.5}$$

From (3.3), $P_{k[r]}$ is an invariant subspace under this representation. Its decomposition (2.3) is thus a special case of the decomposition of $\mathcal{C}^{m^{2f}}$ under (3.5), the theory of which is conveniently discussed, for example, in Boerner [1], Chapter 5. Briefly, the salient points of this theory are as follows.

(a) The centralizer $\mathcal{B}_{2f}$ of the representation is the algebra of linear transformations of $\mathcal{C}^{m^{2f}}$ which commute with all the transformations (3.4). There is a 1–1 correspondence between the right ideals of $\mathcal{B}_{2f}$ and the invariant subspaces of $\mathcal{C}^{m^{2f}}$ under (3.5). The latter are generated by the generating idempotents of the corresponding right ideals. Irreducible invariant subspaces are generated by the primitive idempotents generating minimal right ideals.

(b) Let S_{2f} denote the symmetric group of permutations of $2f$ objects. Any such permutation σ,

$$1,2,\cdots,2f\rightarrow\sigma(1),\cdots,\sigma(2f)\ , \tag{3.6}$$

defines a linear transformation of $\mathcal{C}^{m^{2f}}$,

$$c(i(1),\cdots,i(2f))\rightarrow c(i(\sigma(1)),\cdots,i(\sigma(2f)))\ , \tag{3.7}$$

which is readily seen to commute with all transformations (3.4). More generally, the centralizer $\mathcal{B}_{2f}$ is isomorphic to the group ring $\mathcal{S}_{2f}$, of S_{2f} modulo the two-sided ideal of ring elements which annul $\mathcal{C}^{m^{2f}}$. The idempotents of $\mathcal{B}_{2f}$ thus correspond to idempotents generating right ideals of $\mathcal{S}_{2f}$.

(c) A set of irreducible invariant subspaces spanning $\mathcal{C}^{m^{2f}}$ is generated by essential idempotents of $\mathcal{S}_{2f}$ constructed from the standard Young tableaux with $\leq m$ rows. (Tableaux with $>m$ rows yield idempotents which annul $\mathcal{C}^{m^{2f}}$.)

The invariant subspace $P_{k[r]}$ is seen to be generated by the idempotent

$$e_{k[r]}=\left\{2^f \prod_{i=1}^{r} k(i)!\right\}^{-1} \sum_{u\in U_{\kappa[r]}} u=\prod_{i=1}^{r} e_{k(i)} , \tag{3.8}$$

where $e_{k(i)}$ is the corresponding idempotent for the group $U_{k(i)}$ of permutations which permute with and among the ith set of pairs in (3.2) (corresponding to James [9], equation (13)). Hence we may write

$$P_{k[r]}=e_{k[r]}C^{m^{2f}} , \tag{3.9}$$

and the centralizer of the representation (3.5) is the algebra $e_{k[r]}\mathcal{B}_{2f}e_{k[r]}$. We thus require the primitive idempotents of the latter, since these project onto the irreducible invariant subspaces of $P_{k[r]}$.

4. Application of Young's idempotents

Let $\Phi=[F_1,\cdots,F_g]$ denote an ordered partition of $2f$ into positive integers $F_1\geqq F_2\geqq\cdots\geqq F_g>0$, $\sum_{i=1}^{g} F_i=2f$. A Young's tableau $T(\Phi)$ corresponding to Φ is any arrangement of the integers 1 through $2f$ in successive rows of lengths $F_1,\cdots,F_g$, the first integers in the rows constituting the first column (Boerner [1], Chapter 4).

For each $T(\Phi)$, two elements of the group ring $\mathcal{S}_{2f}$ may be defined. First, the *symmetrizer* $s_{T(\Phi)}=\sum p$, the sum of all permutations p in the subgroup of S_{2f} which leaves the *rows* of T invariant. We also have the *alternator* $a_{T(\Phi)}=\sum \delta_q q$, in which the sum extends over all permutations leaving the *columns* invariant, and $\delta_q=1$ or -1 according as q is an even or odd permutation. Then

$$\varepsilon_{T(\Phi)}=a_{T(\Phi)}s_{T(\Phi)} \tag{4.1}$$

is an essential primitive idempotent (i.e. a nonzero scalar multiple of a primitive idempotent) of $\mathcal{S}_{2f}$. If $g\leqq m$, it generates an irreducible invariant subspace of $C^{m^{2f}}$. Subspaces defined by different tableaux are equivalent if and only if they correspond to the same partition Φ of $2f$.

Any primitive idempotent ε of $\mathcal{S}_{2f}$ has the property that

$$\varepsilon x\varepsilon=\xi_x\varepsilon , \qquad \text{all } x\in\mathcal{S}_{2f} , \tag{4.2}$$

where ξ_x is a scalar which may be zero. Conversely, if ε is idempotent and has the property (4.2) it is a primitive idempotent (Boerner [1], Theorem 3.9). Hence, referring to (3.8), $e_{k[r]}\varepsilon_{T(2\phi)}e_{k[r]}$ is either zero or else an essential primitive idempotent of the centralizer $e_{k[r]}\mathcal{B}_{2f}e_{k[r]}$. It thus generates an irreducible invariant subspace of

$$\mathcal{W}_{\phi}^{k[r]}=\bigoplus_{\kappa[r]}\mathcal{U}_{\phi}^{\kappa[r]} \tag{4.3}$$

(see (2.4)). More specifically

$$e_{k[r]}\left\{\prod_{i=1}^{r} \varepsilon_{T(2\kappa(i))}\right\}\varepsilon_{T(2\phi)}e_{k[r]}\ , \tag{4.4}$$

if it is nonzero, generates an irreducible invariant subspace of $\mathcal{V}_\phi^{\kappa[r]}$. Here, if the integers 1 through $2f$ are regarded as being partitioned into consecutive sets of sizes $k(1),\cdots,k(r)$, $\varepsilon_{T(2\kappa(i))}$ is the essential idempotent for a Young's tableau corresponding to the partition $2\kappa(i)$ constructed from the integers in the ith set.

The above suggests the following construction of the $C_\phi^{\kappa[r]}$.

(1) Project $\prod_{i=1}^{r}(\operatorname{tr} X_i)^{k(i)}$ onto $\mathcal{V}_\phi^{\kappa[r]}$ by means of (4.4), varying the tableaux T until the required number $n_\phi^{\kappa[r]}$ of linearly independent invariant polynomials $\Gamma_\phi^{\kappa[r]}$ in this space is obtained. If necessary, further basis monomials (2.5) may be used.

(2) Select as the first $\tilde{\Gamma}_\phi^{\kappa[r]}$ that unique linear combination of the $\Gamma_\phi^{\kappa[r]}$ which yields the component of $\prod_{i=1}^{r}(\operatorname{tr} X_i)^{k(i)}$ in this space (Davis [7], Section 5). Convenient $\Delta_{k[r]}$-orthonormal linear combinations may then be constructed to complete the set. Multiplication by $z_\phi^{1/2}$ yields a set of $C_\phi^{\kappa[r]}$.

5. Projection of $\prod_{i=1}^{r}(\operatorname{tr} X_i)^{k(i)}$

The tensor corresponding to $\prod_{i=1}^{r}(\operatorname{tr} X_i)^{k(i)}$ is easily seen to be $\prod_{j=1}^{f}\delta(i(2j-1), i(2j))$, where $\delta(k,l)$ is Kronecker's delta. In constructing the required projections, we need to calculate the result of applying $e_{k[r]}\sigma\cdot e_{k[r]}$ to this tensor, where $\sigma\in S_{2f}$ (equation (3.6)). This is clearly

$$\sum_{(i)}\delta(i(\sigma(1)), i(\sigma(2)))\cdots\delta(i(\sigma(2f-1)), i(\sigma(2f)))\cdot x_1(i(1), i(2))\cdots x_r(i(2f-1), i(2f))\ , \tag{5.1}$$

bearing in mind the symmetry of the X_i. A convenient method for reducing (5.1) follows as a direct extension of James [10], Section 6. Write $2f$ dots along a line, and connect successive pairs $(1,2),(3,4),\cdots,(2f-1,2f)$ with loops above the line. Then connect the pairs

$$(\sigma(1),\sigma(2)),\cdots,(\sigma(2f-1),\sigma(2f)) \tag{5.2}$$

with loops below the line, forming closed cycles. Regarding the first $k(1)$ pairs $(1,2),\cdots,(2k(1)-1,2k(1))$ as "X_1-pairs", the next $k(2)$ pairs as "X_2-pairs", etc., each cycle may be interpreted as the trace of a product of powers of $X_1,\cdots,X_r$, the order of terms in the product corresponding to the order in which the cycle passes through the pairs.

The polynomial (5.1) then reduces to the monomial consisting of the product of these traces over all the cycles. Clearly it is a member of the basis (2.5).

Example 2. Take $f=4$, $r=3$, $k(1)=2$, $k(2)=k(3)=1$, and let σ denote the permutation 6 3 5 4 2 8 1 7 of the integers 1 through 8. From Fig. 1, the polynomial (5.1) is given in this case by $\mathrm{tr}\,(X_1X_2)\cdot\mathrm{tr}\,(X_1X_3)$, as is readily confirmed directly.

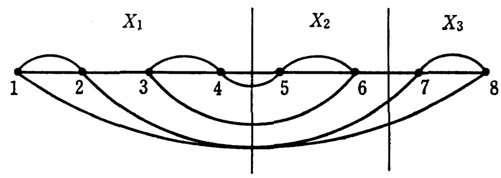

Fig. 1

For computational purposes it is desirable to reduce the number of permutations considered as far as possible. Consideration of the diagram approach suggests that the monomial corresponding to (5.1) is independent of the ordering among or within the pairs; i.e. it depends only on the *left coset* of U_{2f} (the group of permutations among and within the pairs (3.2)) to which σ belongs. In expanding (4.4), we may thus work modulo these cosets. The left cosets of U_{2f} are generated by permutations of the form

$$(5.3)\qquad 1, 2, \cdots, 2f \rightarrow i(1), j(1), i(2), j(2), \cdots, i(2f), j(2f)$$

where

$$(5.4)\qquad i(1)=1\,, \quad i(l)\leqq j(l)\,, \quad (l=1,\cdots,f),\ i(1)<i(2)<\cdots<i(f)\,.$$

Clearly there are $1\times 3\times\cdots\times(2f-1)$ such permutations, noting that U_{2f} has order $2^f f!$. They correspond to the doublets of James.

Example 3. *Construction of* $C_{1^2}^{1,1}(X, Y)$. This polynomial follows directly from $C_{1^2}(X)$ using (2.9). However, we shall derive it directly to illustrate the method. Young tableaux for 2ϕ, $2\kappa(1)$ and $2\kappa(2)$ are respectively

1	2	1	2	3	4
3	4				.

Using cycle notation (Boerner [1], Section 2.2), with ( ) denoting the identical permutation,

$$s_{T(2^2)}=[(\ )+(12)+(34)+(12)(34)]$$
$$a_{T(2^2)}=[(\ )-(13)-(24)+(13)(24)]$$

$$\varepsilon_{T(2\kappa(1))} = [(\ ) + (12)]$$
$$\varepsilon_{T(2\kappa(2))} = [(\ ) + (34)] .$$

All permutations in $s_{T(2^2)}$ are in U_4, and so this term may be replaced by the identity permutation 1234. Hence we obtain successively (working modulo U_4)

$$\varepsilon_{T(2^2)} \to 2[1234 - 1423] ,$$
$$\varepsilon_{T(2\kappa(1))}\varepsilon_{T(2\kappa(2))}\varepsilon_{T(2^2)} \to 4[2 \times 1234 - 1324 - 1423] .$$

Note that the three permutations appearing on the right-hand side are the generators of the three left cosets of U_4 in S_4. Normalizing with respect to $\Delta_{1,1} = \text{diag}\{1, 2\}$, we obtain

$$\tilde{\Gamma}_{1^2}^{1,1}(X, Y) = 3^{-1/2}[(X)(Y) - (XY)] .$$

Since $[2^2]$ has multiplicity one in the decomposition of $[2] \otimes [2]$, this polynomial is unique, and multiplying by $z_{1^2}^{1/2} = 3^{-1/2}$ we obtain

$$C_{1^2}^{1,1}(X, Y) = \frac{1}{3}[(X)(Y) - (XY)] .$$

6. An alternative approach

With increasing $k(i)$ and f, the number of permutations involved in the expansion of (4.4) rapidly becomes astronomical, raising the problem of finding alternative approaches. One approach which appears to be useful, at least in the case of lower degree polynomials, is based on the observation that if $e_{k[r]}\varepsilon_{T(2\phi)}e_{k[r]}$ is nonzero, then it projects onto an irreducible invariant subspace of $\mathcal{W}_\phi^{k[r]}$ (equation (4.3)), and thus may be used to generate an invariant polynomial. Taking various tableaux $T(2\phi)$, it is possible in principle to construct

$$n_\phi^{k[r]} = \sum_{\kappa[r]} n_\phi^{\kappa[r]} \tag{6.1}$$

linearly independent invariant polynomials in $\mathcal{W}_\phi^{k[r]}$. If this is carried out for all $\phi \in \kappa(1) \cdot \cdots \cdot \kappa(r)$, then from (2.7) (i) $\prod_{i=1}^{r} C_{\kappa(i)}(X_i)$ may be expressed as a linear combination of these polynomials, and (ii) the "ϕ-component" of this linear combination is precisely the projection of $\prod C_{\kappa(i)}$, and hence of $\prod (\text{tr } X_i)^{k(i)}$, onto the space $\mathcal{V}_\phi^{\kappa[r]}$. It may thus be taken as the first $\tilde{\Gamma}_\phi^{\kappa[r]}$.

If ϕ has multiplicity 0 or 1 for all $\kappa[r]$ corresponding to a fixed $k[r]$, then all the $C_\phi^{\kappa[r]}$ may be uniquely constructed by this approach. If ϕh as multiplicity 0 or 1 for all but one $\kappa[r]$, say $\kappa_0[r]$, then the $C_\phi^{\kappa[r]}$ may be uniquely constructed for all $\kappa[r]\neq\kappa_0[r]$. Also, the method yields the first $\tilde{\Gamma}_\phi^{\kappa_0[r]}$, leaving $n_\phi^{\kappa_0[r]}-1$ orthonormal polynomials $\Gamma_{\phi'}^{\kappa_0[r]}$ ($\phi'\equiv\phi$) to be constructed in $\mathcal{U}_\phi^{\kappa_0[r]}$. But any $n_\phi^{\kappa_0[r]-1}$ polynomials constructed from the original set of $n_\phi^{k[r]}$, which are orthonormal to the set thus far, can only lie in $\mathcal{U}_\phi^{\kappa_0[r]}$. The method therefore yields the $C_\phi^{\kappa[r]}$ in this case. However, if ϕ has multiplicity >1 for at least two $\kappa[r]$, then at present there appears to be no way of constructing the required orthonormal polynomials in the corresponding $\mathcal{U}_\phi^{\kappa[r]}$ spaces.

This technique has been used in the production of Table 1, which contains the orthonormal three-matrix polynomials $\tilde{\Gamma}_\phi^{\kappa[3]}(X, Y, Z)=z_\phi^{-1/2}C_\phi^{\kappa[3]}(X, Y, Z)$ for partitions ϕ of f, where $f\leqq 5$. In cases where ϕ has multiplicity 3 for a given $\kappa[3]$, the two polynomials required in addition to the component of $(X)^{k(1)}(Y)^{k(2)}(Z)^{k(3)}$ have been constructed using $\Delta_{k[3]}$-orthonormality as described above. The calculations

Table 1. Orthonormal polynomials $\tilde{\Gamma}_\phi^{\kappa[3]}(X, Y, Z)$ for partitions ϕ of $f\leqq 5$. C=multiplicative constant. $(X)=\mathrm{tr}\,X$.

$f=3$, $k[3]=1, 1, 1$

	$\kappa[3]=1, 1, 1$				
$\phi=$	3	21	21	21	1^3
$C^2=$	1/15	1/15	1/3	1	1/3
(XYZ)	8	−6	0	0	2
$(XY)(Z)$	2	1	2	0	−1
$(XZ)(Y)$	2	1	−1	1	−1
$(YZ)(X)$	2	1	−1	−1	−1
$(X)(Y)(Z)$	1	3	0	0	1

$f=4$, $k[3]=2, 1, 1$

	$\kappa[3]=2, 1, 1$						$\kappa[3]=1^2, 1, 1$					
$\phi=$	4	31	31	31	2^2	21^2	31	2^2	21^2	21^2	21^2	1^4
$C^2=$	1/105	2/2079	2/33	2/3	8/135	4/27	2/27	2/27	4/1755	8/39	4/3	2/15
(X^2YZ)	32	−40	8	0	−8	4	−8	4	28	−2	0	−4
$(XYXZ)$	16	−48	−8	0	6	0	0	−6	24	2	0	−2
$(X^2Y)(Z)$	8	32	−2	2	−2	−2	−2	−2	1	−1	−1	2
$(X^2Z)(Y)$	8	32	−2	−2	−2	−2	−2	−2	1	−1	1	2
$(XYZ)(X)$	16	−20	4	0	−4	2	8	−4	−28	2	0	4
$(X^2)(YZ)$	4	2	4	0	4	−2	−2	1	−2	2	0	1
$(XY)(XZ)$	8	−24	−4	0	3	0	0	6	−24	−2	0	2
$(X^2)(Y)(Z)$	2	22	0	0	2	2	−1	−1	−13	0	0	−1
$(XY)(X)(Z)$	4	16	−1	1	−1	−1	2	2	−1	1	1	−2
$(XZ)(X)(Y)$	4	16	−1	−1	−1	−1	2	2	−1	1	−1	−2
$(YZ)(X)^2$	2	1	2	0	2	−1	2	−1	2	−2	0	−1
$(X)^2(Y)(Z)$	1	11	0	0	1	1	1	1	13	0	0	1

Table 1. Continued

$f=5,\quad k[3]=3,1,1$										
	$\kappa[3]=3,1,1$						$\kappa[3]=21,1,1$			
$\phi=$	5	41	41	41	32	31^2	41	32	32	32
$C^2=$	1/945	1/31050	1/115	1/5	8/525	1/30	1/50	2/225	2/45	2/15
(X^3YZ)	192	−336	64	0	−32	16	−32	4	16	0
(X^2YXZ)	192	−768	−64	0	24	0	−16	−28	−16	0
$(X^3Y)(Z)$	48	456	−8	8	−8	−8	−8	−14	−2	2
$(X^3Z)(Y)$	48	456	−8	−8	−8	−8	−8	−14	−2	−2
$(X^2YZ)(X)$	96	−168	32	0	−16	8	24	−28	8	0
$(XYXZ)(X)$	48	−192	−16	0	6	0	16	−22	−4	0
$(X^3)(YZ)$	16	8	16	0	16	−8	−4	−2	4	0
$(X^2Y)(XZ)$	48	−192	−16	0	6	0	−4	18	6	−10
$(X^2Z)(XY)$	48	−192	−16	0	6	0	−4	18	6	10
$(XYZ)(X^2)$	48	−84	16	0	−8	4	−8	26	−16	0
$(X^3)(Y)(Z)$	8	184	0	0	8	8	−2	−6	0	0
$(X^2Y)(X)(Z)$	24	228	−4	4	−4	−4	6	−2	4	−4
$(X^2Z)(X)(Y)$	24	228	−4	−4	−4	−4	6	−2	4	4
$(XYZ)(X)^2$	24	−42	8	0	−4	2	16	−2	−8	0
$(X^2)(XY)(Z)$	12	114	−2	2	−2	−2	−2	9	−3	3
$(X^2)(XZ)(Y)$	12	114	−2	−2	−2	−2	−2	9	−3	−3
$(X^2)(YZ)(X)$	12	6	12	0	12	−6	2	1	−2	0
$(XY)(XZ)(X)$	24	−96	−8	0	3	0	8	14	8	0
$(X^2)(X)(Y)(Z)$	6	138	0	0	6	6	1	3	0	0
$(XY)(X)^2(Z)$	6	57	−1	1	−1	−1	4	7	1	−1
$(XZ)(X)^2(Y)$	6	57	−1	−1	−1	−1	4	7	1	1
$(YZ)(X)^3$	2	1	2	0	2	−1	2	1	−2	0
$(X)^3(Y)(Z)$	1	23	0	0	1	1	1	3	0	0

	$\kappa[3]=21,1,1$							$\kappa[3]=1^3,1,1$					
$\phi=$	31^2	31^2	31^2	2^21	2^21	2^21	21^3	31^2	2^21	21^3	21^3	21^3	1^5
$C^2=$	1/6930	8/99	4/3	1/45	1/9	1/3	1/7	1/21	1/15	1/693	2/27	1	2/45
(X^3YZ)	128	4	0	16	8	0	−8	16	−8	−48	8	0	12
(X^2YXZ)	400	−4	0	−4	−8	0	−4	0	12	−84	−8	0	12
$(X^3Y)(Z)$	−64	−2	2	4	2	2	4	4	4	−12	2	2	−6
$(X^3Z)(Y)$	−64	−2	−2	4	2	−2	4	4	4	−12	2	−2	−6
$(X^2YZ)(X)$	−56	−10	0	8	−8	0	0	−16	8	48	−8	0	−12
$(XYXZ)(X)$	−80	14	0	14	4	0	−2	0	−6	42	4	0	−6
$(X^3)(YZ)$	−4	4	0	−2	−4	0	2	4	−2	2	−6	0	−2
$(X^2Y)(XZ)$	−60	−6	2	−6	0	−4	2	0	−6	42	6	0	−6
$(X^2Z)(XY)$	−60	−6	−2	−6	0	4	2	0	−6	42	6	0	−6
$(XYZ)(X^2)$	−8	8	0	−16	4	0	4	−8	4	24	−6	0	−6
$(X^3)(Y)(Z)$	−66	0	0	−6	0	0	−2	2	2	22	0	0	2
$(X^2Y)(X)(Z)$	98	1	−1	−8	2	2	0	−4	−4	12	−2	−2	6
$(X^2Z)(X)(Y)$	98	1	1	−8	2	−2	0	−4	−4	12	−2	2	6

Table 1. Continued

$f=5,\quad k[3]=3, 1, 1$

	$\kappa[3]=21, 1, 1$							$\kappa[3]=1^3, 1, 1$					
$\phi=$	31^2	31^2	31^2	2^21	2^21	2^21	21^3	31^2	2^21	21^3	21^3	21^3	1^5
$C^2=$	1/6930	8/99	4/3	1/45	1/9	1/3	1/7	1/21	1/15	1/693	2/27	1	2/45
$(XYZ)(X)^2$	−64	−2	0	−8	−4	0	4	8	−4	−24	4	0	6
$(X^2)(XY)(Z)$	−66	0	0	6	−3	−3	−2	−2	−2	6	−1	−1	3
$(X^2)(XZ)(Y)$	−66	0	0	6	−3	3	−2	−2	−2	6	−1	1	3
$(X^2)(YZ)(X)$	2	−2	0	1	2	0	−1	−6	3	−3	6	0	3
$(XY)(XZ)(X)$	−200	2	0	2	4	0	2	0	6	−42	−4	0	6
$(X^2)(X)(Y)(Z)$	33	0	0	3	0	0	1	−3	−3	−33	0	0	−3
$(XY)(X)^2(Z)$	32	1	−1	−2	−1	−1	−2	2	2	−6	1	1	−3
$(XZ)(X)^2(Y)$	32	1	1	−2	−1	1	−2	2	2	−6	1	−1	−3
$(YZ)(X)^3$	2	−2	0	1	2	0	−1	2	−1	1	−2	0	−1
$(X)^3(Y)(Z)$	33	0	0	3	0	0	1	1	1	11	0	0	1

$f=5,\quad k[3]=2, 2, 1$

	$\kappa[3]=2, 2, 1$								
$\phi=$	5	41	41	41	32	32	32	31^2	2^21
$C^2=$	1/945	2/135	2/45	2/45	8/31185	8/495	8/45	2/81	16/405
(X^2Y^2Z)	128	8	16	0	−272	16	0	8	8
(X^2YZY)	64	−8	0	16	32	−20	4	−8	4
$(XYXYZ)$	128	−40	−16	0	120	24	0	24	−12
(XY^2XZ)	64	−8	0	−16	32	−20	−4	−8	4
$(X^2Y^2)(Z)$	32	20	−8	0	−40	−8	0	4	−8
$(X^2YZ)(Y)$	64	4	8	0	−136	8	0	4	4
$(XYXY)(Z)$	16	4	−8	0	−6	12	0	−12	6
$(XYXZ)(Y)$	32	−4	0	−8	16	−10	−2	−4	2
$(XY^2Z)(X)$	64	4	8	0	−136	8	0	4	4
$(XYZY)(X)$	32	−4	0	8	16	−10	2	−4	2
$(X^2Y)(YZ)$	32	−4	0	8	16	−10	2	−4	2
$(X^2Z)(Y^2)$	16	4	4	−4	64	4	4	−4	−4
$(XY^2)(XZ)$	32	−4	0	−8	16	−10	−2	−4	2
$(XYZ)(XY)$	64	−20	−8	0	60	12	0	12	−6
$(Y^2Z)(X^2)$	16	4	4	4	64	4	−4	−4	−4
$(X^2Y)(Y)(Z)$	16	10	−4	0	−20	−4	0	2	−4
$(X^2Z)(Y)^2$	8	2	2	−2	32	2	2	−2	−2
$(XY^2)(X)(Z)$	16	10	−4	0	−20	−4	0	2	−4
$(XYZ)(X)(Y)$	32	2	4	0	−68	4	0	2	2
$(Y^2Z)(X)^2$	8	2	2	2	32	2	−2	−2	−2
$(X^2)(Y^2)(Z)$	4	4	0	0	44	0	0	4	4
$(X^2)(YZ)(Y)$	8	2	2	2	32	2	−2	−2	−2
$(XY)^2(Z)$	8	2	−4	0	−3	6	0	−6	3
$(XY)(XZ)(Y)$	16	−2	0	−4	8	−5	−1	−2	1

Table 1. Continued

$f=5,\quad k[3]=2, 2, 1$

$\kappa[3]=2, 2, 1$

$\phi=$	5	41	41	41	32	32	32	31^2	2^21
$C^2=$	1/945	2/135	2/45	2/45	8/31185	8/495	8/45	2/81	16/405
$(XY)(YZ)(X)$	16	−2	0	4	8	−5	1	−2	1
$(XZ)(Y^2)(X)$	8	2	2	−2	32	2	2	−2	−2
$(X^2)(Y)^2(Z)$	2	2	0	0	22	0	0	2	2
$(XY)(X)(Y)(Z)$	8	5	−2	0	−10	−2	0	1	−2
$(XZ)(X)(Y)^2$	4	1	1	−1	16	1	1	−1	−1
$(Y^2)(X)^2(Z)$	2	2	0	0	22	0	0	2	2
$(YZ)(X)^2(Y)$	4	1	1	1	16	1	−1	−1	−1
$(X)^2(Y)^2(Z)$	1	1	0	0	11	0	0	1	1

$\kappa[3]=2, 1^2, 1$

$\phi=$	41	32	31^2	31^2	31^2	2^21	21^3
$C^2=$	1/90	8/405	1/100926	16/12015	64/135	4/81	4/63
(X^2Y^2Z)	−32	8	352	64	2	8	−8
(X^2YZY)	0	−20	512	−4	−2	4	−4
$(XYXYZ)$	0	0	0	0	0	0	0
(XY^2XZ)	−16	4	560	−60	0	−8	0
$(X^2Y^2)(Z)$	−8	−8	−232	−26	2	4	4
$(X^2YZ)(Y)$	32	−8	−352	−64	−2	−8	8
$(XYXY)(Z)$	0	0	0	0	0	0	0
$(XYXZ)(Y)$	16	−4	−560	60	0	8	0
$(XY^2Z)(X)$	−16	4	176	32	1	4	−4
$(XYZY)(X)$	0	−10	256	−2	−1	2	−2
$(X^2Y)(YZ)$	0	20	−512	4	2	−4	4
$(X^2Z)(Y^2)$	−8	2	−200	−4	−2	2	2
$(XY^2)(XZ)$	−8	2	280	−30	0	−4	0
$(XYZ)(XY)$	0	0	0	0	0	0	0
$(Y^2Z)(X^2)$	−4	−4	76	30	0	−4	4
$(X^2Y)(Y)(Z)$	8	8	232	26	−2	−4	−4
$(X^2Z)(Y)^2$	8	−2	200	4	2	−2	−2
$(XY^2)(X)(Z)$	−4	−4	−116	−13	1	2	2
$(XYZ)(X)(Y)$	16	−4	−176	−32	−1	−4	4
$(Y^2Z)(X)^2$	−2	−2	38	15	0	−2	2
$(X^2)(Y^2)(Z)$	−2	−2	−178	0	0	−2	−2
$(X^2)(YZ)(Y)$	4	4	−76	−30	0	4	−4
$(XY)^2(Z)$	0	0	0	0	0	0	0
$(XY)(XZ)(Y)$	8	−2	−280	30	0	4	0
$(XY)(YZ)(X)$	0	10	−256	2	1	−2	2
$(XZ)(Y^2)(X)$	−4	1	−100	−2	−1	1	1
$(X^2)(Y)^2(Z)$	2	2	178	0	0	2	2
$(XY)(X)(Y)(Z)$	4	4	116	13	−1	−2	−2

Table 1. Continued

$f=5$, $k[3]=2, 2, 1$

$\kappa[3]=2, 1^2, 1$

$\phi=$	41	32	31^2	31^2	31^2	2^21	21^3
$C^2=$	1/90	8/405	1/100926	16/12015	64/135	4/81	4/63
$(XZ)(X)(Y)^2$	4	−1	100	2	1	−1	−1
$(Y^2)(X)^2(Z)$	−1	−1	−89	0	0	−1	−1
$(YZ)(X)^2(Y)$	2	2	−38	−15	0	2	−2
$(X)^2(Y)^2(Z)$	1	1	89	0	0	1	1

$\kappa[3]=1^2, 1^2, 1$

$\phi=$	32	31^2	2^21	2^21	2^21	21^3	21^3	21^3	1^5
$C^2=$	2/81	32/567	4/5265	8/117	4/9	4/189	8/27	4/9	2/45
(X^2Y^2Z)	8	8	−76	2	0	−4	2	0	8
(X^2YZY)	4	−2	22	−4	2	−6	0	2	4
$(XYXYZ)$	−24	12	84	6	0	−20	−2	0	8
(XY^2XZ)	4	−2	22	−4	−2	−6	0	−2	4
$(X^2Y^2)(Z)$	4	1	28	2	0	−4	2	0	−4
$(X^2YZ)(Y)$	−8	−8	76	−2	0	4	−2	0	−8
$(XYXY)(Z)$	−6	3	−6	−6	0	2	2	0	−2
$(XYXZ)(Y)$	−4	2	−22	4	2	6	0	2	−4
$(XY^2Z)(X)$	−8	−8	76	−2	0	4	−2	0	−8
$(XYZY)(X)$	−4	2	−22	4	−2	6	0	−2	−4
$(X^2Y)(YZ)$	−4	2	−22	4	−2	6	0	−2	−4
$(X^2Z)(Y^2)$	2	2	−1	−1	−1	−1	−1	1	−2
$(XY^2)(XZ)$	−4	2	−22	4	2	6	0	2	−4
$(XYZ)(XY)$	24	−12	−84	−6	0	20	2	0	−8
$(Y^2Z)(X^2)$	2	2	−1	−1	1	−1	−1	−1	−2
$(X^2Y)(Y)(Z)$	−4	−1	−28	−2	0	4	−2	0	4
$(X^2Z)(Y)^2$	−2	−2	1	1	1	1	1	−1	2
$(XY^2)(X)(Z)$	−4	−1	−28	−2	0	4	−2	0	4
$(XYZ)(X)(Y)$	8	8	−76	2	0	−4	2	0	8
$(Y^2Z)(X)^2$	−2	−2	1	1	−1	1	1	1	2
$(X^2)(Y^2)(Z)$	1	1	13	0	0	3	0	0	1
$(X^2)(YZ)(Y)$	−2	−2	1	1	−1	1	1	1	2
$(XY)^2(Z)$	6	−3	6	6	0	−2	−2	0	2
$(XY)(XZ)(Y)$	4	−2	22	−4	−2	−6	0	−2	4
$(XY)(YZ)(X)$	4	−2	22	−4	2	−6	0	2	4
$(XZ)(Y^2)(X)$	−2	−2	1	1	1	1	1	−1	2
$(X^2)(Y)^2(Z)$	−1	−1	−13	0	0	−3	0	0	−1
$(XY)(X)(Y)(Z)$	4	1	28	2	0	−4	2	0	−4
$(XZ)(X)(Y)^2$	2	2	−1	−1	−1	−1	−1	1	−2
$(Y^2)(X)^2(Z)$	−1	−1	−13	0	0	−3	0	0	−1
$(YZ)(X)^2(Y)$	2	2	−1	−1	1	−1	−1	−1	−2
$(X)^2(Y)^2(Z)$	1	1	13	0	0	3	0	0	1

were performed by Mrs. J. A. Rover on a PDP 11/10 computer at CSIRO, Adelaide. Polynomials for $r=2$, $f=6$ are available from the author.

Example 4. *Construction of polynomials for* $k(1)=2$, $k(2)=1$. From the binomial theorem,

$$C_{3}^{2,1}(X, Y)=\frac{1}{15}[(X)^2(Y)+4(XY)(X)+2(X^2)(Y)+8(X^2Y)]$$

$$C_{1^3}^{1^2,1}(X, Y)=\frac{1}{3}[(X)^2(Y)-2(XY)(X)-(X^2)(Y)+2(X^2Y)] .$$

As in Example 1, $\Delta_{2,1}=\operatorname{diag}\{1, 4, 2, 8\}$. It remains to construct $C_{21}^{2,1}$ and $C_{21}^{1^2,1}$ ($\phi=21$). Projecting $(X)^2(Y)$ using the 2ϕ tableaux

1	2	3	4		1	2	5	6
5	6				3	4		

the following polynomials in $\mathcal{W}_{21}^{2,1}$ are obtained,

$$p_1=(X)^2(Y)+(X^2)(Y)-2(X^2Y)$$

$$p_2=(X)^2(Y)+2(XY)(X)-(X^2)(Y)-2(X^2Y)$$

respectively. Writing

$$C_2(X)C_1(Y)=\alpha C_3^{2,1}+\beta p_1+\gamma p_2$$

we readily obtain $\alpha=3$, $\beta=6/5$, $\gamma=-2/5$, so that $C_{21}^{2,1}$ is proportional to $3p_1-p_2$, i.e. to

$$(X)^2(Y)-(XY)(X)+2(X^2)(Y)-2(X^2Y) .$$

Similarly, $C_{21}^{1^2,1}$ is found to be proportional to p_2. All four polynomials are seen to be mutually $\Delta_{2,1}$-orthogonal.

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