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STABILITY THEOREMS FOR SOME CHARACTERIZATIONS OF THE EXPONENTIAL DISTRIBUTION

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Summary

Stability theorems are derived for various characterizations of the exponential distribution. In particular, we utilize a method which, to some extent, unifies the proof of stability for a wide class of characterizations.

1. Introduction

In their book, Galambos and Kotz [3] pointed out many open problems concerning the stability of various characterizations of the exponential distribution. Very recently, Shimizu [8] has produced results in this area, as a by-product of the study of functional equations.

Here, we also consider these problems, and derive stability theorems using a straightforward approach. In particular, we use a method, reminiscent of the proof of Gronwall's inequality (cf. Hartman [4]), which seems to unify the derivation of stability theorems for a fairly large class of characterizations.

We should also mention that as regards the characterizations which both Shimizu [8] and ourselves consider, the results do not appear to be directly comparable since our assumptions seem to be weaker than his, as are our conclusions.

2. Stability of characterizations involving order statistics

Throughout the paper, ε is a positive real number. X will be a non-negative random variable having a continuous distribution function $F(x)$ where $F(0)=0$; invariably, we use $G(x)\equiv 1-F(x)$. Also, $X_{k,n}$ denotes the kth order statistic in a random sample $X_1, X_2, \cdots, X_n$ from X.

We first need a

DEFINITION. The random variables (or vectors) U and V are ε-independent (in the sense of Ornstein), if

$$|\mathrm{P}(U \in A \mid V \in B) - \mathrm{P}(U \in A)| < \varepsilon \tag{2.1}$$

for all measurable subsets A and B of the probability spaces on which U and V are defined respectively.

This definition is motivated by the work of Ornstein [7] (cf. Smorodinsky [9], pp. 22–23). It is worth noting that the property of ε-independence is not symmetric in U and V; this obstacle can be easily removed if necessary.

Our first result is to establish the stability of a theorem of Fisz [2] (cf. [3], p. 46).

THEOREM 1. *Suppose that $F(x)$ is strictly increasing for $x \geqq 0$, and that $X_{2,2} - X_{1,2}$ and $X_{1,2}$ are ε-independent. Then, there exists $b > 0$ such that*

$$\sup_{x \geqq 0} |G(x) - \exp(-bx)| < 4\varepsilon + (2\varepsilon)^{1/2}. \tag{2.2}$$

PROOF. Since

$$\mathrm{P}(X_{2,2} - X_{1,2} < x \mid X_{1,2} = z) = (G(z) - G(x+z))/G(z)$$

for all $x \geqq 0$ and almost all $z > 0$, then the ε-independence of $X_{2,2} - X_{1,2}$ and $X_{1,2}$ amounts to saying that

$$|H(x) - 1 + (G(x+z)/G(z))| < \varepsilon \tag{2.3}$$

for all $x \geqq 0$, almost all $z > 0$, where $H(x)$ is the distribution function of $X_{2,2} - X_{1,2}$. Letting $z \to 0$ in (2.3), we obtain $|H(x) - 1 + G(x)| \leqq \varepsilon$ for all $x \geqq 0$, which when combined with (2.3) using the triangle inequality shows that

$$\left| G(x) - \frac{G(x+z)}{G(z)} \right| < 2\varepsilon, \tag{2.4}$$

that is, that $G(x)$ 'almost' satisfies the lack of memory property. The conclusion follows from Azlarov ([1], Theorem 1).

This result can be generalized (cf. Kagan, Linnik and Rao [5]) in several ways.

THEOREM 2. *For $1 \leqq k < n$, if $X_{k+1,n} - X_{k,n}$ and $X_{k,n}$ are ε-independent, where $F(x)$ is strictly increasing, then there exists $b > 0$ such that*

$$\sup_{x \geqq 0} |G(x) - \exp(-bx)| < ((2\varepsilon)^{1/2} + 4\varepsilon)^{1/(n-k)}. \tag{2.5}$$

PROOF. Since for all $x \geqq 0$, and almost all $z > 0$,

$$\mathrm{P}\,(X_{k+1,n}-X_{k,n}<x \mid X_{k,n}=z)=1-(G(x+z)/G(z))^{n-k}\,,$$

then we can proceed as before to deduce that

$$|(G(x))^{n-k}-(G(x+z)/G(z))^{n-k}|<2\varepsilon\,,$$

so that there exists $b>0$ satisfying

$$\sup_{x\geqq 0}|(G(x))^{n-k}-\exp(-(n-k)bx)|<(2\varepsilon)^{1/2}+4\varepsilon\,. \tag{2.6}$$

From (2.6), (2.5) follows easily.

COROLLARY 1. *Let $U_1=X_{1,n}$, $U_k=X_{k,n}-X_{1,n}$, $k=2,\cdots,n$. With the same assumptions on F, if the vector $\boldsymbol{U}=(U_2,U_3,\cdots,U_n)$ and U_1 are ε-independent, then there exists $b>0$ such that*

$$\sup_{x\geqq 0}|G(x)-\exp(-bx)|<(4\varepsilon+(2\varepsilon)^{1/2})^{1/(n-1)}\,. \tag{2.7}$$

PROOF. If $\boldsymbol{U}$ and U_1 are ε-independent, then so are U_2 and U_1. Thus, (2.7) holds by Theorem 2.

3. Stability of characterizations via conditional expectations

Our main result in this section is to determine the stability of a theorem of Laurent [6] (cf. Galambos and Kotz [3], p. 31) under hypotheses weaker than those in the literature. For the rest of the paper, we will assume that the random variable X has a finite mean μ.

THEOREM 3. *Let $g(x)$ be a continuous decreasing function for $0\leqq x<\infty$, and further, let there be a positive constant c such that $g(x)\geqq c$ for all $x\geqq 0$. If for all $x\geqq 0$,*

$$|\mathrm{E}\,(X-x \mid X\geqq x)-g(x)|<\varepsilon\,, \tag{3.1}$$

where $0<\varepsilon<c$, then

$$\sup_{x\geqq 0}\left|G(x)-\frac{\mu}{g(x)}\exp\left(-\int_0^x\frac{dy}{g(y)}\right)\right|<\frac{\varepsilon\mu(\mu+2\varepsilon)}{c^2(c-\varepsilon)}\,. \tag{3.2}$$

PROOF. We note that to choose $\varepsilon<c$ is not unnatural, since we are only interested in small ε. To start the proof, since

$$\mathrm{E}\,(X-x \mid X\geqq x)=\int_x^\infty G(y)dy/G(x)\,,$$

then (3.1) is equivalent to

$$(g(x)-\varepsilon)G(x)<H(x)=\int_x^\infty G(y)dy<(g(x)+\varepsilon)G(x) \tag{3.3}$$

for all $x \geqq 0$. As G is continuous, then H is differentiable, and $H'(x) = -G(x)$. Thus, from (3.3), we get

$$-(g(x)-\varepsilon)^{-1} < H'(x)/H(x) < -(g(x)+\varepsilon)^{-1}, \qquad x \geqq 0 \tag{3.4}$$

which when integrated over the interval $(0, x)$, shows that for $x \geqq 0$,

$$\mu \exp\left(-\int_0^x (g(y)-\varepsilon)^{-1} dy\right) < H(x) < \mu \exp\left(-\int_0^x (g(y)+\varepsilon)^{-1} dy\right). \tag{3.5}$$

Combining (3.3) and (3.5), we have

$$\frac{\mu}{g(x)+\varepsilon} \exp\left(-\int_0^x \frac{dy}{g(y)-\varepsilon}\right) < G(x) < \frac{\mu}{g(x)-\varepsilon} \exp\left(-\int_0^x \frac{dy}{g(y)+\varepsilon}\right). \tag{3.6}$$

At this point, some remarks are in order. A quick perusal of the proof so far shows that it is possible to set $\varepsilon = 0$; then (3.6) becomes an equality, and we have proved Laurent's theorem without assuming as in [6] and [3], that g is differentiable. To continue, we need an upper bound for the function

$$v(x) = \frac{\mu}{g(x)-\varepsilon} \exp\left(-\int_0^x \frac{dy}{g(y)+\varepsilon}\right) - \frac{\mu}{g(x)} \exp\left(-\int_0^x \frac{dy}{g(y)}\right), \qquad x \geqq 0.$$

Since

$$v(x) = \frac{\mu}{g(x)} \exp\left(-\int_0^x \frac{dy}{g(y)+\varepsilon}\right)\left\{\frac{g(x)}{g(x)-\varepsilon} - \exp\left(-\varepsilon \int_0^x \frac{dy}{g(y)(g(y)+\varepsilon)}\right)\right\}$$

and by $g(x)$ being decreasing,

$$\begin{aligned} \exp\left(-\varepsilon \int_0^x \frac{dy}{g(y)(g(y)+\varepsilon)}\right) &\geqq \exp\{-\varepsilon x / g(x)(g(x)-\varepsilon)\} \\ &\geqq 1 - (\varepsilon x / g(x)(g(x)-\varepsilon)), \end{aligned}$$

then we find that

$$v(x) \leqq \varepsilon\mu \exp\left(-\int_0^x \frac{dy}{g(y)+\varepsilon}\right)(x+g(x))/g(x)^2(g(x)-\varepsilon). \tag{3.7}$$

By (3.1), $\mu + \varepsilon > g(0) \geqq g(x) \geqq c$ for all x; hence, from (3.7),

$$\begin{aligned} v(x) &\leqq \varepsilon\mu(x+\mu+\varepsilon) \exp(-(\mu+2\varepsilon)^{-1}x)/c^2(c-\varepsilon) \\ &< \varepsilon\mu(\mu+2\varepsilon)/c^2(c-\varepsilon). \end{aligned} \tag{3.8}$$

Similarly, it is easy to check that

$$\begin{aligned} u(x) &= \frac{\mu}{g(x)+\varepsilon} \exp\left(-\int_0^x \frac{dy}{g(y)-\varepsilon}\right) - \frac{\mu}{g(x)} \exp\left(-\int_0^x \frac{dy}{g(y)}\right), \qquad x \geqq 0 \\ &> -\varepsilon\mu(\mu+2\varepsilon)/c^2(c-\varepsilon), \end{aligned}$$

which together with (3.8) proves (3.2).

The special case $g(x)\equiv c$ has been treated using different methods by Azlarov [1].

COROLLARY 2. *Suppose that* $\hat{X}=\sum_1^n \alpha_i X_i$, $n>1$, *where* $\alpha_1, \cdots, \alpha_n$ *are real numbers with* $\sum_1^n \alpha_i=1$, *and*

$$\sup_{x\geq 0} |\mathrm{E}\,(\hat{X}-x|X_{1,n}=x)-c|<(n-1)\varepsilon/n\,, \qquad 0<\varepsilon<c\,. \tag{3.9}$$

Then

$$\sup_{x\geq 0}\left|G(x)-\exp\left(-\frac{(n-1)x}{nc}\right)\right|<\frac{2\varepsilon\mu(\mu+2\varepsilon)}{c^2(c-\varepsilon)}\,. \tag{3.10}$$

PROOF. It is easy to check as in [3] (Theorem 3.4.6), that

$$\mathrm{E}\,(\hat{X}|X_{1,n}=x)=\frac{x}{n}+\frac{(n-1)}{nG(x)}\int_x^\infty ydF(y)\,.$$

Thus, (3.9) becomes

$$\left|\frac{nc}{(n-1)}G(x)-\int_x^\infty G(y)dy\right|<\varepsilon G(x) \tag{3.11}$$

for all $x\geqq 0$. Comparing (3.11) with (3.3), we find that

$$\sup_{x\geq 0}\left|G(x)-\frac{(n-1)\mu}{nc}\exp\left(-\frac{(n-1)x}{nc}\right)\right|<\frac{\varepsilon\mu(\mu+2\varepsilon)}{c^2(c-\varepsilon)}\,. \tag{3.12}$$

Since $G(0)=1$, then (3.12) implies that

$$\left|1-\frac{(n-1)\mu}{nc}\right|<\frac{\varepsilon\mu(\mu+2\varepsilon)}{c^2(c-\varepsilon)}\,, \tag{3.13}$$

which along with (3.12) and the triangle inequality leads to (3.10).

Finally, we note that the methods used in this section can also be used to obtain stability theorems for other characterizations such as those in [5] (Theorem 13.6.4), [3] (Theorem 3.4.5) and [8] (Theorem 4) since in all these cases, the hypotheses imply an inequality similar to (3.3). We omit the details.

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