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THE GENERALIZED HYPERGEOMETRIC FAMILY OF DISTRIBUTIONS

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Summary

This is an expository summary of the authors' report on classification of the generalized hypergeometric (GHg for short) family of distributions (Sibuya and Shimizu (1981), *Keio Science and Technology Report*, to appear). Emphasis is laid on the definition of the distributions based on some conventional rules, and on the complete classification of the multivariate GHg distributions, whose types are found to be rather limited in spite of their quite general definition. Previous classifications and namings are summarized and compared with the new one.

1. Generalization of the Hg distributions

We start with the analysis of simple univariate hypergeometric (Hg for short) distributions. The ordinary Hg distributions are defined by

$$p(x)=\binom{M}{x}\binom{N-M}{n-x}\Big/\binom{N}{n}=\binom{n}{x}\binom{N-n}{M-x}\Big/\binom{N}{M}, \tag{1.1}$$

where N, M and n are positive integers and x is an integer such that $\max(0, n+M-N)\leqq x\leqq \min(n, M)$. It is called hypergeometric because $p(x)$ is written as

$$p(x)=\frac{(N-M)!(N-n)!}{(N-M-n)!N!}\,\frac{(-M)_x(-n)_x}{(N-M-n+1)_x x!}, \tag{1.2}$$

where $(a)_x$ is the ascending factorial product

$$(a)_x=a(a+1)\cdots(a+x-1),$$

and the factor depending on x is a term of the Hg series

$$F(\alpha, \beta; \gamma; z)=\sum_{x=0}^{\infty}\frac{(\alpha)_x(\beta)_x}{(\gamma)_x x!}z^x,$$

with $z=1$. For real parameters $F(\alpha, \beta; \gamma; 1)$ is absolutely convergent if and only if $\gamma-\alpha-\beta>0$ and in this case we have

$$F(\alpha, \beta; \gamma; 1)=\frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}. \tag{1.3}$$

If α or β is equal to $-m$, a negative integer (when both α and β are negative integers as in (1.2), $-m$ is the larger one), then the series is finite and defined unless γ is a negative integer smaller than $-m$, and (1.3) is valid for this case also provided that a rule (1.9) in later discussion is applied where necessary.

The parameter values in expression (1.1) are easily extended since the binomial coefficient is usually defined, for any real a and any integer x, by

$$\binom{a}{x}=\frac{a^{(x)}}{x!}=\frac{a(a-1)\cdots(a-x+1)}{x!}, \quad \text{if } x\geqq 0; \qquad \text{and } =0, \text{ if } x<0.$$

So that

$$p(x)=\binom{a}{x}\binom{b}{n-x}\Big/\binom{a+b}{n}, \qquad x=0, 1, \cdots, n, \tag{1.4}$$

is a formal generalization of Hg distributions. If $a, b>n-1$, then this is the positive Hg distribution, which includes the ordinary Hg as a special case, and

$$\begin{aligned} p(x)&=\binom{-a}{x}\binom{-b}{n-x}\Big/\binom{-a-b}{n} \\ &=\binom{a+x-1}{x}\binom{b+n-x-1}{n-x}\Big/\binom{a+b+n-1}{n}, \\ &\qquad a, b>0;\ x=0, 1, \cdots, n, \end{aligned} \tag{1.5}$$

is the negative Hg distribution. On the other hand, expressions (1.2) and (1.3) suggest another generalization of Hg based on the form

$$p(x)=\frac{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}\frac{(\alpha)_x(\beta)_x}{(\gamma)_x x!}, \tag{1.6}$$

making the parameters integer and real, and positive and negative. If (1.6) is formally put in the form of (1.4) following the correspondence between (1.1) and (1.2), then

$$p(x)=\binom{-\alpha}{x}\binom{\gamma-\beta-1}{-\beta-x}\Big/\binom{\gamma-\alpha-\beta-1}{-\beta}, \tag{1.7}$$

where $\alpha=-a$, $\beta=-n$ and $\gamma=b-n+1$ of (1.4).

Now we observe the difference between the two forms (1.4) and

(1.6). The case of (1.6) where all the parameters are positive and $\gamma-\alpha-\beta>0$, and $x=0, 1, 2, \cdots$, provides a natural definition of distributions, but in the form of (1.4) or (1.7) this case requires an extension of the binomial coefficient allowing a negative lower argument. So (1.6) looks general enough comparing (1.7) without such an extension. Moreover the symmetry of (1.6) with respect to α and β is not well reflected in (1.7). In the negative Hg (1.5), however, if $-b$ is a negative integer, then its form in (1.6) has gamma functions with negative integer argument, $\Gamma(\gamma-\beta)$ and $\Gamma(\gamma)$. The same trouble occurs if we try to extend the binomial coefficient using the expression

$$\binom{a}{x}=\frac{a(a-1)\cdots(a-x+1)}{x!}=\frac{\Gamma(a)}{\Gamma(a-x)\Gamma(1+x)}, \tag{1.8}$$

and allowing x to be negative. The trouble is solved at least for the negative Hg case if we define, for nonnegative integers m and n,

$$\frac{\Gamma(-m)}{\Gamma(-n)}=\lim_{\varepsilon\to 0}\frac{\Gamma(-m+\varepsilon)}{\Gamma(-n+\varepsilon)}=(-1)^{m-n}\frac{\Gamma(n+1)}{\Gamma(m+1)}. \tag{1.9}$$

The first expression is equal to the last one if m and n are nonintegers and $m-n$ is an integer.

Further, in the negative Hg (1.5), if $b=1$ or $\beta=\gamma=-n$ in (1.6), then the probability function becomes, with the help of (1.9),

$$p(x)=\frac{\Gamma(-n-\alpha)\Gamma(0)}{\Gamma(-\alpha)\Gamma(-n)}\frac{(\alpha)_x(-n)_x}{(-n)_x x!}=\frac{\Gamma(\alpha+1)n!}{\Gamma(\alpha+n+1)}\frac{(\alpha)_x}{x!},$$

and the factor depending on x in the last expression is independent of n, which means that the range of the distribution is not determined naturally unless the factor $(-n)_x/(-n)_x$ is kept uncanceled. This fact will be reconsidered in Definition 1. Returning to the ordinary Hg, we find another gap between its two forms (1.1) and (1.2). The ordinary Hg is naturally determined by (1.1) when $n+M-N>0$ and the range of distribution is $[n+M-N, \min(n, M)]$. But the factor $(N-M-n+1)_x$ in the denominator of (1.2) vanishes in the range. Form (1.6) defines distributions on the intervals $[0, n]$ or $[0, \infty)$, and the possible parameter values are easily specified as we shall see later. To make (1.6) cover the distributions on the other intervals as above, we have to extend $(a)_x$ allowing x to be a negative integer by applying (1.9) to the relation

$$(a)_x=\Gamma(a+x)/\Gamma(a).$$

Based on the above observations, we define the univariate GHg family of distributions covering typical ones which are naturally defined by (1.4) or (1.6).

DEFINITION 1. A univariate GHg distribution, denoted by $F(\alpha, \beta; \gamma)$, is the one satisfying the following three conditions (i)–(iii).
(i) It has the probabilities of the form

$$p(x)=\frac{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}{\Gamma(\gamma-\alpha-\beta)\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+x)\Gamma(\beta+x)}{\Gamma(\gamma+x)\Gamma(1+x)}, \tag{1.10}$$

or

$$p(x)=\frac{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(\gamma-\alpha-\beta)\Gamma(1+x)\Gamma(\gamma+x)\Gamma(1-\alpha-x)\Gamma(1-\beta-x)}. \tag{1.11}$$

If there is a pair of gamma functions with negative integer argument one in the numerator and the other in the denominator, then we let the arguments positive applying the rule of (1.9). If the numbers of gamma functions with negative integer argument in the numerator and the denominator are unbalanced, then the probability is undefined.
(ii) The probabilities $p(x)$'s are defined and positive on a finite or infinite integer interval $[\mu, \nu]$ and their sum over it is one. $p(x)$ is zero or undefined on $x=\mu-1$ or $\nu+1$ if they are finite.
(iii) If there are two parameters with the same integer value ν, or there is a parameter with the value $\nu=1$, then the distribution might be defined for both $x\leqq-\nu$ and $x\leqq-\nu+1$. We regard it, however, as a distribution on either side and not on both sides.

Condition (ii) excludes the range $[\mu, \nu]$ containing $[-1, 0]$, that is, integers of the interval must be all nonnegative or all negative. In fact, if the factor $1/\Gamma(1+x)$ in (1.10) or (1.11) is defined for $x\leqq-1$ applying (1.9) in pair with a gamma function with negative integer argument in the numerator, then the $p(x)$ remains undefined for $x\geqq0$. If there is no such gamma function in the numerator, then $1/\Gamma(1+x)$ is defined for $x\geqq0$ but not for $x\leqq-1$. Similar argument shows that one of the parameters must be $-\nu$ unless $\nu=\infty$, and that another parameter must be $-\mu+1$ unless $\mu=-\infty$ or $\mu=0$. Condition (iii) overrides Condition (ii) to include $F(\alpha, -n; -n)$ on $[0, n]$ in our family.

THEOREM 1. *All the univariate* GHg *distributions of Definition* 1 *are classified as in Table* 1, *and only the types of distributions of Table* 1 *are possible.*

Basically, there are five types of GHg: A1, A2, B1, B2 and B3. Type A's have finite range $[0, n]$, A1 is the positive Hg and A2 the negative Hg, which is also called Markov-Pólya, etc. Type B's have infinite range $[0, \infty)$. B1 is unimodal and its variance is smaller than

Table 1. Classification of the univariate GHg family of distributions, Definition 1.

Range	Type	Distribution	Restriction	Name (Kemp and Kemp type)
$[0, n]$	A1	$F(-\xi, -n; \zeta)$	$\xi > n-1$	Positive Hg (I A)
	A2	$F(\xi, -n; -\zeta)$	$\zeta > n-1$	Negative Hg, Markov-Pólya, Pólya-Eggenberger, binomial beta (II A, III A)
$[0, \infty)$	B1	$F(-n+\varepsilon, -n+\delta; \zeta)$	—	— (I B)
	B2	$F(\varepsilon, -n+\delta; -n+\rho)$	$\rho > \varepsilon+\delta$	— (II B, III B)
	B3	$F(\xi, \eta; \zeta)$	$\zeta > \xi+\eta$	Inverse Markov-Pólya, Inverse Pólya-Eggenberger, generalized Waring, negative binomial beta (IV)
				Relation with the above types
$[m, n]$	$A1^+$	$F(-\xi, -n; -m+1)$	$\xi > n-1$	Right m shift of A1: $F(-\xi+m, -n+m; 1+m)$ on $[0, n-m]$ or inversion $(Y=n-X)$ of A1: $F(-n, -n+m; \xi-n+1)$ on $[0, n-m]$
$[m, \infty)$	$B1^+$	$F(-m-n+\delta, -m-n+\varepsilon; -m+1)$		Right m shift of B1: $F(-n+\delta, -n+\varepsilon; 1+m)$ on $[0, \infty)$
	$B3^+$	$F(\xi-m, \eta-m; -m+1)$	$m+1 > \xi+\eta$	Right m shift of B3: $F(\xi, \eta; 1+m)$ on $[0, \infty)$
$[-n, -m]$	$A2^-$ *	$F(\xi, m; n+1)$	$\xi > n$	Left n shift of A2: $F(\xi-n, -n+m; -n+1)$ on $[0, n-m]$ or inversion $(Y=-X-m)$ of A2: $F(m, -n+m; -\xi+m+1)$ on $[0, n-m]$
$(-\infty, -m]$	$B3^-$ *	$F(-\zeta+m+1, m; m+1-\xi)$	$\zeta > \xi+m$	Inversion $(Y=-X-m)$ of B3: $F(\xi, m; \zeta)$ on $[0, \infty)$
	$B3^-$	$F(1, 1; m)$	$m=3, 5, \cdots$	Inversion $(Y=-X-m)$ of B3: $F(1, 1; m)$ on $[0, \infty)$
$[m, \infty)$	C	$F(\varepsilon, -m+1; -k+\varepsilon)$; $k=1, 2, 3, \cdots$; $m=k+3, k+5, \cdots$ for (1.10), $m=k+2, k+4, \cdots$ for (1.11), $\varepsilon=\varepsilon(k, m)$, $0<\varepsilon<1$, is uniquely determined as follows.		

$m-k$ \ k	1	2	3	4	5
2	.56155	.43484	.37228	.33406	.30784
3	.5	.33333	.25	.2	.16667
4	.46293	.27164	.17843	.12568	.09297
5	.43775	.23027	.13328	.08257	.05400
⋮	⋯	⋯	⋯	⋯	⋯

m, n, k: Positive integers

ξ, η, ζ : Positive real numbers

$\delta, \varepsilon, \rho$: Real numbers on $(0, 1)$

* The form of (1.11) only.

or equal to its mean if they exist, B2 is unimodal or bimodal and its mean is infinite, and B3 is the type called inverse Markov-Pólya, etc. The distributions on the other intervals, $A1^+$, $B1^+$, $B3^+$, $A2^-$ and $B3^-$, are shift or inversion of the basic five types. A random variable of these types is expressed as $X \pm m$ or $-X \pm m$ using a random variable X of the basic types. Type C is a very special one.

The classification into the basic five types is essentially due to Kemp and Kemp [4] who disregarded, however, other possibilities. Our classification is simpler and complete, but there are some unpleasant facts in Table 1: (i) There are Type C distributions on $[m, \infty)$ defined only for isolated special values of parameters. (ii) For some parameter values, forms (1.10) and (1.11) differ by a factor -1, and (1.10) cannot be probabilities while (1.11) defines a distribution. (iii) $F(\varepsilon, m; \rho)$, $0<\varepsilon<\rho<1$ and $F(1, 1; m)$, $m=3, 5, 7, \cdots$ are distributions on both $[0, \infty)$ and $(-\infty, -m]$. Our definition is a compromise to cover the distributions of two closely related but different forms, is based on conventional rules, and cannot be free from such defects.

2. Multivariate GHg distributions

The multivariate ordinary Hg distributions are defined by

$$(2.1) \quad p(\boldsymbol{x})=p(x_1, \cdots, x_q)=\left[\prod_{j=1}^{q}\binom{M_j}{x_j}\right]\binom{N-\sum M_j}{n-\sum x_j}\Big/\binom{N}{n}$$

$$=\frac{(N-n)!(N-\sum M_j)!}{N!(N-n-\sum M_j)!}\frac{(-n)_{\sum x_j}}{(N-n-\sum M_j)_{\sum x_j}}\prod_{j=1}^{q}\frac{(-M_j)_{x_j}}{x_j!},$$

where M_j's, n and N are positive integers and x_j's are integers such that $0 \leqq x_j \leqq M_j$ and $\max(0, n+\sum M_j - N) \leqq \sum x_j \leqq n$. Therefore, multivariate version of (1.4) and (1.6) are

$$(2.2) \quad p(\boldsymbol{x})=\left[\prod_{j=1}^{q}\binom{a_j}{x_j}\right]\binom{b}{n-\sum x_j}\Big/\binom{\sum a_j+b}{n},$$

and

$$(2.3) \quad p(\boldsymbol{x})=\frac{\Gamma(\omega-\lambda)\Gamma(\omega-\sum \alpha_j)(\lambda)_{\sum x_j}}{\Gamma(\omega-\lambda-\sum \alpha_j)\Gamma(\omega)(\omega)_{\sum x_j}}\prod_{j=1}^{q}\frac{(\alpha_j)_{x_j}}{x_j!},$$

respectively. Discussions on these forms are quite parallel to those in the univariate case, and our definition of multivariate GHg is similar to that of univariate GHg.

DEFINITION 2. A q-variate GHg distribution, denoted by $F(\boldsymbol{\alpha}; \lambda; \omega)$, $\boldsymbol{\alpha}=(\alpha_1, \cdots, \alpha_q)$, is the one satisfying the following three conditions (i)–(iii).

(i) It has probabilities of the forms

$$(2.4)\qquad p(\boldsymbol{x})=\frac{\Gamma(\omega-\lambda)\Gamma(\omega-\sum\alpha_j)\Gamma(\lambda+\sum x_j)}{\Gamma(\omega-\lambda-\sum\alpha_j)\Gamma(\lambda)\Gamma(\omega+\sum x_j)}\prod_{j=1}^{q}\frac{\Gamma(\alpha_j+x_j)}{\Gamma(\alpha_j)\Gamma(1+x_j)},$$

or

$$(2.5)\qquad p(\boldsymbol{x})=\frac{\Gamma(\omega-\lambda)\Gamma(\omega-\sum\alpha_j)\Gamma(1-\lambda)}{\Gamma(\omega-\lambda-\sum\alpha_j)\Gamma(\omega+\sum x_j)\Gamma(1-\lambda-\sum x_j)}\times\prod_{j=1}^{q}\frac{\Gamma(1-\alpha_j)}{\Gamma(1-\alpha_j-x_j)\Gamma(1+x_j)}.$$

The gamma functions with negative integer argument are treated in the same way as in Definition 1.
(ii) The probabilities $p(\boldsymbol{x})$'s are defined and positive on a connected discrete region of an orthant, and undefined or zero on its neighboring outside points, and their sum over it is one.
(iii) The distribution range of (ii) is not degenerate into a lower dimensional space. This implies that the range consists of at least two adjacent values of $\sum x_j=\nu$ and $\nu+1$, say, and of at least q adjacent $(q-1)$-dimensional points on $\sum x_j=\nu$ or $\nu+1$.

Condition (ii) means that the range of a distribution cannot cover the points with $x_j=0$ and $x_j=-1$ because of the factor $1/\Gamma(1+x_j)$, therefore the range must be within an orthant (the axes are regarded as positive side points). The boundaries of the range are, by similar observation, $0\leqq x_j$, $x_j\leqq m_j$ $(m_j>0)$ or $x_j\leqq -m_j$ $(m_j>0)$, $j=1,2,\cdots,q$; or $0<n\leqq\sum x_j$, $\sum x_j\leqq m$ $(m>0)$ or $\sum x_j\leqq -m$ $(m>0)$, if there are corresponding integer valued parameters.

The sum of the components is a univariate GHg $\mathrm{F}(\sum\alpha_j,\lambda;\omega)$ and the distribution of any component X_i, given their sum equal to s, is also a univariate GHg $\mathrm{F}(\alpha_i,-s;\sum\alpha_j-\alpha_i+s-1)$. Thus from earlier considerations as given in Table 1, the possible parameter values and possible distribution ranges of multivariate GHg are quite limited.

THEOREM 2. *All the multivariate* GHg *distributions of Definition* 2 *are classified as in Table* 2, *and only the types of distributions of Table* 2 *are possible.*

There are four kinds of distribution ranges; finite or infinite ones in the nonnegative or the negative orthant. The distributions in the negative orthant are inverses of those in the nonpositive orthant. Figure 1 shows distribution ranges of the types in Table 2.

Types #1–#4 are generalization of the univariate positive Hg (Types A1 or A1$^+$), but a little more complicated. A multivariate ordinary Hg (2.1) belongs to the intersection of #1 and #2 if $n+\sum M_j<N$, and

Table 2. Classification of the multivariate GHg family of distributions, Definition 2.

No.	Type of distributions				F $(\boldsymbol{\alpha}; \lambda; \omega)$	Condition	Distribution range	Name
	$S=\sum X_j$	$X_i \mid S=s$	X_i	$X_i \mid X_j=x_j$				
							[finite]	
# 1	A1	A1	A1	A1	$F(-\boldsymbol{\xi}; -n; \zeta)$	$\xi_j>n-1$ or pos. integer	$0\leqq \sum x_j \leqq n$	MHg/MP
# 2	A1	A1/A1$^+$	A1	A1	$F(-\boldsymbol{k}; -\lambda; \zeta)$	$\lambda>\sum k_j-1$	$0\leqq x_j \leqq k_j$	—
# 3	A1$^+$	A1	A1	A1	$F(-\boldsymbol{\xi}; -n; 1-m)$	$\xi_j>n-1$ or pos. integer	$m\leqq \sum x_j \leqq n$	MHg/MP
# 4	A1$^+$	A1/A1$^+$	A1	A1	$F(-\boldsymbol{k}, -\lambda; 1-m)$	$\lambda>\sum k_j-1$	$0\leqq x_j\leqq k_j,\ m\leqq \sum x_j$	—
# 5	A2	A2	A2	A2	$F(\boldsymbol{\xi}; -n; -\zeta)$	$\zeta>n-1$	$0\leqq \sum x_j\leqq n$	MNHg/MP
# 6	A2	A1/A1$^+$	A2	A2	$F(-\boldsymbol{k}; \lambda; -\zeta)$	$\zeta>\sum k_j-1$	$0\leqq x_j\leqq k_j$	MIHg/MIP
							[infinite]	
# 7	B2	A2	B2	B2	$F(\boldsymbol{\delta}; -n+\sigma; -n+\rho)$	$\sum \delta_j+\sigma<\rho$	$0\leqq x_j<\infty$	—
# 8	B3	A2	B3	B3	$F(\boldsymbol{\xi}; \lambda; \zeta)$	$\sum \xi_j+\lambda<\zeta$	$0\leqq x_j<\infty$	MNIHg/MIP
# 9	B3$^+$	A2	B2	B3$^+$/B3	$F(\boldsymbol{\delta}; -m+\sigma; -m+1)$	$\sum \delta_j+\sigma<1$	$m\leqq \sum x_j<\infty$	—
#10	C	A2	*	*/B3	$F(\boldsymbol{\delta}; -m+1; -k+\sum \delta_j)$	$\sum \delta_i=\varepsilon(m, k)$	$m\leqq \sum x_j<\infty$	—
							[negative finite]	
#11**	A2$^-$	A2$^-$	A2$^-$	A2$^-$	$F(\boldsymbol{k}; \eta; n+1)$	$\sum k_j<n<\eta$	$x_j\leqq -k_j,\ -n\leqq \sum x_j$	—
							[negative infinite]	
#12**	B3$^-$	A2$^-$	B3$^-$	B3$^-$	$F(\boldsymbol{k}; -\lambda+\sum k_j+1; -\zeta+\sum k_j+1)$	$\lambda>\zeta+\sum k_j$	$x_j\leqq -k_j$	—

See Table 1 for A1, A2, B2, etc.

$\boldsymbol{\xi}=(\xi_j)$: positive real vector; $\boldsymbol{k}=(k_j)$: positive integer vector; $\boldsymbol{\delta}=(\delta_j)$: real vector $0<\delta_j<1$.

λ, ξ: positive real numbers; m, n: positive integers; ρ, σ: real numbers on $(0, 1)$.

*: Not GHg; $\varepsilon(m, k)$: cf. Type C distribution of Table 1.

**: The form (2.5) is valid only for odd-dimensional distributions.

Names are those in Janardan-Patil [2].

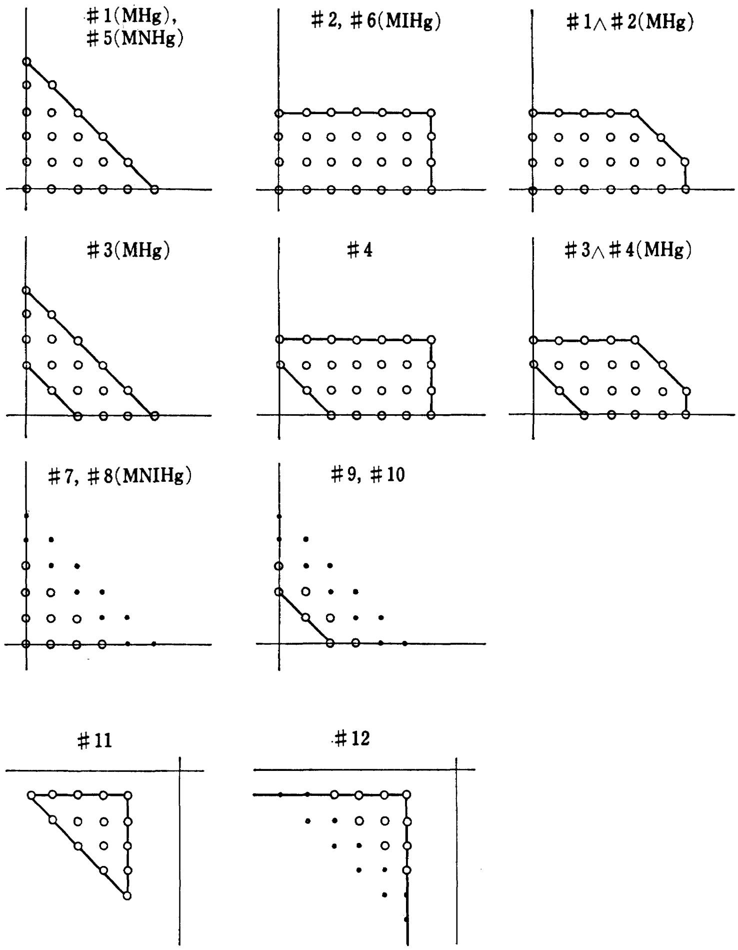

Figure 1. Distribution ranges of the Types of distributions of Table 2 (bivariate GHg).

to the intersection of #3 and #4 otherwise. More generally, a distribution of (2.2) with $a_j > n-1$ $(j=1, \cdots, q)$ belongs to #1 if $b > n-1$, and to #3 if $b = n-m$ and m is a positive integer. Let $S = \sum_{j=1}^{q} X_j$ be a positive Hg variable having probabilities (1.4) or $\mathrm{F}(-a, -n; b-n+1)$ with $a, b > n-1$ (A1 or $A1^+$). If the conditional probabilities of $\boldsymbol{X} = (X_1, \cdots, X_q)$ given $S=s$ is

$$\left[\prod_{j=1}^{q} \binom{a_j}{x_j}\right] \Big/ \binom{a}{s}, \qquad a = \sum_{j=1}^{q} a_j ,$$

that is X_i has $\mathrm{F}(-a_i, -s; a-a_i-s+1)$, then $\boldsymbol{X}$ has (2.2) or $\mathrm{F}(-\boldsymbol{a}, -n; b-n+1)$, $\boldsymbol{a} = (a_i, \cdots, a_q)$, of Type #1 or #3. While if the conditional probabilities of $\boldsymbol{X}$ given $S=s$ is

$$\left[\prod_{j=1}^{q}\binom{n_j}{x_j}\right]\Big/\binom{n}{s}, \qquad n=\sum_{j=1}^{q} n_j,$$

where n_j's are positive integers, then $\boldsymbol{X}$ has the probabilities

$$\left[\prod_{j=1}^{q}\binom{n_j}{x_j}\right]a^{(s)}b^{(n-s)}/(a+b)^{(n)}, \tag{2.6}$$

or $\mathrm{F}(-\boldsymbol{n};\,-a;\,b-n+1)$, $\boldsymbol{n}=(n_1,\cdots,n_q)$, which is of Type #2 or #4, and cannot be written in the form of (2.2). Another difference between #1 and #2, or #3 and #4, is the shape of distribution ranges: Typically, the range for Type #1 is a simplex (a triangle), and for Type #2 an interval (a quadruple). Some corners may be cut off for some parameter values, and for Types #3 and #4, a simplex including the origin $(0,\cdots,0)$ is always cut off. Moreover, Types 1 and 2, or Types 3 and 4, are not exclusive of each other as we have seen in studying multivariate ordinary Hg. Still, this is a characteristic difference between them. Types #2 and #4 are not studied much in literature.

Type #5 is a generalization of the negative Hg distributions, Type A2. It can be written as

$$p(\boldsymbol{x})=\left[\prod_{j=1}^{q}\binom{-a_j}{x_j}\right]\binom{-b}{n-\sum x_j}\Big/\binom{-\sum a_j-b}{n}, \tag{2.7}$$

and defined on a simplex $0\leqq\sum x_j\leqq n$. Type #6 is also a generalization of Type A2 to another direction. It can be written as

$$\begin{aligned}p(\boldsymbol{x})&=\left[\prod_{j=1}^{q}\binom{k_j}{x_j}\right]\frac{(b)_s(c)_{\sum k_j-s}}{(b+c)_{\sum k_j}} \\ &=\left[\binom{-b}{s}\binom{-c}{\sum k_j-s}\Big/\binom{-b-c}{\sum k_j}\right]\times\left[\prod_{j=1}^{q}\binom{k_j}{x_j}\right]\Big/\binom{\sum k_j}{s}, \\ &\qquad b,\,c>0;\; s=\sum x_j;\; k_j=1,2,\cdots;\end{aligned} \tag{2.8}$$

and defined on an interval $0\leqq x_j\leqq k_j$. The differences between Types #5 and #6 are their distribution ranges and the conditional distributions of a component X_i when the sum $\sum X_j=s$ is given, which correspond to Types A2 and A1 respectively. See also discussions in Section 3.

Types #8 and #9 are natural generalizations of Types B3 and B3$^+$ respectively. Type #7 has no good model and Type #10 is very special.

In the negative orthant, a distribution with finite (infinite) range is obtained by the inversion $Y_i=m_i-X_i$, where $(X_1,\cdots,X_q)$ is a Type #5 (Type #8) variable. If the dimension q is odd, then Types #11 and #12 are valid for the form (2.5) only.

3. Comments on classification and naming

In the univariate GHg family, Types A1, A2 and B3 are typical, and in the multivariate GHg family, #1, #3, #5, #6 and #8 are typical. There are many chance mechanisms generating these as discussed by Janardan [1], Janardan and Schaeffer [3], and the authors [5]. Some well known ones are summarized in diagrams of Figures 2 and 3 for better view of the GHg and for later discussion on naming.

Figure 2 shows how Types A1, A2 and B3 distributions are related to the binomial or the negative binomial distributions. (A): In drawing balls out of an urn with white and red balls, we delete further $c-1$ balls of the same color as sampled, or just replace each sampled ball, or return the sampled ball adding further $c-1$ balls of the same color. We draw out balls a fixed number of times and count the number of red ball samplings. (B): For the urn of (A), we continue the procedure until a fixed number of white balls are observed, and the variable is the number of red ball samplings. (C): X and Y are binomial (or negative binomial) variables of the same probability parameter and consider the distribution of X under the condition that $X+Y=s$ is given. (D): Assume the probability parameter of a binomial (or a negative binomial) distribution to be a beta variable, and consider the

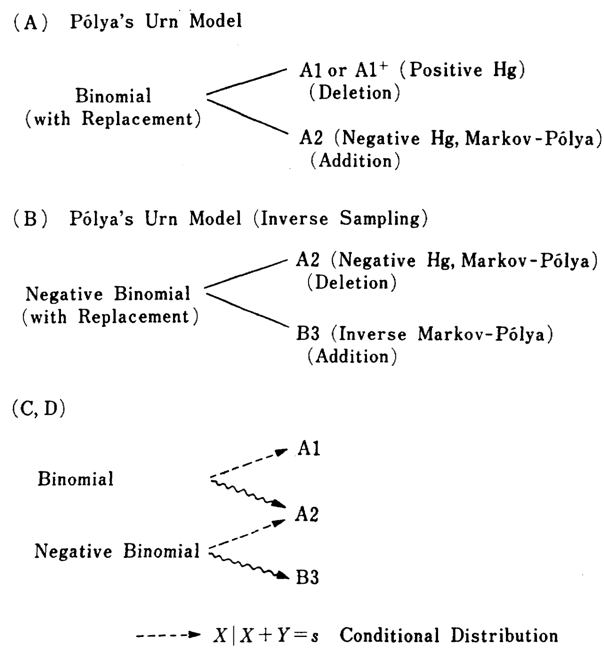

Figure 2. Generation schemata of A1, A2 and B3.

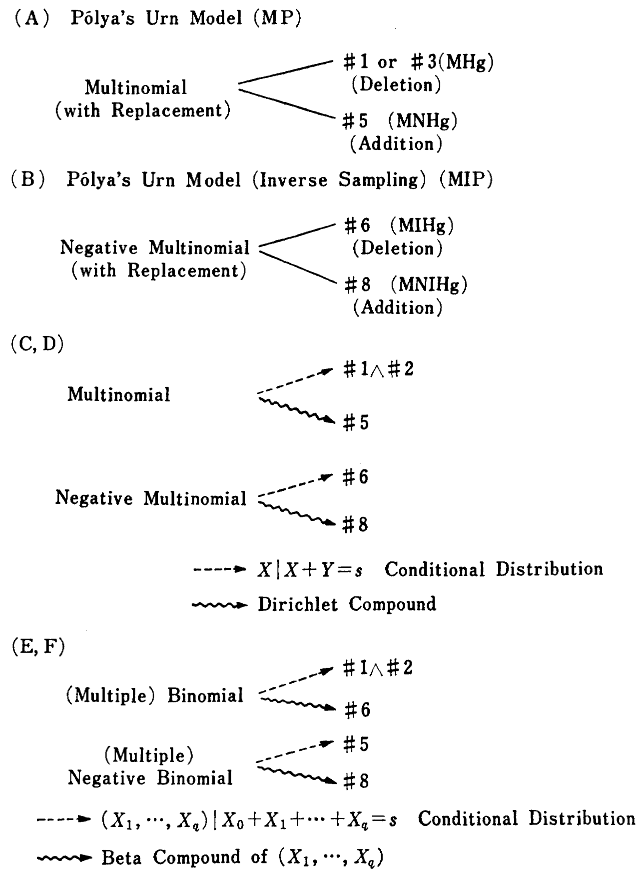

Figure 3. Generation schemata of #1, #3, #5, #6 and #8.

compound distribution. Remark parallelism between (A, B) and (C, D).

Extending Figure 2 to the multivariate case, we get Figure 3 which is a little more complicated. In (A) and (B), the urn has now white and other multiple color balls, and observe the number of balls of each color. In (C) and (D), binomial or negative binomial and beta distributions are extended to multinomial or negative multinomial and Dirichlet distributions. In all of these cases, the difference between Types #5 and #6 should be noticed, and except for this the schemata are parallel to those of Figure 1. (C) and (D) of Figure 2, however, can be generalized to another direction, that is, (E) and (F) of Figure 3, where a number of independent binomial or negative binomial distributions of the same probability parameter are considered. In (E), we consider the simultaneous distribution of components $(X_1, \cdots, X_q)$ of independent variables $(X_0, X_1, \cdots, X_q)$ under the condition that $X_0 + X_1 + \cdots + X_q = s$ is given. In (F), we consider the compound simultaneous distribution of $(X_1, \cdots, X_q)$ when the probability parameter is a beta variable. Comparing (E) and (F) with (C) and (D), we notice that the roles of ♯5 and ♯6 are exchanged.

Because of the various models and forms, the GHg distributions have many names as written in Tables and Figures. Moreover, since Pólya's urn model is initially studied by A. A. Markov and by F. Eggenberger and G. Pólya, Type A2 distributions are sometimes called after one or two of these names, and Type B3 distributions have these names preceded by 'inverse.' See discussions in Janardan and Schaeffer [3].

In Table 2 and Figures 1 and 3, names in Janardan and Patil [2] and Janardan [1] are written. Their Multivariate Pólya (Inverse Multivariate Pólya) distributions with negative or positive contagion correspond to Types ♯1$\vee$♯3 or ♯5 (♯6 or ♯8) respectively, and if a parameter value in these is limited to integer, then they are named MHg or MNHg (MIHg or MNIHg) respectively. So they propose two ways of classification and naming corresponding to each other. The latter naming is not consistent with univariate case, since the popular univariate names are inconsistent. The negative binomial distributions have a negative parameter value and correspond to inverse sampling. The negative Hg distributions have negative parameters in a form (2.2), correspond to both positive contagion and inverse sampling as shown in Figure 2. Thus, neither the term 'inverse' nor 'negative' has a definite meaning.

We do not propose here another naming system, but we believe that the whole picture of GHg and the relationship among names are now made clear.

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REFERENCES

[1] Janardan, K. G. (1973). Chance mechanisms for multivariate hypergeometric models, *Sankhyā*, A, **35**, 465-478.

[2] Janardan, K. G. and Patil, G. P. (1972). A unified approach for a class of multivariate hypergeometric models, *Sankhyā*, A, **34**, 363-376.

[3] Janardan, K. G. and Schaeffer, D. J. (1977). A generalization of Markov-Pólya distributions, its extensions and applications, *Biometrical J.*, **19**, 87-106.

* Now at Keio University.

[4] Kemp, C. D. and Kemp, A. W. (1956). Generalized hypergeometric distributions, *J. Roy. Statist. Soc.*, B, **18**, 202-211.
[5] Sibuya, M. and Shimizu, R. (1981). Classification of the generalized hypergeometric family of distributions, *Keio Science and Technology Report*, to appear.