Ann. Inst. Statist. Math.
33 (1981), Part A, 155-164

ON THE COMPARISON OF PBIB DESIGNS WITH TWO ASSOCIATE CLASSES

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(Received Dec. 25, 1979; revised Sept. 18, 1980)

Abstract

A method to compare two-associate-class PBIB designs is discussed. As an application, it is shown that if d^* is a group-divisible design with $\lambda_2=\lambda_1+1$, a group-divisible design with group size two and $\lambda_1=\lambda_2+1>1$, a design based on a triangular scheme and $\lambda_2=\lambda_1+1$, a design based on a triangular scheme, $v=10$, and $\lambda_1=\lambda_2+1$, a design with an L_2 scheme and $\lambda_2=\lambda_1+1$, a design with an L_s scheme, $v=(s+1)^2$, and $\lambda_1=\lambda_2+1$, where s is a positive integer, or a design with a cyclic scheme, $v=5$, and $\lambda_1=\lambda_2\pm1$, then d^* is optimum with respect to a very general class of criteria over all the two-associate-class PBIB designs with the same values of v, b and k as d^*. The best two-associate-class PBIB design, however, is not necessarily optimal over all designs.

1. Introduction

To control the variations in experiments for comparing several treatments, one often uses block designs. Suppose v treatments are to be compared via b blocks of size k with $k<v$. Any arrangement of the v treatments into the bk experimental units is called a *design*. For convenience, let $\Omega_{v,b,k}$ denote the collection of all such designs. The usual additive model specifies the expectation of an observation on treatment i in block j to be $\alpha_i+\beta_j$ (treatment effect+block effect), where α_i and β_j are unknown constants, $1\leq i\leq v$, $1\leq j\leq b$. Furthermore, the bk observations are assumed to be uncorrelated with common variance.

For each $d\in\Omega_{v,b,k}$, various optimality criteria are defined in terms of the coefficient matrix of the reduced normal equation for estimating the treatment effects (also called C-matrix):

This paper was prepared with the support of Office of Naval Research Contract No. N00014-75-C-0444/NR 042-036 and National Science Foundation Grant No. MCS-79-09502.

(1.1) $$C_d = \operatorname{diag}(r_{d1}, \cdots, r_{dv}) - k^{-1} N_d N_d' ,$$

where r_{di} is the number of replications of treatment i, and N_d is the treatment-block incidence matrix. Let $\mu_{d1} \geqq \mu_{d2} \geqq \cdots \geqq \mu_{d,v-1} \geqq \mu_{dv} = 0$ be the eigenvalues of C_d. This paper is mainly concerned with optimality criteria Φ_f defined by

(1.2) $$\Phi_f(C_d) = \sum_{i=1}^{v-1} f(\mu_{di}) ,$$

where f is a real-valued function defined on $[0, \max_{d \in \Omega_{v,b,k}} \operatorname{tr} C_d]$ such that

(i) f is continuous, strictly convex, and strictly decreasing on $[0, \max_{d \in \Omega_{v,b,k}} \operatorname{tr} C_d]$ (the possibility that $\lim_{x \to 0^+} f(x) = f(0) = \infty$ is allowed);

(ii) f is continuously differentiable on $(0, \max_{d \in \Omega_{v,b,k}} \operatorname{tr} C_d)$ and f' is strictly concave on $(0, \max_{d \in \Omega_{v,b,k}} \operatorname{tr} C_d)$.

Such a criterion Φ_f is called a *type 1 criterion* in Cheng [2]. A function Φ^* of C_d is called a *type 1 criterion in the wide sense* if it is of the form $F \circ \Phi_f$ where F is nondecreasing and Φ_f is a type 1 criterion. If Φ is the pointwise limit of a sequence of type 1 criteria in the wide sense, then it is called a *generalized type 1 criterion*. A design which minimizes $\Phi(C_d)$ is called *Φ-optimal*. It can be shown that the commonly used A- and D-criteria are of type 1, and the E-criterion is a generalized type 1 criterion. Note that a D-, A-, or E-optimum design minimizes $\prod_{i=1}^{v-1} \mu_{di}^{-1}$, $\sum_{i=1}^{v-1} \mu_{di}^{-1}$, or $\mu_{d,v-1}^{-1}$, respectively. Our definition of generalized type 1 criteria is slightly different from that given in Cheng [2]. Under the earlier definition, the generalized type 1 criteria did not really cover the E-criterion. We are grateful to the referee for pointing out this deficiency.

Kiefer [7] proved that if a balanced incomplete block design (BIBD) exists, then it is optimal over $\Omega_{v,b,k}$ with respect to a very general class of criteria including all the generalized type 1 criteria mentioned above. Bose and Nair [1] proposed the use of partially balanced incomplete block designs (PBIBD) when a BIBD does not exist. Among the PBIB designs, a group divisible (GD) design with two groups and $\lambda_2 = \lambda_1 + 1$ has been shown by Cheng [2] to be optimal over $\Omega_{v,b,k}$ with respect to any generalized type 1 criterion. When such designs also fail to exist, the problem of finding an optimal design in $\Omega_{v,b,k}$ turns out to be very difficult. However, since practically the PBIBD's with two associate classes are the most popular designs next to BIBD's, and quite often there are more than one such design in $\Omega_{v,b,k}$, it is worthwhile investigating how to choose the best one when they are not unique. As will be seen later, this kind of comparison is made possible by the fact that for a PBIBD with two associate classes, C_d has only two distinct nonzero eigenvalues. Note that tables of PBIBD's with two associate classes have been prepared by Clatworthy [4].

As usual, we write $r \equiv bk/v$, the number of replications of each treatment in a PBIBD. For a two-associate-class PBIB design d, define λ_{di} as the number of times a treatment appears together with any ith associate of it, and write n_{di} as the number of ith associates of each treatment, $i=1, 2$. These notations are the same as the commonly used ones except that the letter d is inserted to emphasize that they depend on the design being considered. Sometimes it is suppressed if there is no ambiguity. For convenience, $\max\{\lambda_{d1}, \lambda_{d2}\}$ and $\min\{\lambda_{d1}, \lambda_{d2}\}$ are denoted as $\lambda_{d[1]}$ and $\lambda_{d[2]}$, respectively.

In this paper, it is shown that if $d^* \in \Omega_{v,b,k}$ is one of the following designs:

(i) a GD design with $\lambda_2=\lambda_1+1$ (not necessarily with two groups),
(ii) a GD design with group size two and $\lambda_1=\lambda_2+1>1$,
(iii) a design with a triangular scheme and $\lambda_2=\lambda_1+1$,
(iv) a design with a triangular scheme, $v=10$, and $\lambda_1=\lambda_2+1$,
(v) a design with an L_2 scheme and $\lambda_2=\lambda_1+1$,
(vi) a design with an L_s scheme, $v=(s+1)^2$, and $\lambda_1=\lambda_2+1$, where s is a positive integer,
(vii) a design with a cyclic scheme, $v=5$, and $\lambda_1=\lambda_2\pm 1$,

then d^* is optimum over all the two-associate-class PBIB designs in $\Omega_{v,b,k}$ with respect to any generalized type 1 criterion.

We first discuss the E-criterion in Section 2. The results for the E-criterion are then used to treat the general criteria in Section 3.

2. E-optimality

Recall that an E-optimum design maximizes $\mu_{d,v-1}$. In studying the performance of a PBIB design with two associate classes under the E-criterion, it is more convenient to look at the following matrix

$$\phi(C_d)=\{kC_d-\{r(k-1)+\lambda_{d[1]}\}I_v+\lambda_{d[1]}J_v\}/(\lambda_{d[1]}-\lambda_{d[2]}) , \tag{2.1}$$

where I_v is the $v\times v$ identity matrix, and J_v is the $v\times v$ matrix of ones. This matrix $\phi(C_d)$ actually is the adjacency matrix of a strongly regular graph with v vertices and degree n_{d1} or n_{d2} depending on whether $\lambda_{d1}<\lambda_{d2}$ or $\lambda_{d1}>\lambda_{d2}$. (See Raghavarao [11], p. 187.) For convenience, we denote this graph by $G(d)$.

It is easily seen that $\mu_{d,v-1}$ is related to the smallest eigenvalue of $\phi(C_d)$, *say* $\tilde{\mu}_d$, by

$$\mu_{d,v-1}=k^{-1}\{(\lambda_{d[1]}-\lambda_{d[2]})\tilde{\mu}_d+r(k-1)+\lambda_{d[1]}\} . \tag{2.2}$$

So if d_1 and d_2 are two PBIB designs with two associate classes in $\Omega_{v,b,k}$, then d_1 is E-better than d_2 if and only if d_1 has a bigger value of $(\lambda_{d[1]}-\lambda_{d[2]})\tilde{\mu}_d+\lambda_{d[1]}$. Since $\tilde{\mu}_d$ is always negative (actually $\tilde{\mu}_d\leqq -1$), it seems wise to make $\lambda_{d[1]}-\lambda_{d[2]}$ as small as possible and also make $\tilde{\mu}_d$ as big as possible. Note that the latter has to do with the structure of the associated graph.

It is known that a design $d\in\Omega_{v,b,k}$ is E-optimal over $\Omega_{v,b,k}$ if d is

(i) a GD design with $\lambda_{d2}=\lambda_{d1}+1$ (see Takeuchi [12]),

or

(ii) a GD design with group size 2 and $\lambda_{d1}=\lambda_{d2}+1>1$ (see Cheng [3]).

For design (i), the smallest eigenvalue of $\phi(C_d)$ is equal to -1 which achieves the biggest possible value, and $\lambda_{d[1]}-\lambda_{d[2]}=1$, also achieving the minimum. It is known that the smallest eigenvalue of $\phi(C_d)$ is equal to -1 if and only if d is GD with $\lambda_{d2}>\lambda_{d1}$. This can be proved by the same argument as in Lemma 3 of Takeuchi [12]. However, when $\lambda_{d2}-\lambda_{d1}$ is too big, the design turns out to be bad.

For design (ii), we still have $\lambda_{d[1]}-\lambda_{d[2]}=1$; however, the smallest eigenvalue of $\phi(C_d)$ is now equal to -2, not -1. Unlike the case $\tilde{\mu}_d=-1$, the condition that $\tilde{\mu}_d=-2$ does not characterize the structure of the graph $G(d)$. The following are some other examples with $\tilde{\mu}_d=-2$:

(i) a design with a triangular scheme and $\lambda_{d2}>\lambda_{d1}$,

(ii) a design with a triangular scheme, $v=10$, and $\lambda_{d1}>\lambda_{d2}$,

(iii) a design with an L_2 scheme and $\lambda_{d2}>\lambda_{d1}$,

(iv) a design with an L_s scheme, $v=(s+1)^2$, and $\lambda_{d1}>\lambda_{d2}$, where s is a positive integer.

For example, let d be a design with a triangular scheme and $\lambda_{d2}>\lambda_{d1}$. By the definition of a triangular scheme, $v=n(n-1)/2$ for some integer $n\geqq 2$. Since N_dN_d' has eigenvalues rk, $r+(n-4)\lambda_{d1}-(n-3)\lambda_{d2}$, and $r-2\lambda_{d1}+\lambda_{d2}$ (see Raghavarao [11], p. 128), it follows that the eigenvalues of $\phi(C_d)$ are n_{d1} $(=2(n-2))$, -2, and $n-4$. Thus $\tilde{\mu}_d=-2$. Cases (ii), (iii) and (iv) can be similarly checked.

Unlike the group-divisible case, it is not known whether designs (i)-(iv) are E-optimum over $\Omega_{v,b,k}$ when it is also true that $\lambda_{d[1]}=\lambda_{d[2]}+1$. However, the following can be proved:

PROPOSITION 2.1. Let $d^*\in\Omega_{v,b,k}$ be a PBIBD with two associate classes such that $|\lambda_1-\lambda_2|=1$, and the smallest eigenvalue of the adjacency matrix of the graph $G(d)$ is equal to -2. If there is no GD design with $\lambda_2=\lambda_1+1$ in $\Omega_{v,b,k}$, then d^* is E-optimum among the PBIB designs with two associate classes in $\Omega_{v,b,k}$.

PROOF. Let d^* be a design with the stated properties. For any two-associate-class PBIB design d in $\Omega_{v,b,k}$, we have $\lambda_{d[2]}\leqq\lambda_{d^*[2]}$, since

$\lambda_{d^*[1]}=\lambda_{d^*[2]}+1$. If $\tilde{\mu}_d \leqq -2$, then

$$\begin{aligned}(\lambda_{d[1]}-\lambda_{d[2]})\tilde{\mu}_d+\lambda_{d[1]} &\leqq -2(\lambda_{d[1]}-\lambda_{d[2]})+\lambda_{d[1]} \\ &= -(\lambda_{d[1]}-\lambda_{d[2]})+\lambda_{d[2]} \\ &\leqq -(\lambda_{d^*[1]}-\lambda_{d^*[2]})+\lambda_{d^*[2]} \\ &= (\lambda_{d^*[1]}-\lambda_{d^*[2]})\tilde{\mu}_{d^*}+\lambda_{d^*[1]} .\end{aligned}$$

It follows that d^* is at least as good as d under the E-criterion.

If $\tilde{\mu}_d=-1$, then d is a group-divisible design with $\lambda_{d2}>\lambda_{d1}$. By assumption, $\lambda_{d2}\geqq\lambda_{d1}+2$. To check $(\lambda_{d[1]}-\lambda_{d[2]})\tilde{\mu}_d+\lambda_{d[1]}\leqq(\lambda_{d^*[1]}-\lambda_{d^*[2]})\tilde{\mu}_{d^*}+\lambda_{d^*[1]}$, we need $\lambda_{d1}\leqq\lambda_{d^*[1]}-2$, i.e., $\lambda_{d1}\leqq\lambda_{d^*[2]}-1$. We now claim that this is true. Otherwise, λ_{d1} must be equal to $\lambda_{d^*[2]}$, which implies that $n_{d2}(\lambda_{d2}-\lambda_{d1})=n_{d^*2}$ or n_{d^*1}. However, this is impossible, since for a group-divisible design, $n_{d2}\geqq v/2$, and hence $n_{d2}(\lambda_{d2}-\lambda_{d1})\geqq v>n_{d^*2}$ and n_{d^*1}. Thus we conclude that d^* is at least as good as d under the E-criterion.

To complete the proof, we show that there is no $d\in\Omega_{v,b,k}$ which is a PBIBD with two associate classes and $-1>\tilde{\mu}_d>-2$.

For a two-associate-class PBIBD d, C_d has two distinct nonzero eigenvalues, say α_d and β_d. It is well-known that both $k\alpha_d+k\beta_d$ and $k^2\alpha_d\beta_d$ are integers. Therefore $k\alpha_d$ and $k\beta_d$ are both integers or are conjugate irrationals. If they are conjugate irrationals, then they must have the same multiplicity as eigenvalues of kC_d. In this case, v is odd. Accordingly, by a result of Connor and Clatworthy [5] (Theorem 8.11.1 of Raghavarao [11], p. 160), if $\tilde{\mu}_d$ is not an integer, then d must have a pseudocyclic scheme. In the latter case v is of the form $4t+1$, and $n_{d1}=n_{d2}=2t$. Therefore the graph $G(d)$ has degree $2t$ and its adjacency matrix $\phi(C_d)$ has eigenvalues $2t$ (with multiplicity 1), $\tilde{\mu}_d$ (with multiplicity $2t$) and the conjugate of $\tilde{\mu}_d$, say $\tilde{\nu}_d$ (with multiplicity $2t$).

Counting $\operatorname{tr}\phi(C_d)$ and $\operatorname{tr}[\phi(C_d)]^2$, we have

$$2t+2t\tilde{\mu}_d+2t\tilde{\nu}_d=0 \tag{2.3}$$

and

$$4t^2+2t\tilde{\mu}_d^2+2t\tilde{\nu}_d^2=2t(4t+1) . \tag{2.4}$$

Solving (2.3) and (2.4), one gets

$$\tilde{\mu}_d=\{-1-(4t+1)^{1/2}\}/2 . \tag{2.5}$$

If there exists a design $d\in\Omega_{v,b,k}$ which is a PBIBD with two associate classes and $-2<\tilde{\mu}_d<-1$, then $\tilde{\mu}_d$ is not an integer. In this case we need $\{-1-(4t+1)^{1/2}\}/2>-2$. This is true if and only if $t=1$, i.e., $v=5$. However, a two-associate-class PBIBD with $v=5$ must be of the pseudocyclic type (see Mesner [8]). This contradicts the existence of d^* which has $\tilde{\mu}_{d^*}=-2$ and is not of the pseudocyclic type.

If there exists a $d^* \in \Omega_{v,b,k}$ which is a design with a triangular scheme and $\lambda_2=\lambda_1+1$, a design with a triangular scheme, $v=10$, and $\lambda_1=\lambda_2+1$, a design with an L_2 scheme and $\lambda_2=\lambda_1+1$, or a design with an L_s scheme, $v=(s+1)^2$, and $\lambda_1=\lambda_2+1$, then it can be shown that there is no GD design with $\lambda_2=\lambda_1+1$ in $\Omega_{v,b,k}$. For example, assume that there exists a PBIB design $d^* \in \Omega_{v,b,k}$ which has an L_2 scheme, $v=s^2$, and $\lambda_2=\lambda_1+1$. Then $n_{d^*1}=2(s-1)$, $n_{d^*2}=(s-1)^2$. For a GD design with $\lambda_2=\lambda_1+1$, one has $n_{d1}+1 \mid n_{d2}$ since $n_{d1}+1$ equals the group size. So if there is a GD design with $\lambda_2=\lambda_1+1$ in $\Omega_{v,b,k}$, then we must have $2(s-1)+1 \mid (s-1)^2$ or $(s-1)^2+1 \mid 2(s-1)$, both of which never hold.

Accordingly, if $d^* \in \Omega_{v,b,k}$ is one of the above designs, then it is E-optimum over the two-associate-class PBIB designs in $\Omega_{v,b,k}$.

From the proof of Proposition 2.1, $v=5$ is the only case where there is a non-group-divisible two-associate-class PBIBD with $\tilde{\mu}_d>-2$. As a matter of fact, a two-associate-class PBIBD with $v=5$, and $\lambda_2=\lambda_1+1$ or $\lambda_1=\lambda_2+1$ has also been shown by Cheng [3] to be E-optimal over the whole $\Omega_{5,b,k}$.

3. The general criteria

We need some notations from Cheng [2]. Given any two positive numbers A and B such that $A^2 \geqq B \geqq A^2/(v-1)$, and any positive integer n, $1 \leqq n \leqq v-2$, the solution of

$$nR_1+(v-1-n)R_2=A$$

$$nR_1^2+(v-1-n)R_2^2=B$$

and

$$R_1 \geqq R_2$$

is given by

$$\mathrm{R}_1(n;\, A,\, B)=(A+[n^{-1}(v-1)(v-1-n)]^{1/2}P)/(v-1)\,, \tag{3.1}$$

and

$$R_2(n;\, A,\, B)=(A-[(v-1-n)^{-1}n(v-1)]^{1/2}P)/(v-1)\,, \tag{3.2}$$

where $P=[B-A^2/(v-1)]^{1/2}$; see Cheng [2].

As in Cheng [2], for any real-valued function f, we define $F(n;\, A,\, B)$ to be $nf(R_1(n;\, A,\, B))+(v-1-n)f(R_2(n;\, A,\, B))$. Finally, we denote $\operatorname{tr} C_d$ and $\operatorname{tr} C_d^2$ by A_d and B_d respectively. Then for a connected design d whose C-matrix has two distinct nonzero eigenvalues, $\Phi_f(C_d)=F(n;\, A_d,$

B_d), where n is the multiplicity of the biggest eigenvalue of C_d.

LEMMA 3.1. *Let d_1 and d_2 be two designs such that both C_{d_1} and C_{d_2} have two distinct nonzero eigenvalues, $A_{d_1}=A_{d_2}$, d_1 is at least as good as d_2 under the E-criterion, and $B_{d_1}\leqq B_{d_2}$. Then $\Phi(C_{d_1})\leqq\Phi(C_{d_2})$ for any generalized type* 1 *criterion Φ.*

PROOF. Let $\mu_{d_1 1}=\cdots=\mu_{d_1 n_1}=\alpha_1>\mu_{d_1,n_1+1}=\cdots=\mu_{d_1,v-1}=\beta_1$ be the nonzero eigenvalues of C_{d_1}, and $\mu_{d_2 1}=\cdots=\mu_{d_2 n_2}=\alpha_2>\mu_{d_2,n_2+1}=\cdots=\mu_{d_2,v-1}=\beta_2$ be the nonzero eigenvalues of C_{d_2}. Then by assumption, $\beta_1\geqq\beta_2$.

If $n_1\geqq n_2$, then clearly $(\mu_{d_1 1},\cdots,\mu_{d_1,v-1})$ is majorized by $(\mu_{d_2 1},\cdots,\mu_{d_2,v-1})$. As a matter of fact, $\mu_{d_2 i}\geqq\mu_{d_1 1}$ for all $i=1,\cdots,n_2$ and $\mu_{d_2 i}\leqq\mu_{d_1,v-1}$, for all $i=n_2+1,\cdots,v-1$. This implies that for any type 1 criterion Φ_f,

$$\Phi_f(C_{d_1})=\sum_{i=1}^{v-1}f(\mu_{d_1 i})\leqq\sum_{i=1}^{v-1}f(\mu_{d_2 i})=\Phi_f(C_{d_2}) .$$

Note that the majorization of $(y_1,y_2,\cdots,y_n)$ by $(x_1,x_2,\cdots,x_n)$ is equivalent to $\sum_{i=1}^{n}f(x_i)\geqq\sum_{i=1}^{n}f(y_i)$ for each real continuous convex function f defined on some real interval (see, e.g., Mirsky [9]). On the other hand, if $n_1<n_2$, then by Lemma A3 (iii) of Cheng [2],

$$\Phi_f(C_{d_2})=F(n_2;A_{d_2},B_{d_2})\geqq F(n_1;A_{d_2},B_{d_2}) . \tag{3.3}$$

Since $B_{d_2}\geqq B_{d_1}$ and $A_{d_2}=A_{d_1}$, by Lemma A3 (i) of Cheng [2],

$$F(n_1;A_{d_2},B_{d_2})\geqq F(n_1;A_{d_1},B_{d_1})=\Phi_f(C_{d_1}) ,$$

which, combined with (3.3), implies that $\Phi_f(C_{d_2})\geqq\Phi_f(C_{d_1})$.

COROLLARY 3.1. *Let d_1 and d_2 be two* PBIB *designs with two associate classes in $\Omega_{v,b,k}$. If d_1 is at least as good as d_2 under the E-criterion, and* $\operatorname{tr}C_{d_1}^2\leqq\operatorname{tr}C_{d_2}^2$, *then d_1 is at least as good as d_2 under any generalized type* 1 *criterion.*

PROOF. Clearly $A_{d_1}=A_{d_2}$ holds. By assumption, $B_{d_1}\leqq B_{d_2}$. Corollary 3.1 follows from Lemma 3.1 and the assumption that d_1 is at least as good as d_2 under the E-criterion.

Note that $\operatorname{tr}C_d^2$ can be made small by reducing the difference between λ_{d1} and λ_{d2} since minimizing $\operatorname{tr}C_d^2$ is equivalent to minimizing $n_{d1}\lambda_{d1}^2+(v-1-n_{d1})\lambda_{d2}^2$ and $n_{d1}\lambda_{d1}+(v-1-n_{d1})\lambda_{d2}$ is a constant. When $|\lambda_{d1}-\lambda_{d2}|=1$, the minimum value of $\operatorname{tr}C_d^2$ is achieved. Therefore we conclude our main theorem :

THEOREM 3.1. *Let $d^*\in\Omega_{v,b,k}$ be a two-associate-class* PBIB *design* *such that* $|\lambda_1-\lambda_2|=1$, *and the smallest eigenvalue of the adjacency matrix of the graph* $G(d)$ *is* $\geqq -2$. *If* d^* *itself is a* GD *design with* $\lambda_2=\lambda_1+1$ *or there is no* GD *design with* $\lambda_2=\lambda_1+1$ *in* $\Omega_{v,b,k}$, *then* d^* *is optimum over the two-associate-class* PBIB *designs in* $\Omega_{v,b,k}$ *with respect to any generalized type* 1 *criterion.*

The theorem follows immediately from Proposition 2.1 and Corollary 3.1. Thus the E-optimality result in Section 2 is extended to any generalized type 1 criterion. For example, when $v=16$, $k=4$, $b=36$, Clatworthy [4] listed seven two-associate-class PBIB designs: a GD design with 4 groups, $\lambda_1=5$, $\lambda_2=1$, a GD design with 4 groups, $\lambda_1=1$, $\lambda_2=2$, a design with an L_2 scheme and $\lambda_1=3$, $\lambda_2=1$, a design with an L_2 scheme and $\lambda_1=0$, $\lambda_2=3$, a design with an L_3 scheme and $\lambda_1=3$, $\lambda_2=0$, a design with an L_3 scheme and $\lambda_1=1$, $\lambda_2=3$, and a design of pseudo Latin square type with $\lambda_1=0$ and $\lambda_2=3$. Among these, by Theorem 3.1, a GD design with 4 groups and $\lambda_1=1$, $\lambda_2=2$ is the best one. Another example, when $v=9$, $k=3$, $b=30$, an L_2 type design with $\lambda_1=2$ and $\lambda_2=3$ is the best one among the six designs listed in Clatworthy [4] with respect to any generalized type 1 criterion.

Corollary 3.1 is particularly useful for comparing two PBIB designs with two associate classes without doing much computation. A rule of thumb is to make $|\lambda_1-\lambda_2|$ as small as possible, and to choose a design for which the smallest eigenvalue of the associated graph is as large as possible.

When the association scheme is fixed, whether the design has $\lambda_1>\lambda_2$ or $\lambda_2>\lambda_1$ is important. It was pointed out before that if d is a GD design with $\lambda_2>\lambda_1$, then the smallest eigenvalue of $G(d)$ is -1. Also, if d is a design with a triangular scheme or an L_2 scheme with $\lambda_2>\lambda_1$, then $\tilde{\mu}_d$ is -2. However, when the roles of λ_1 and λ_2 are interchanged, things change a lot. For a GD design with $\lambda_1>\lambda_2$, $\tilde{\mu}_d=(-1)\times$the group size. Thus $\tilde{\mu}_d=-2$ if the group size is 2 and it decreases steadily as the group size increases. For a design with a triangular scheme, $v=n(n-1)/2$, and $\lambda_1>\lambda_2$, $\tilde{\mu}_d=-(n-3)$, which is equal to -2 if $n=5$, i.e., $v=10$. (When $n=4$, it is the same as a GD design with three groups.) When v increases, $\tilde{\mu}_d$ also decreases. For example, when $v=21$, $k=2$, $b=105$, there are two designs with the same triangular scheme. One has $\lambda_1=1$, $\lambda_2=0$, and the other has $\lambda_1=0$, $\lambda_2=1$ (designs T_7 and T_8 of Clatworthy [4]). In this case, the latter is better. The same thing happens for a design of L_2 type with $\lambda_1>\lambda_2$ too.

The results obtained in this paper also hold when the phrases "triangular scheme" and "Latin square scheme" are replaced by "pseudo triangular scheme" and "pseudo Latin square scheme," since only the parameters, not the structure, of the association scheme are relevant to the definition of our optimality criteria.

Although a GD design with two groups and $\lambda_2=\lambda_1+1$ is optimum over $\Omega_{v,b,k}$ with respect to any generalized type 1 criterion, the best two-associate-class PBIBD need not be optimum over $\Omega_{v,b,k}$. The following are two examples:

(i) When $v=9$, $k=3$, $b=18$, the best two-associate-class PBIB design is one based on an L_2 scheme, $\lambda_1=1$ and $\lambda_2=2$. It is A- and D-worse than design #8 listed in Mitchell and John [10]. The two designs have the same performance under the E-criterion.

(ii) When $v=10$, $k=2$, $b=30$, the best two-associate-class PBIB design is one based on a triangular scheme, $\lambda_1=1$ and $\lambda_2=0$. This design is A- and D-worse than design #12 listed in Mitchell and John [10]. The two designs also have the same performance under the E-criterion.

Both designs #8 and #12 mentioned above are not PBIB designs. So the story of designs based on triangular and Latin square schemes seems to be quite discouraging. However, group-divisible designs with $\lambda_2=\lambda_1+1$ are very promising to be optimum over all designs under quite general criteria. This class of designs deserves more study.

Acknowledgment

The author wishes to thank the referee for pointing out an error in the earlier draft of the paper.

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