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SOME BOUNDS FOR PARTIALLY BALANCED BLOCK DESIGNS

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Abstract

Bounds on eigenvalues of the C-matrix for a partially balanced block (PBB) design are given together with some bounds on the number of blocks. Furthermore, a certain equiblock-sized PBB design is characterized. These results contain, as special cases, the known results for variance-balanced block designs and so on.

1. Introduction

A block design $N=\|n_{ij}\|$ here is an arrangement of v treatments in b blocks of the j-th block size k_j $(j=1, 2, \cdots, b)$ such that the i-th treatment occurs r_i times $(i=1, 2, \cdots, v)$ and the i-th treatment occurs in the j-th block n_{ij} times, where $n_{ij}=0$ or 1 (binary case). In the adjusted intrablock normal equations for estimating the vector of treatment effects under the usual assumptions, the following C-matrix plays an important role:

$$C=D_r-ND_k^{-1}N'$$

where D_r and D_k stand for diagonal matrices with diagonal elements $r_1, r_2, \cdots, r_v$ and $k_1, k_2, \cdots, k_b$, respectively, and A' is the transpose of the matrix A. Throughout this note, we shall deal only with connected designs (i.e., the rank of C is $v-1$) in which all elementary contrasts of treatment effects are estimable (cf. [2]).

Suppose (cf. [4]) that the association matrices $A_0, A_1, \cdots, A_p$ are defined in a usual sense. Furthermore, $A_i^{\#}$, $i=0, 1, \cdots, p$, rank$(A_i^{\#})=\alpha_i$, are the mutually orthogonal idempotents of the association algebra. Following Ishii and Ogawa [3], a block design is said to be partially balanced with p associate classes if the C-matrix of the design has the eigenvalues $0, \rho_1, \rho_2, \cdots, \rho_p$ with multiplicities $1, \alpha_1, \alpha_2, \cdots, \alpha_p$ and the

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linear space spanned by the eigenvectors corresponding to a root ρ_i is equal to the linear space spanned by the column vectors of $A_i^\sharp$, $i=1,2,\cdots,p$ (by a suitable change of order of ρ_i). It is well known (cf. [4]) that a partially balanced block (PBB) design with parameters v, b, r_i, k_j ($i=1,2,\cdots,v$; $j=1,2,\cdots,b$) based on an association scheme of p associate classes is given by an incidence matrix N satisfying

$$(1.1)\qquad (C=)D_r-ND_k^{-1}N'=\rho_1A_1^\sharp+\rho_2A_2^\sharp+\cdots+\rho_pA_p^\sharp \\ (=a_0A_0+a_1A_1+\cdots+a_pA_p)\,,$$

where

$$a_0=\left(\sum_{i=1}^{v} r_i-b\right)\Big/v \quad \text{and} \quad a_i\leqq 0\,, \quad i=1,2,\cdots,p\,.$$

Note that if $\rho_1=\rho_2=\cdots=\rho_p$, the PBB design essentially becomes a variance-balanced block (BB) design in the sense of the definition of Rao [10]. Further note (cf. [4], Theorem 13.2) that a PBB design with a constant block size based on an association scheme is a partially balanced incomplete block (PBIB) design based on the same association scheme. PBB designs are useful as generalizations of PBIB designs and BB designs.

In general, it is known (cf. [5]) that the eigenvalues of the C-matrix for a block design are bounded above by the maximum of replication numbers r_i's. From a point of view of an improvement of this upper bound, Kageyama and Tsuji [8] have discussed in detail some bounds on the eigenvalue of the C-matrix for a BB design. Little attention has been given to discussions in the form of specifying PBB designs. The main purpose of this note is to give other upper and lower bounds on the largest eigenvalue and the smallest eigenvalue of the C-matrix for a PBB design. Furthermore, other observations relating to these bounds are obtained. These results contain the well-known results for BB designs and so on, as special cases. The idea of approaches here is the almost same as that of Kageyama and Tsuji [8].

Finally, since a design uniquely determines its incidence matrix and vice versa, both a design and its incidence matrix are denoted by the same symbol throughout this note. For the convenience of notation, we further let $\max r_i=\max_{1\leqq i\leqq v} r_i$, $\min r_i=\min_{1\leqq i\leqq v} r_i$, $\max k_j=\max_{1\leqq j\leqq b} k_j$ and $\min k_j=\min_{1\leqq j\leqq b} k_j$.

2. General derivations of bounds

Let $\rho_{\max}=\max_{1\leq i\leq p}\rho_i$ and $\rho_{\min}=\min_{1\leq i\leq p}\rho_i$ for (1.1). Then $\rho_{\min}\leqq\rho_i\leqq\rho_{\max}$ for $i=1, 2, \cdots, p$. We here derive bounds on the extreme eigenvalues $\rho_{\min}$ and $\rho_{\max}$ of the C-matrix of a PBB design.

We first present two lemmas which are useful to later discussions.

LEMMA 2.1. *For a* PBB *design with parameters* v, b, r_i, k_j $(i=1, 2, \cdots, v;\ j=1, 2, \cdots, b)$,

$$\frac{v}{v-1}(\max r_i)\left(1-\frac{1}{\min k_j}\right)\leqq\frac{\sum_{i=1}^{v} r_i-b}{v-1}\leqq\frac{v}{v-1}(\min r_i)\left(1-\frac{1}{\max k_j}\right).$$

PROOF. From the definition of a PBB design N, we have

$$C=D_r-ND_k^{-1}N'=a_0A_0+a_1A_1+\cdots+a_pA_p\,,$$

where $a_0=\left(\sum_{i=1}^{v} r_i-b\right)\Big/v$ and $a_i\leqq 0$ $(i=1, 2, \cdots, p)$. The i-th diagonal element of the above relation yields for $n=\sum_{i=1}^{v} r_i$,

$$r_i-\left(\frac{n_{i1}}{k_1}+\cdots+\frac{n_{ib}}{k_b}\right)=\frac{n-b}{v}\,.$$

Hence

$$\begin{aligned} r_i&=\frac{n-b}{v}+\left(\frac{n_{i1}}{k_1}+\cdots+\frac{n_{ib}}{k_b}\right)\\ &\geqq\frac{n-b}{v}+\frac{n_{i1}+\cdots+n_{ib}}{\max k_j}\\ &=\frac{n-b}{v}+\frac{r_i}{\max k_j}\end{aligned}$$

which implies

$$r_i\left(1-\frac{1}{\max k_j}\right)\geqq\frac{n-b}{v}\,,\qquad i=1, 2, \cdots, v\,,$$

so

$$(\min r_i)\left(1-\frac{1}{\max k_j}\right)\geqq\frac{n-b}{v}$$

which is equivalent to

$$\frac{n-b}{v-1}\leqq\frac{v}{v-1}(\min r_i)\left(1-\frac{1}{\max k_j}\right).$$

On the other hand,

$$r_i=\frac{n-b}{v}+\left(\frac{n_{i1}}{k_1}+\cdots+\frac{n_{ib}}{k_b}\right)$$
$$\leqq\frac{n-b}{v}+\frac{r_i}{\min k_j},$$

which implies

$$r_i\left(1-\frac{1}{\min k_j}\right)\leqq\frac{n-b}{v}, \qquad i=1, 2, \cdots, v,$$

so

$$(\max r_i)\left(1-\frac{1}{\min k_j}\right)\leqq\frac{n-b}{v}$$

which is equivalent to

$$\frac{v}{v-1}(\max r_i)\left(1-\frac{1}{\min k_j}\right)\leqq\frac{n-b}{v-1}.$$

This completes the proof.

Remark 2.1. The inequality of Lemma 2.1 is obviously equivalent to

$$\text{(2.1)} \quad \sum_{i=1}^{v} r_i-\left(1-\frac{1}{\max k_j}\right)(\min r_i)v\leqq b\leqq\sum_{i=1}^{v} r_i-\left(1-\frac{1}{\min k_j}\right)(\max r_i)v.$$

In particular, when the design is equireplicated (i.e., $r_1=r_2=\cdots=r_v=r$, say),

$$\left(\frac{r}{\max k_j}\right)v\leqq b\leqq\left(\frac{r}{\min k_j}\right)v.$$

These are already given for a BB design by Kageyama and Tsuji ([8], Theorem 3.3). Thus, it should also be noted that Lemma 2.1 holds for both a PBB design and a BB design. Though bounds on the number of blocks are variously known, a mathematical expression of an inequality for a block design is given by Kageyama [6] in the form of including all of the known results.

LEMMA 2.2. *For a* PBB *design with parameters* v, b, r_i, k_j $(i=1, 2, \cdots, v;\ j=1, 2, \cdots, b)$,

$$\min r_i\geqq\frac{\sum_{i=1}^{v} r_i-b}{v-1}.$$

PROOF. From Lemma 2.1 it follows that for $n=\sum_{i=1}^{v} r_i$,

$$\min r_i \geqq \frac{\max k_j}{\max k_j - 1} \cdot \frac{n-b}{v} .$$

Furthermore, from $v \geqq \max k_j$ it holds that

$$\frac{\max k_j}{\max k_j - 1} \cdot \frac{n-b}{v} \geqq \frac{n-b}{v-1} .$$

Thus, we obtain $\min r_i \geqq (n-b)/(v-1)$.

Remark 2.2. Lemma 2.2 implies

$$b \geqq \sum_{i=1}^{v} r_i - (\min r_i)(v-1) . \tag{2.2}$$

However, since $v \geqq \max k_j$, bound (2.1) is more stringent than bound (2.2).

We next derive some bounds on the extreme eigenvalues of the C-matrix.

THEOREM 2.1. *For a* PBB *design* N *with parameters* v, b, r_i, k_j $(i=1, 2, \cdots, v;\ j=1, 2, \cdots, b)$ *in which* $C = D_r - N D_k^{-1} N' = \rho_1 A_1^{\#} + \rho_2 A_2^{\#} + \cdots + \rho_p A_p^{\#}$,

$$\rho_{\min} \leqq \frac{\sum_{i=1}^{v} r_i - b}{v-1} \leqq \rho_{\max} .$$

PROOF. Since $\operatorname{tr}(C) = \sum_{i=1}^{v} r_i - b$ and $\sum_{i=1}^{p} \operatorname{tr}(A_i^{\#}) = v-1$, the required bounds clearly follow from the defining relation.

Remark 2.3. It is obvious that two bounds in Theorem 2.1 are attainable only when the PBB design is a BB design. Furthermore, it is known (cf. [4], [5]) that in general

$$\rho_{\max} \leqq \max_{1 \leqq i \leqq v} r_i . \tag{2.3}$$

From Lemma 2.2 and Theorem 2.1, it follows that

$$\rho_{\min} \leqq \min_{1 \leqq i \leqq v} r_i .$$

This is inferior to Theorem 2.1, but this expression is very simple and practical.

In general, it holds that $\rho_{\min} > 0$. As another lower bound reflecting certain block structure, we have

THEOREM 2.2. *For a* PBB *design* $N = \|n_{ij}\|$ *with parameters* v, b, r_i, k_j $(i=1, 2, \cdots, v;\ j=1, 2, \cdots, b)$ *in which* $C = D_r - N D_k^{-1} N' = \rho_1 A_1^{\#} +$

$\rho_2 A_2^{\#} + \cdots + \rho_p A_p^{\#}$,

$$\rho_{\min} \geqq \frac{2\left(\sum_{i=1}^{v} r_i - b\right)}{v} - \frac{(v-2)(\max_{i,i'} \lambda_{ii'})}{\min k_j},$$

where $\lambda_{ii'} = \sum_{j=1}^{b} n_{ij} n_{i'j}$.

PROOF. From Frobenius' theorem (cf. [1], p. 66), we have

$$\rho_{\min} \geqq 2 \min_{1 \leqq i \leqq v} c_{ii} + (v-2)d,$$

where c_{ii} is the i-th diagonal element of the C-matrix and d is the numerically largest absolute value of off-diagonal elements of C. Now,

$$c_{ii} = \left(\sum_{i=1}^{v} r_i - b\right) \Big/ v,$$

$$\begin{aligned} |d| &= \max_{1 \leqq i, i' \leqq v} \left\{ \frac{n_{i1} n_{i'1}}{k_1} + \cdots + \frac{n_{ib} n_{i'b}}{k_b} \right\} \\ &\leqq \max_{i,i'} \frac{n_{i1} n_{i'1} + \cdots + n_{ib} n_{i'b}}{\min k_j} \\ &= (\max_{i,i'} \lambda_{ii'}) / (\min k_j), \end{aligned}$$

where $\lambda_{ii'} = \sum_{j=1}^{b} n_{ij} n_{i'j}$. Hence, we can get the required bound of $\rho_{\min}$.

A lower bound of $\rho_{\max}$ is given in Theorem 2.1. As other bounds reflecting certain block structure, we have

THEOREM 2.3. *For a* PBB *design* $N = \|n_{ij}\|$ *with parameters* v, b, r_i, k_j $(i=1, 2, \cdots, v;\ j=1, 2, \cdots, b)$ *in which* $C = D_r - N D_k^{-1} N' = \rho_1 A_1^{\#} + \cdots + \rho_p A_p^{\#}$,

$$\rho_{\max} \geqq \max_{i,i'} \left\{ \frac{r_i + r_{i'}}{2} \left(1 - \frac{1}{\min k_j}\right) + \frac{\lambda_{ii'}}{\min k_j} \right\},$$

$$\rho_{\min} \leqq \min_{i,i'} \left\{ \frac{r_i + r_{i'}}{2} \left(1 - \frac{1}{\max k_j}\right) + \frac{\lambda_{ii'}}{\max k_j} \right\},$$

where $\lambda_{ii'} = \sum_{j=1}^{b} n_{ij} n_{i'j}$.

PROOF. From a property of the C-matrix for the PBB design, it holds that, for some orthogonal matrix T of order v,

$$TCT' = \begin{pmatrix} \rho_1 & & & & \\ & \rho_2 & & 0 & \\ & & \ddots & & \\ & 0 & & \rho_p & \\ & & & & 0 \end{pmatrix},$$

and so

$$x'TCT'x \leqq \rho_{\max} x'x$$

for any x. Letting $y = T'x$, we get

$$y'Cy \leqq \rho_{\max} y'y \,. \tag{2.4}$$

Choosing for y the column vector $(0, \cdots, 0, \overset{i}{\check{1}}/\sqrt{2}, 0, \cdots, 0, -\overset{i'}{\check{1}}/\sqrt{2}, 0, \cdots, 0)'$, (2.4) yields

$$\begin{aligned}
\rho_{\max} &\geqq \frac{r_i + r_{i'}}{2} - \frac{1}{2}\left\{\frac{(n_{i1}-n_{i'1})^2}{k_1} + \cdots + \frac{(n_{ib}-n_{i'b})^2}{k_b}\right\} \\
&\geqq \frac{r_i + r_{i'}}{2} - \frac{1}{2}\left\{\frac{(n_{i1}-n_{i'1})^2 + \cdots + (n_{ib}-n_{i'b})^2}{\min k_j}\right\} \\
&= \frac{r_i + r_{i'}}{2} - \frac{1}{2}\left\{\frac{r_i + r_{i'} - 2\lambda_{ii'}}{\min k_j}\right\} \\
&= \frac{r_i + r_{i'}}{2}\left(1 - \frac{1}{\min k_j}\right) + \frac{\lambda_{ii'}}{\min k_j} \,,
\end{aligned}$$

where $\lambda_{ii'} = \sum_{j=1}^{b} n_{ij} n_{i'j}$. Thus, we get the required bound of $\rho_{\max}$. On the other hand, it holds that for any $x' = (x_1, \cdots, x_{v-1}, 0)$

$$\rho_{\min} x'x \leqq x'TCT'x \,.$$

Letting $y = T'x$, we get

$$\rho_{\min} y'y \leqq y'Cy \tag{2.5}$$

for some y. Now, since we can choose for y the column vector $(0, \cdots, 0, \overset{i}{\check{1}}/\sqrt{2}, 0, \cdots, 0, -\overset{i'}{\check{1}}/\sqrt{2}, 0, \cdots, 0)'$, (2.5) yields

$$\begin{aligned}
\rho_{\min} &\leqq \frac{r_i + r_{i'}}{2} - \frac{1}{2}\left\{\frac{(n_{i1}-n_{i'1})^2}{k_1} + \cdots + \frac{(n_{ib}-n_{i'b})^2}{k_b}\right\} \\
&\leqq \frac{r_i + r_{i'}}{2} - \frac{1}{2}\left\{\frac{(n_{i1}-n_{i'1})^2 + \cdots + (n_{ib}-n_{i'b})^2}{\max k_j}\right\} \\
&= \frac{r_i + r_{i'}}{2} - \frac{1}{2}\left\{\frac{r_i + r_{i'} - 2\lambda_{ii'}}{\max k_j}\right\} \\
&= \frac{r_i + r_{i'}}{2}\left(1 - \frac{1}{\max k_j}\right) + \frac{\lambda_{ii'}}{\max k_j} \,.
\end{aligned}$$

Thus, we get the required bound of $\rho_{\min}$.

The bounds in Theorem 2.3 are not practical in the sense that they include parameters reflecting block structure. As bounds free from these parameters of block structure, the proof of Theorem 2.3 obviously yields

COROLLARY 2.1. *For a* PBB *design with parameters* v, b, r_i, k_j $(i=1, 2, \cdots, v;\ j=1, 2, \cdots, b)$ *in which* $C=\rho_1 A_1^\sharp+\cdots+\rho_p A_p^\sharp$,

$$\rho_{\max}\geqq\max_{i,i'}\left\{\frac{r_i+r_{i'}}{2}-\frac{b}{2\min k_j}\right\},$$

$$\rho_{\min}\leqq\min_{i,i'}\frac{r_i+r_{i'}}{2}.$$

It is clear that the bounds in Corollary 2.1 are inferior to those in Theorem 2.3. However, the expression of Corollary 2.1 is more practical.

We can give the best possible bound from Theorems 2.1, 2.2 and 2.3 and a bound (2.3).

PROPOSITION 2.1. For a PBB design with parameters v, b, r_i, k_j $(i=1, 2, \cdots, v;\ j=1, 2, \cdots, b)$ and $n=\sum_{i=1}^{v} r_i=\sum_{j=1}^{b} k_j$,

$$\mathrm{Max}\left\{0,\ \frac{2(n-b)}{v}-(v-2)\frac{\max\lambda_{ii'}}{\min k_j}\right\}$$
$$\leqq\rho_{\min}\leqq\mathrm{Min}\left\{\frac{n-b}{v-1},\ \min_{i,i'}\left\{\frac{r_i+r_{i'}}{2}\left(1-\frac{1}{\max k_j}\right)+\frac{\lambda_{ii'}}{\max k_j}\right\}\right\},$$

$$\mathrm{Max}\left\{\frac{n-b}{v-1},\ \max_{i,i'}\left\{\frac{r_i+r_{i'}}{2}\left(1-\frac{1}{\min k_j}\right)+\frac{\lambda_{ii'}}{\min k_j}\right\}\right\}\leqq\rho_{\max}\leqq\max r_i,$$

where $\lambda_{ii'}=\sum_{j=1}^{b} n_{ij}n_{i'j}$. Note that $\frac{n-b}{v-1}\leqq\min r_i$ and $\rho_{\min}>0$.

Remark 2.4. For a BB design, we have $\rho_1=\cdots=\rho_p=(n-b)/(v-1)$. In this case Proposition 2.1 yields several results given in Kageyama and Tsuji [8].

We finally compare bounds in Proposition 2.1 by several examples all of which are quoted from Kageyama [4].

(1) An example (an example on p. 596 in [4]) with $\rho_{\min}=1$ and $\rho_{\max}=3$ implies

$$0=\mathrm{Max}\left\{0,\ -\frac{61}{6}\right\}<\rho_{\min}\leqq\mathrm{Min}\left\{\frac{29}{11},\ \frac{17}{6}\right\}=\frac{29}{11},$$

$$3=\mathrm{Max}\left\{\frac{29}{11},\ 3\right\}\leqq\rho_{\max}\leqq 4.$$

(2) An example (Example 15.3 in [4]) with $\rho_{\min}=(9-\sqrt{5})/4\ (\fallingdotseq 1.7)$ and $\rho_{\max}=(9+\sqrt{5})/4\ (\fallingdotseq 2.81)$ implies

$$\frac{3}{5}=\mathrm{Max}\left\{0,\ \frac{3}{5}\right\}\leqq\rho_{\min}\leqq\mathrm{Min}\left\{\frac{9}{4},\ \frac{13}{5}\right\}=\frac{9}{4},$$

$$\frac{5}{2}=\mathrm{Max}\left\{\frac{9}{4}, \frac{5}{2}\right\}\leqq\rho_{\max}\leqq 3 .$$

(3) An example (Example 17.1 in [4]) with $\rho_{\min}=1$ and $\rho_{\max}=2$ implies

$$\frac{1}{3}=\mathrm{Max}\left\{0, \frac{1}{3}\right\}\leqq\rho_{\min}\leqq\mathrm{Min}\left\{\frac{7}{5}, \frac{4}{3}\right\}=\frac{4}{3},$$

$$\frac{3}{2}=\mathrm{Max}\left\{\frac{7}{5}, \frac{3}{2}\right\}\leqq\rho_{\max}\leqq 2 .$$

(4) An example (an example on p. 606 in [4]) with $\rho_{\min}=1$ and $\rho_{\max}=2$ implies

$$0=\mathrm{Max}\left\{0, -\frac{4}{3}\right\}<\rho_{\min}\leqq\mathrm{Min}\left\{\frac{8}{5}, \frac{11}{6}\right\}=\frac{8}{5},$$

$$2=\mathrm{Max}\left\{\frac{8}{5}, 2\right\}\leqq\rho_{\max}\leqq 2 .$$

(5) An example (Example 18.6 in [4]) with $\rho_{\min}=2$ and $\rho_{\max}=3$ implies

$$0=\mathrm{Max}\,\{0, 0\}<\rho_{\min}\leqq\mathrm{Min}\left\{\frac{12}{5}, \frac{8}{3}\right\}=\frac{12}{5},$$

$$\frac{5}{2}=\mathrm{Max}\left\{\frac{12}{5}, \frac{5}{2}\right\}\leqq\rho_{\max}\leqq 3 .$$

Thus, the bounds in Proposition 2.1 are stringent in general. Furthermore, if we are considering expressions free from parameters reflecting block structure, then simple bounds

$$0<\rho_{\min}\leqq\frac{\sum_{i=1}^{v} r_i-b}{v-1}\ (\leqq\min r_i)$$

$$\frac{\sum_{i=1}^{v} r_i-b}{v-1}\leqq\rho_{\max}\leqq\max r_i$$

appear to be the best for our purpose. In this sense Theorem 2.1 is fundamental and powerful.

Incidentally, some known results on matrix theory in linear algebra give bounds on eigenvalues ρ_i of a symmetric C-matrix. However, they may not be useful for this problem.

3. Special derivations of bounds

We here consider bounds on eigenvalues of the C-matrix of PBB designs based on some association schemes. Two methods of derivation of bounds are used.

METHOD I. For association matrices $A_0, A_1, \cdots, A_p$ of an association scheme of p associate classes, we derive

$$\begin{aligned} D_r - ND_k^{-1}N' &= \rho_1 A_1^{\#} + \rho_2 A_2^{\#} + \cdots + \rho_p A_p^{\#} \\ &= a_0 A_0 + a_1 A_1 + \cdots + a_p A_p \end{aligned}$$

where a_i's are functions of some ρ_i's. So it holds that for any column vector $\boldsymbol{x}$,

$$\boldsymbol{x}'(D_r - ND_k^{-1}N')\boldsymbol{x} = \sum_{i=0}^{p} a_i \boldsymbol{x}' A_i \boldsymbol{x} ,$$

which yields a relation on ρ_i's by choosing an appropriate $\boldsymbol{x}$. This relation gives some bounds on ρ_i's. Note that $\boldsymbol{x}$ should be chosen according to the form of association matrices A_i's.

METHOD II. For association matrices $A_0, A_1, \cdots, A_p$ of an association scheme of p associate classes, we first derive

$$ND_k^{-1}N' = D_r - a_0 A_0 - a_1 A_1 - \cdots - a_p A_p ,$$

as in the same notation as Method I. In this case, some bounds on ρ_i's are derived from the fact that the i-th diagonal element of $ND_k^{-1}N'$ is greater than or equal to the elements of the i-th row or the i-th column of $ND_k^{-1}N'$ for $i=1, 2, \cdots, v$.

We now use these methods to derive bounds on eigenvalues of the C-matrix of PBB designs based on concrete association schemes. For description of association schemes, we refer to Raghavarao [9].

3.1. *Group divisible association scheme*

For a group divisible association scheme of $v=mn$ treatments with m groups of n treatments each,

$$A_1^{\#} = \frac{1}{mn}\{(m-1)A_0 + (m-1)A_1 - A_2\} ,$$

$$A_2^{\#} = \frac{1}{n}\{(n-1)A_0 - A_1\} ,$$

$$A_0 = I_v ,$$

$$A_1 = \text{diag}\{G_n, G_n, \cdots, G_n\} - I_v ,$$

$$A_2 = G_v - I_v - A_1 ,$$

where I_v is the identity matrix of order v and G_s is an $s \times s$ matrix with positive unit elements everywhere. Hence, for a group divisible PBB design $N=\|n_{ij}\|$ with parameters $v=mn$, b, r_i, k_j $(i=1, 2, \cdots, v; j=1, 2, \cdots, b)$ in which $C=D_r-ND_k^{-1}N'=\rho_1 A_1^* + \rho_2 A_2^* = \{(m-1)\rho_1/v+(n-1)\cdot\rho_2/n\}A_0+\{(m-1)\rho_1/v-\rho_2/n\}A_1-(\rho_1/v)A_2$, we apply Methods I and II.

(i) Method I with $x'=(\underbrace{1, 1, \cdots, 1}_{n}, 0, \cdots, 0)$ yields

$$\sum_{i=1}^{n} r_i-\sum_{j=1}^{b}(n_{1j}+n_{2j}+\cdots+n_{nj})^2\Big/k_j=\frac{n(m-1)}{m}\rho_1$$

which implies

$$\frac{n(m-1)}{m}\rho_1 \leqq \sum_{i=1}^{n} r_i-\frac{1}{\max k_j}\sum_{j=1}^{b}\left(\sum_{i=1}^{n} n_{ij}\right)^2$$
$$\leqq \sum_{i=1}^{n} r_i\left(1-\frac{1}{\max k_j}\right),$$

so

$$\rho_1 \leqq \left(\frac{1}{n}\sum_{i=1}^{n} r_i\right)\left\{\frac{m}{m-1}\left(1-\frac{1}{\max k_j}\right)\right\}.$$

Note that $\{m/(m-1)\}\{1-1/(\max k_j)\}\leqq 1$ if and only if $\max k_j \leqq m$.

(ii) Method I with $x'=(\underbrace{1, 1, \cdots, 1}_{n}, \underbrace{-1, -1, \cdots, -1}_{n}, 0, \cdots, 0)$ yields

$$\sum_{i=1}^{2n} r_i-\sum_{j=1}^{b}\left\{\frac{\left(\sum_{i=1}^{n} n_{ij}-\sum_{i=n+1}^{2n} n_{ij}\right)^2}{k_j}\right\}=2n\rho_1$$

which implies

$$\rho_1 \leqq \frac{1}{2n}\sum_{i=1}^{2n} r_i .$$

(iii) Method II: The i-th diagonal element of $ND_k^{-1}N'$ is given by $r_i-\{(m-1)/(mn)\}\rho_1-\{(n-1)/n\}\rho_2$. On the other hand, the off-diagonal elements of $ND_k^{-1}N'$ are given by $-\{(m-1)/(mn)\}\rho_1+(1/n)\rho_2$ (corresponding to the position of the first associates) or $\{1/(mn)\}\rho_1$ (corresponding to the position of the second associates). Then the form of association matrix A_1 implies

$$r_i-\frac{m-1}{mn}\rho_1-\frac{n-1}{n}\rho_2 \geqq -\frac{m-1}{mn}\rho_1+\frac{1}{n}\rho_2 \qquad \text{for } all \ i=1, 2, \cdots, v$$

which yields

$$\rho_2 \leqq \min_{1\leqq i\leqq v} r_i .$$

Thus, we get

THEOREM 3.1. *For a group divisible* PBB *design* N *with parameters* $v=mn$ *(*m *groups of* n *treatments each),* b, r_i, k_j $(i=1,2,\cdots,v;\ j=1,2,\cdots,b)$ *in which* $C=\rho_1A_1^{\#}+\rho_2A_2^{\#}$,

$$\rho_1\leqq \mathrm{Min}\left\{\frac{1}{2n}\sum_{i=1}^{2n}r_i,\ \left(\frac{1}{n}\sum_{i=1}^{n}r_i\right)\left\{\frac{m}{m-1}\left(1-\frac{1}{\max k_j}\right)\right\}\right\},$$

$$\rho_2\leqq \min_{1\leqq i\leqq v} r_i\,.$$

Remark 3.1. We may not see whether Theorem 3.1 gives an improvement of Proposition 2.1 for a group divisible case. Because, in general, we may not evaluate eigenvalues ρ_1 and ρ_2. However, it seems that Theorem 3.1 is more stringent under a "group divisible association scheme".

3.2. *Other association schemes*

Applications of Methods I and II to other association schemes (a triangular association scheme of two associate classes, a rectangular association scheme of three associate classes) do not yield fine bounds in the sense of an improvement of the bounds in Proposition 2.1, so far as the author investigates. From various experience, the general bounds in Proposition 2.1 appear to be relatively stringent.

Especially, block structures of equireplicated $(r_1=r_2=\cdots=r_v=r$, say) PBB designs based on various association schemes satisfying $\rho_i=r$ for some i are characterized in Kageyama [7] by relating blocks with association schemes.

Incidentally, for a group divisible PBB design N with parameters v, b, r_i, k_j $(i=1,2,\cdots,v;\ j=1,2,\cdots,b)$ in which $C=D_r-ND_k^{-1}N'=\rho_1A_1^{\#}+\rho_2A_2^{\#}$, if $\max_{i=1,2}\rho_i<\min_{1\leqq i\leqq v} r_i$, we can show $|ND_k^{-1}N'|\neq 0$ and so rank $(ND_k^{-1}N')=v$ which yields the Fisher's inequality $b\geqq v$. It is conjectured that this condition "$\max\rho_i<\min r_i$" gives a sufficient condition for the validity of the Fisher's inequality for any PBB design.

As a characterization of a binary PBB design with a constant block size, as already mentioned in the introduction, Kageyama [4] showed that the PBB design is a PBIB design. Finally, we generally consider an n-ary case of such situation. An "n-ary" case in this note means that the elements n_{ij} of incidence matrix $N=\|n_{ij}\|$ of a PBB design can take any of the values, $0,1,\cdots$, or $n-1$ for some positive integer n. In this case we get

PROPOSITION 3.1. An equiblock-sized n-ary PBB design is pairwise partially balanced of index λ_p.

PROOF. For a PBB design $N=\|n_{ij}\|$ with parameters v, b, r_i $(i=1,2,\cdots,v)$ and k, we have for (1.1)

$$D_r - \frac{1}{k} NN' = a_0 A_0 + a_1 A_1 + \cdots + a_p A_p \tag{3.1}$$

where $a_0 = [bk - \{\mathrm{tr}(NN')\}/k]/v$ and a_i $(i=1, 2, \cdots, p)$ is some negative constant. A comparison of off-diagonal elements in (3.1) yields

$$\sum_{l=1}^{b} n_{il} n_{jl} = -k a_f$$

provided the i-th and j-th treatments are f-th associates $(f=1, 2, \cdots, p)$. Setting $\lambda_f = -k a_f$ gives the required result.

Remark 3.2. In Proposition 3.1, when $n=2$ (i.e., $n_{ij}=0$ or 1) the i-th diagonal element of (3.1) implies

$$r_i - \frac{r_i}{k} = a_0 = \frac{\sum_{i=1}^{v} r_i - b}{v}, \qquad i=1, 2, \cdots, v$$

which show that replication numbers are constant. Thus we get a PBIB design.

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