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ON A "LACK OF MEMORY" PROPERTY

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Abstract

For two independent nonnegative random variables X and Y we say that X is ageless relative to Y if the conditional probability $\mathrm{P}[X>Y+x \mid X>Y]$ is defined and is equal to $\mathrm{P}[X>x]$ for all $x>0$. Suppose that X is ageless relative to a nonlattice Y with $\mathrm{P}[Y=0]<\mathrm{P}[Y<X]$. We show that the only such X is the exponential variable. As a corollary it follows that exponential variable is the only one which possesses the ageless property relative to a continuous variable.

The "ageless" or "lack of memory" property of the exponential distribution is a well-known characteristic property: see, for instance, Feller ([4], pp. 458–460), Aczel [1] for the associated Cauchy functional equation, and, for a refinement of the above property, Marsaglia and Tubilla [7].

If now X and Y be two independent nonnegative random variables (on some pr. space), we say that X is "ageless relative to Y" if $\mathrm{P}[X>Y]>0$ and

$$(1) \qquad \mathrm{P}[X>Y+x \mid X>Y]=\mathrm{P}[X>x], \qquad x>0,$$

equivalently,

$$(2) \qquad \int_{[0,\infty)} \bar{F}(x+y)dG(y)=\int_{[0,\infty)} \bar{F}(x)\bar{F}(y)dG(y), \qquad x>0,$$

where $F(x)=1-\bar{F}(x)=\mathrm{P}[X\leqq x]$ and $G(y)=\mathrm{P}[Y\leqq y]$. (2) obviously holds for *any* G if F is exponential. The question arises: is the weaker condition (2), true for some fixed G, already enough to characterize

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exponentiality for F? Krishnaji [5], [6] obtained some partial results. Noting that the basic functional equation (2) can essentially be dealt with in the same way as the basic integral equation (10) of Rossberg [10], we are able to establish the following:

THEOREM. *Let X be ageless relative to Y. If Y is non-lattice and if*

$$\mathrm{P}\,[Y=0]<\mathrm{P}\,[X>Y] \tag{3}$$

then X is distributed exponentially.

Remark. Regarding condition (3), see the discussion after the proof outlined below. In view of the definition of memorylessness, (3) is satisfied if $\mathrm{P}\,[Y=0]=0$ and in particular if Y has a continuous distribution.

PROOF. (1) is equivalent to

$$a^{-1}\int_{-\infty}^{\infty} G[(t-x)^{-}]dF(t)=\bar{F}(x)\,, \qquad x>0\,, \tag{4}$$

where $a=\mathrm{P}\,[X>Y]$ is positive, by definition (1). In fact $0<a<1$, since $a=1$ would imply, by (1), that $\mathrm{P}\,[X>Y+x]=\mathrm{P}\,[X>x]$ for all $x>0$, which is impossible unless Y degenerates at zero. Let then $K(y)=1-a^{-1}G[(-y)^{-}]$, so that (4) becomes (cf. relation (10) of Rossberg, [10])

$$\int_{-\infty}^{\infty} K(x-t)dF(t)=F(x)\,, \qquad x>0\,. \tag{5}$$

If we introduce H on R^1 according to (cf. relation (11) of Rossberg, [10])

$$\int_{-\infty}^{\infty} K(x-t)dF(t)=F(x)+H(x)\,, \qquad x\in R^1\,,$$

then K and H are non-decreasing functions of bounded variation of R^1 and we may therefore speak of the Laplace-Stieltjes transforms of F, K and H:

$$f(s)=\int_{[0,\infty)} e^{-sx}dF(x)\,,\quad k(s)=\int_{(-\infty,0]} e^{-sx}dK(x)\,,\quad h(s)=\int_{(-\infty,0]} e^{-sx}dH(x)\,.$$

Note that (i) $k(0)=a^{-1}>1$, (ii) as $\sigma\to-\infty$, $k(\sigma)\to K(0)-K(0^{-})=\dfrac{\mathrm{P}\,[Y=0]}{\mathrm{P}\,[X>Y]}<1$ by assumption (3), and (iii) k is strictly increasing on $(-\infty,0)$ in view of k' being positive there, so that $k(\sigma^*)=1$ for a *unique negative* number σ^*. Using *inter alia* the assumed non-latticeness of G also, we arrive, as in Rossberg [10], at the conclusion that for some constants c and p,

$$f(s)=\frac{c}{s-\sigma^*}+p=p+(1-p)\frac{\sigma^*}{\sigma^*-s}$$

in view of $f(0)=1$. F is thus the mixture of a degenerate and an exponential distribution, and we check on substituting this form for $\bar{F}$ in (2) that $p=0$, whence the theorem.

Discussion. Condition (3), which is needed to ensure the existence of σ^* above, is rather awkward in that it involves X as well as Y. Without (3), however, the theorem fails, as shown by the example: let X and Y be independent; $F(x)=x$ for $0\leqq x\leqq 1$; $G(y)=0$ for $y<0$, $1-e^{-1}$ for $0\leqq y<1$ and $1-e^{-y}$ for $y\geqq 1$. Then X is ageless relative to Y, though X is not exponential. In fact, any X supported on $[0, 1]$ is ageless relative to any Y, provided $0<\mathrm{P}[Y=0]=\mathrm{P}[Y\leqq 1]<1$.

For connections of our result with reliability theory, we refer to Cinlar and Jagers [3], and to Barlow and Proschan [2].

With slight modification in our proof, we can obtain the following variant of the Theorem:

Let X be a nonnegative random variable. If there exists a nonlattice nonnegative random variable Y, independent of X, with $\mathrm{P}[Y=0]<\mathrm{P}[X\geqq Y]$ and $\mathrm{P}[X\geqq Y+x\,|\,X\geqq Y]=\mathrm{P}[X\geqq x]$ for $x\geqq 0$, then X is exponential.

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