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AN ORDERING RELATION OF THE BLOCKING TWO-STAGE TANDEM QUEUEING SYSTEM TO THE REDUCED SINGLE SERVER QUEUEING SYSTEM

GENJI YAMAZAKI

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1. Introduction

This paper is concerned with the following two-stage tandem queueing system (TQ for short). There are two service facilities (or servers for short) arranged in tandem. Each customer arriving at the system receives the service from the first server (server 1), and then the second (server 2), before leaving the system. The queue before the server 1 may be allowed to grow unlimitedly, whereas no queue before the server 2 is allowed. If the server 2 is busy, therefore, when a service is completed to a customer by the server 1, this customer stays at the first stage and blocks further service until the server 2 becomes free. The service discipline is defined on FCFS basis. The nth customer C_n arrives at time T_n and has a service time $S_{k,n}$ by the server k ($k=1, 2$), and we define $A_n=T_n-T_{n-1}$ for $n=1, 2, \cdots$, where $T_0=0$. It is assumed that $S_{1,1}, S_{1,2}, \cdots, S_{2,1}, S_{2,2}, \cdots, A_1, A_2, \cdots$ are mutually independent, the $S_{k,n}$'s are identically distributed random variables (r.v.'s) with distribution function (d.f.) G_k ($k=1, 2$), and A_n's are also identically distributed r.v.'s. For such a TQ, a notation $GI/G_1\rightarrow G_2$ is employed.

For the $GI/G_1\rightarrow G_2$ queue, accurate analysis of the d.f.'s of such characteristic quantities as the sojourn and delay times of a customer and the number of customers in the system is extremely difficult and even their expectations cannot be computed analytically, except for some special cases, e.g. in Suzuki [10], Avi-Itzhak and Yadin [1], Tumura and Ishikawa [11].

Therefore, bounds for these d.f.'s and expectations are of value. Special interest lies in the bounds given by characteristics of other queueing systems which are relatively analyzed easily. For usual $GI/G/m$ queues there exist such useful bounds included in Brumelle [4], H. Stoyan and D. Stoyan [9] and Miyazawa [6].

From this viewpoint Sakasegawa and Yamazaki [7] tried to compare the $GI/G_1\rightarrow G_2$ queue with the following single server queueing system

(SQ), which was called as a reduced SQ (RSQ) of the TQ. There is a single server queue with unlimited waiting room. Customers are served individually and the service discipline is defined as FCFS. The arrival process is identical with that of the $GI/G_1 \to G_2$ queue, i.e., the inter-arrival time between C_{n-1} and C_n is A_n. Let's denote the service time of C_n by $\tilde{S}_n$ ($n=1, 2, \cdots$). It is assumed that the $\tilde{S}_n$'s are independent of the arrival process, and that they form a sequence of independent identically distributed (i.i.d.) r.v.'s with d.f. $\tilde{G}$:

$$\tilde{G}(t)=\Pr(\tilde{S}_n \leqq t)=\Pr(S_{1,n} \vee S_{2,n} \leqq t)=G_1(t)\cdot G_2(t) ,$$

where $X \vee Y$ denotes the maximum of r.v.'s X and Y. We denote this system by the notation $GI/\tilde{G}/1$.

In [7], by a comparison between the 2-stage TQ and its RSQ, upper bounds on a mean delay time in the TQ were derived, and the extension of their results to K-stage TQ ($\geqq 3$) was tried. Furthermore, the authors gave three conjectures from a series of simulation experiments, one of which presented the fact that [the mean delay time in the 2-stage TQ]$\geqq$[the mean waiting time in its RSQ] held in the steady state.

The present paper is intended for more extended studies of the last result, i.e., a semi-ordering relationship between the delay time in the 2-stage TQ and the waiting time in its RSQ, which gives a positive answer to the above conjecture (Section 3). In Section 2, some notations and lemmas used in Section 3 are given.

2. Preliminaries

In Section 3 we use the following semi-orderings of r.v.'s.

DEFINITION. Let X and Y make r.v.'s with d.f.'s F_X and F_Y, respectively. Then

(i) $X \overset{(1)}{\leqq} Y \Longleftrightarrow \bar{F}_X(x) \leqq \bar{F}_Y(x)$ for all x, where $\bar{F}(x)=1-F(x)$,

(ii) $X \overset{(2)}{\leqq} Y \Longleftrightarrow \int_x^\infty \bar{F}_X(t)dt \leqq \int_x^\infty \bar{F}_Y(t)dt$ for all x, where $\int_0^\infty \bar{F}_Y(t)dt < \infty$.

The semi-order $\overset{(1)}{\leqq}$, of course, is the well-known stochastic ordering. The semi-order $\overset{(2)}{\leqq}$ was introduced by Bessler and Veinott [2] and has been used by H. Stoyan and D. Stoyan [9], Borovkov [3] and others in order to investigate the order relationship between some queueing systems. In the case of the equal means, it formalizes the notion of one distribution's being more variable or more spread out than the other (cf. Lemma 3 below).

The following properties of $\overset{(i)}{\leqq}$ are used.

LEMMA 1 ([9]).

(a) *If* $X \overset{(i)}{\leqq} Y$ *and* $Y \overset{(i)}{\leqq} Z$, *then* $X \overset{(i)}{\leqq} Z$ *for* $i=1, 2$.

(b) *If* Z *is independent of* X *and* Y *and* $X \overset{(i)}{\leqq} Y$, *then* $X+Z \overset{(i)}{\leqq} Y+Z$ *for* $i=1, 2$.

(c) *If* $X \overset{(i)}{\leqq} Y$, *then* $\mathrm{E}(X) \leqq \mathrm{E}(Y)$ *for* $i=1, 2$.

(d) *If* $X \overset{(i)}{\leqq} Y$, *then* $0 \vee X \overset{(i)}{\leqq} 0 \vee Y$ *for* $i=1, 2$.

(e) *If* $X \overset{(1)}{\leqq} Y$, *then* $X \overset{(2)}{\leqq} Y$.

LEMMA 2 (*Borovkov* [3]). $X \overset{(2)}{\leqq} Y$ *is equivalent to* $\mathrm{E}\{0 \vee (X-x)\} \leqq \mathrm{E}\{0 \vee (Y-x)\}$ *for all* x.

LEMMA 3 (*Stidham* [8]). *Suppose that* $\int_0^\infty \bar{F}_Y(t)dt < \infty$ *and* $\mathrm{E}(X) = \mathrm{E}(Y)$. *If there exists a number* x_0 *such that*

$$F_X(x) \begin{cases} \leqq F_Y(x) & x < x_0 \\ \geqq F_Y(x) & x > x_0 , \end{cases} \qquad \text{then } X \overset{(2)}{\leqq} Y .$$

From Lemma 3 we have the following.

LEMMA 4. *Let* X_1, X_2 *and* Y *be mutually independent r.v.'s and let* $X_1 \overset{d}{\sim} X_2$, *where the sign* $\overset{d}{\sim}$ *denotes the equality of distribution. If* $\int_0^\infty t d \Pr(X_1 \vee Y \leqq t) < \infty$, *then*

$$X_1 \vee Y \overset{(2)}{\leqq} X_1 \vee Y - X_1 + X_2 . \tag{2.1}$$

PROOF. Let Z_1 and Z_2 be r.v.'s defined as $Z_1 = X_1 \vee Y$, $Z_2 = X_1 \vee Y - X_1 + X_2$. Because of Lemma 2, in order to prove (2.1) it is sufficient to show that

$$\mathrm{E}\{0 \vee (Z_1 - x) \mid Y\} \leqq \mathrm{E}\{0 \vee (Z_2 - x) \mid Y\} \tag{2.2}$$

holds. For any realization value y of Y, let us define $Z_{1,y} = X_1 \vee y$ and $Z_{2,y} = X_1 \vee y - X_1 + X_2$. Clearly,

$$\Pr(Z_{1,y} \leqq t) = \begin{cases} 0 & t < y \\ \Pr(X_1 \leqq t) & t \geqq y \end{cases} \tag{2.3}$$

and

$$\Pr(Z_{2,y} \leqq t) \begin{cases} \geqq 0 & t < y \\ \leqq \Pr(X_2 \leqq t) & t \geqq y . \end{cases} \tag{2.4}$$

Since $X_1 \overset{d}{\sim} X_2$, a comparison of (2.3) with (2.4) immediately gives

$$\Pr(Z_{1,y} \leqq t) \begin{cases} \leqq \Pr(Z_{2,y} \leqq t) & t < y \\ \geqq \Pr(Z_{2,y} \leqq t) & t \geqq y . \end{cases} \tag{2.5}$$

Furthermore,

$$\mathrm{E}(Z_{2,y}) = \mathrm{E}(Z_{1,y}) - \mathrm{E}(X_1) + \mathrm{E}(X_2) = \mathrm{E}(Z_{1,y}) .$$

Hence, from Lemma 3 we can obtain

$$Z_{1,y} \overset{(2)}{\leqq} Z_{2,y} , \tag{2.6}$$

so that (2.2) follows by Lemma 2.

We need the following lemma to prove Lemma 6 below.

LEMMA 5 (*Kawashima* [5]). *If X_1 and X_2 are i.i.d. r.v.'s, then for any real numbers x_1, x_2 and x_3*

$$\Pr(X_1 \leqq x_2,\ X_1 + X_2 \vee x_1 \leqq x_3) \geqq \Pr(X_1 \leqq x_2,\ X_1 \vee x_1 + X_2 \leqq x_3) . \tag{2.7}$$

In the case where the constants x_1, x_2 and x_3 are replaced by r.v.'s Y, Z_1 and Z_2, this lemma reads as follows:

LEMMA 5′. *If X_1 and X_2 are i.i.d. r.v.'s and if r.v.'s Y, Z_1 and Z_2 are independent of X_1 and X_2, then*

$$\Pr(X_1 \leqq Z_1,\ X_1 + X_2 \vee Y \leqq Z_2) \geqq \Pr(X_1 \leqq Z_1,\ X_1 \vee Y + X_2 \leqq Z_2) . \tag{2.7′}$$

LEMMA 6. *Suppose that $\{X_i;\ i=1, 2, \cdots\}$ and $\{Y_i;\ i=1, 2, \cdots\}$ are independent sequences of i.i.d. r.v.'s and they are independent of each other. Then, for any $\{Z_i;\ i=1, 2, \cdots\}$ which are independent of $\{X_i\}$ and $\{Y_i\}$, the following inequality holds.*

$$\begin{aligned} &\Pr\Big(X_1 \leqq Z_1,\ X_1 + \sum_{j=2}^{i} X_j \vee Y_j \leqq Z_i\ (i=2, 3, \cdots, n)\Big) \\ &\quad \geqq \Pr\Big(X_1 \leqq Z_1,\ \sum_{j=1}^{i-1} X_j \vee Y_{j+1} + X_i \leqq Z_i\ (i=2, 3, \cdots, n)\Big) . \end{aligned} \tag{2.8}$$

PROOF. A direct application of Lemma 5′ to the left-hand side of (2.8) gives

$$\begin{aligned} &\Pr\Big(X_1 \leqq Z_1,\ X_1 + \sum_{j=2}^{i} X_j \vee Y_j \leqq Z_i\ (i=2, \cdots, n)\Big) \\ &= \Pr\Big(X_1 \leqq Z_1,\ X_1 + X_2 \vee Y_2 \leqq \min_{2 \leqq i \leqq n}\Big(Z_i - \sum_{j=3}^{i} X_j \vee Y_j\Big)\Big) \\ &\geqq \Pr\Big(X_1 \leqq Z_1,\ X_1 \vee Y_2 + X_2 \leqq \min_{2 \leqq i \leqq n}\Big(Z_i - \sum_{j=3}^{i} X_j \vee Y_j\Big)\Big) \\ &= \Pr\Big(X_1 \leqq Z_1,\ X_1 \vee Y_2 + X_2 \leqq Z_2, \end{aligned} \tag{2.9}$$

$$X_1 \vee Y_2 + X_2 + \sum_{j=3}^{i} X_j \vee Y_j \leqq Z_i \ (i=3,\cdots,n)\Big) .$$

By applying Lemma 5′ again to the conditional probability of the right-hand side of (2.9), we can obtain the conditional result

$$\text{(2.10)} \quad \Pr\Big(X_1 \leqq Z_1,\ X_1 \vee Y_2 + X_2 \leqq Z_2,$$
$$X_1 \vee Y_2 + X_2 + \sum_{j=3}^{i} X_j \vee Y_j \leqq Z_i \ (i=3,\cdots,n) \,|\, X_1 \leqq Z_1\Big)$$
$$= \Pr\Big(X_2 \leqq Z_2 - X_1 \vee Y_2,$$
$$X_2 + X_3 \vee Y_3 \leqq \min_{3 \leqq i \leqq n} \Big(Z_i - \sum_{j=4}^{i} X_j \vee Y_j - X_1 \vee Y_2\Big) \Big| X_1 \leqq Z_1\Big)$$
$$\geqq \Pr\Big(X_2 \leqq Z_2 - X_1 \vee Y_2,$$
$$X_2 \vee Y_3 + X_3 \leqq \min_{3 \leqq i \leqq n} \Big(Z_i - \sum_{j=4}^{i} X_j \vee Y_j - X_1 \vee Y_2\Big) \Big| X_1 \leqq Z_1\Big)$$
$$= \Pr\Big(X_1 \leqq Z_1,\ X_1 \vee Y_2 + X_2 \leqq Z_2,\ X_1 \vee Y_2 + X_2 \vee Y_3 + X_3 \leqq Z_3,$$
$$X_1 \vee Y_2 + X_2 \vee Y_3 + X_3 + \sum_{j=4}^{i} X_j \vee Y_j \leqq Z_i$$
$$(i=4,\cdots,n) \,|\, X_1 \leqq Z_1\Big) .$$

Unconditioning, we have

$$\text{(2.11)} \quad \Pr\Big(X_1 \leqq Z_1,\ X_1 \vee Y_2 + X_2 \leqq Z_2,$$
$$X_1 \vee Y_2 + X_2 + \sum_{j=3}^{i} X_j \vee Y_j \leqq Z_i \ (i=3,\cdots,n)\Big)$$
$$\geqq \Pr\Big(X_1 \leqq Z_1,\ X_1 \vee Y_2 + X_2 \leqq Z_2,\ X_1 \vee Y_2 + X_2 \vee Y_3 + X_3 \leqq Z_3,$$
$$X_1 \vee Y_2 + X_2 \vee Y_3 + X_3 + \sum_{j=4}^{i} X_j \vee Y_j \leqq Z_i \ (i=4,\cdots,n)\Big) .$$

Continuation of this procedure yields (2.8).

3. Order relationship between $GI/G_1 \to G_2$ and $GI/\tilde{G}/1$ queues

For the $GI/G_1 \to G_2$ queue we introduce the following notation.

$W_n(v)$ = the sum of the waiting (in front of the server 1) and blocking times of C_n (i.e., the delay time of C_n) when the system starts from the initial condition that the delay time of C_1 is v.

For the $GI/\tilde{G}/1$ queue, we denote the waiting time of C_n by $\tilde{W}_n(v)$ with the initial condition that the waiting time of C_n is v.

Sakasegawa and Yamazaki [7] showed that

(3.1) $$W_n(v)+S_{1,n}\overset{(1)}{\leqq}\tilde{W}_n(v)+\tilde{S}_n\ ,$$

so that,

(3.2) $$\mathrm{E}\,(W_n(v))\leqq\mathrm{E}\,(\tilde{W}_n(v))+\mathrm{E}\,(\tilde{S}_n)-\mathrm{E}\,(S_{1,n})\ .$$

They conjectured that

(3.3) $$\mathrm{E}\,(\tilde{W})\leqq\mathrm{E}\,(W)^{1)}\ .$$

The objective in this section is to prove the following stronger result for (3.3).

THEOREM. *If* $\mathrm{E}\,(W_n(v))<\infty$, *then*

(3.4) $$\tilde{W}_n(v)\overset{(2)}{\leqq}W_n(v)\ .$$

PROOF. It is well-known that for the $GI/\tilde{G}/1$ queue

(3.5) $$\tilde{W}_n(v)=0\vee(\tilde{U}_{n-1}+\tilde{W}_{n-1}(v))\ ,$$

where $\tilde{U}_{n-1}=\tilde{S}_{n-1}-A_n$. We also have the following recurrence relation for the $GI/G_1\to G_2$ queue (cf. [7]).

(3.6) $$W_n(v)=0\vee(U_{n-1}+S_{1,n-1}-S_{1,n}+W_{n-1}(v))\ ,$$

where $U_{n-1}=S_{1,n}\vee S_{2,n-1}-A_n$. We note that the recurrence relation (3.5) gives

(3.7) $$\begin{aligned}\tilde{W}_n(v)&=0\vee\tilde{U}_{n-1}\vee(\tilde{U}_{n-1}+\tilde{U}_{n-2})\vee\cdots\vee(\tilde{U}_{n-1}+\cdots+\tilde{U}_2)\\&\quad\vee(\tilde{U}_{n-1}+\cdots+\tilde{U}_1+v)\\&=\max\left[0,\sum_{j=i}^{n-1}(S_{1,j}\vee S_{2,j}-A_{j+1}+\delta_{1i}v)\ \ (i=n-1,n-2,\cdots,1)\right]\\&\overset{d}{\sim}\max\left[0,\sum_{j=2}^{i}(S_{1,j}\vee S_{2,j}-A_{j-1}+\delta_{in}v)\ \ (i=2,3,\cdots,n)\right]\ ,\end{aligned}$$

where the symbol $\delta.$ is the Kronecker delta, and, similarly, (3.6) leads

(3.8) $$\begin{aligned}W_n(v)&=0\vee(U_{n-1}+S_{1,n-1}-S_{1,n})\vee(U_{n-1}+U_{n-2}+S_{1,n-2}-S_{1,n})\vee\cdots\\&\quad\vee(U_{n-1}+U_{n-2}+\cdots+U_2+S_{1,2}-S_{1,n})\vee(U_{n-1}+U_{n-2}+\cdots\\&\quad+U_1+S_{1,1}+v-S_{1,n})\\&=\max\left[0,\sum_{j=i}^{n}(S_{1,j}\vee S_{2,j-1}-A_j)+S_{1,i-1}-S_{1,n}+\delta_{2i}v\right.\end{aligned}$$

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1) We often use an r.v. X without a subscript n, which indicates an r.v. with a limiting d.f. of X_n (e.g., W instead of W_n).

$$(i=n, n-1, \cdots, 2)\Big]$$
$$\overset{d}{\sim} \max\Big[0, \sum_{j=1}^{i}(S_{1,j}\vee S_{2,j+1}-A_j)+S_{1,i+1}-S_{1,1}+\delta_{i(n-1)}v$$
$$(i=1, 2, \cdots, n-1)\Big] .$$

From (3.7), (3.8) and (d) of Lemma 1, to prove (3.4) it suffices to show that

$$\text{(3.9)} \qquad \max\Big[\sum_{j=2}^{i}(S_{1,j}\vee S_{2,j}-A_{j-1})+\delta_{in}v \ (i=2, 3, \cdots, n)\Big]$$
$$\overset{(2)}{\leqq} \max\Big[\sum_{j=1}^{i}(S_{1,j}\vee S_{2,j+1}-A_j)+S_{1,i+1}-S_{1,1}+\delta_{i(n-1)}v$$
$$(i=1, 2, \cdots, n-1)\Big] .$$

We now show (3.9). By applying Lemma 4 to the left-hand side of (3.9), we can obtain

$$\text{(3.10)} \quad \max\Big[\sum_{j=2}^{i}(S_{1,j}\vee S_{2,j}-A_{j-1})+\delta_{in}v \ (i=2, 3, \cdots, n)\Big]$$
$$=S_{1,2}\vee S_{2,2}-A_1+\max\Big[0, \sum_{j=3}^{i}(S_{1,j}\vee S_{2,j}-A_{j-1})+\delta_{in}v$$
$$(i=3, \cdots, n)\Big]$$
$$\overset{(2)}{\leqq} S_{1,2}\vee S_{2,2}-S_{1,2}+S_{1,1}-A_1+\max\Big[0, \sum_{j=3}^{i}(S_{1,j}\vee S_{2,j}-A_{j-1})$$
$$+\delta_{in}v \ (i=3, \cdots, n)\Big]$$
$$\overset{d}{\sim} S_{1,1}\vee S_{2,2}-S_{1,1}+S_{1,2}-A_1+\max\Big[0, \sum_{j=3}^{i}(S_{1,j}\vee S_{2,j}-A_{j-1})$$
$$+\delta_{in}v \ (i=3, \cdots, n)\Big]$$
$$=S_{1,1}\vee S_{2,2}-S_{1,1}-A_1+B_1 ,$$

where $B_1=\max\Big[S_{1,2}, \sum_{j=3}^{i}(S_{1,j}\vee S_{2,j}-A_{j-1})+S_{1,2}+\delta_{in}v \ (i=3, \cdots, n)\Big]$. Lemma 6 gives

$$\text{(3.11)} \quad \Pr(B_1\leqq x)=\Pr\Big(S_{1,2}\leqq x,\ S_{1,2}+\sum_{j=3}^{i}S_{1,j}\vee S_{2,j}\leqq x+\sum_{j=3}^{i}A_{j-1}-\delta_{in}v$$
$$(i=3, \cdots, n)\Big)$$
$$\geqq \Pr\Big(S_{1,2}\leqq x,\ \sum_{j=2}^{i-1}S_{1,j}\vee S_{2,j+1}+S_{1,i}\leqq x+\sum_{j=2}^{i-1}A_j-\delta_{in}v$$

$$(i=3,\cdots,n)\Big)=\Pr(B_2\leqq x)\,,$$

where $B_2=\max\left[S_{1,2},\ \sum_{j=2}^{i-1}(S_{1,j}\vee S_{2,j+1}-A_j)+S_{1,i}+\delta_{in}v\ (i=3,\cdots,n)\right]$. From (b) of Lemma 1, (3.10) and (3.11) we have

$$\text{(3.12)}\qquad \begin{aligned}&\max\left[\sum_{j=2}^{i}(S_{1,j}\vee S_{2,j}-A_{j-1})+\delta_{in}v\ (i=2,\cdots,n)\right]\\ &\overset{(2)}{\leqq} S_{1,1}\vee S_{2,2}-S_{1,1}-A_1+B_1\\ &\overset{(1)}{\leqq} S_{1,1}\vee S_{2,2}-S_{1,1}-A_1+B_2\,.\end{aligned}$$

Because the right-hand side of (3.12) is identical with that of (3.9), (3.9) follows by Lemma 1.

It is well-known that both $\tilde{W}_n(v)$ and $W_n(v)$ converge for $n\to\infty$ in distribution if $\mathrm{E}(U)<0$. If, moreover, $\mathrm{E}(\tilde{S}^2)$ and $\mathrm{E}(A)$ are finite in addition, expectations of the limiting d.f.'s of $\tilde{W}_n(v)$ and $W_n(v)$ exist. Hence we can obtain from the above theorem

COROLLARY. *If* $\mathrm{E}(U)<0$, $\mathrm{E}(A)<\infty$ *and* $\mathrm{E}((S_1\vee S_2)^2)<\infty$, *then*

$$\text{(3.13)}\qquad \tilde{W}=\lim_{n\to\infty}\tilde{W}_n(v)\overset{(2)}{\leqq}\lim_{n\to\infty}W_n(v)=W\,,$$

so that (3.3) *follows.*

Finally, in order to show that $\mathrm{E}(\tilde{W})$ is the best as lower bounds on the mean delay time concerning the TQ indicated in this paper, we consider a special case where the service time by the server 1 is constant, i.e., $GI/D\to G$ queue. For this TQ, in [7]

$$\text{(3.14)}\qquad \tilde{W}_n(v)\overset{d}{\sim}W_n(v)$$

was derived and therefore,

$$\text{(3.15)}\qquad \mathrm{E}(\tilde{W})=\mathrm{E}(W)\,.$$

(3.15) proves the above.

We can obtain the following remark by a combination of (3.4) and (3.14).

Remark. Let's denote by $GI/G\to D$ a dual TQ of the $GI/D\to G$ queue which is obtained by interchanging two servers. Then,

$$\text{(3.16)}\qquad [W_n(v)\text{ in the }GI/D\to G\text{ queue}]\overset{(2)}{\leqq}[W_n(v)\text{ in the }GI/G\to D\text{ queue}]^{2)}\,.$$

2) For a departure time epoch of C_n, a stronger result, i.e.,
[the departure epoch of C_n from $GI/D\to G$ queue]
$\overset{(1)}{\leqq}$[that from the $GI/G\to D$ queue] with the same initial condition
has been derived by Kawashima [5].

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KOGAKUIN UNIVERSITY

REFERENCES

[1] Avi-Itzhak, B. and Yadin, M. (1965). A sequence of two servers with no intermediate queue, *Management Sci.*, **11**, 553-564.

[2] Bessler, S. and Veinott, A. F. (1966). Optimal policy a dynamic multiechelon inventory model, *Naval Res. Logist. Quart.*, **13**, 355-389.

[3] Borovkov, A. A. (1970). Factorization identities and properties of the distribution of the supremum of sequential sums, *Theory Prob. Appl.*, **15**, 359-402.

[4] Brumelle, S. L. (1971). Some inequalities for parallel-server queues, *Operat. Res.*, **19**, 402-413.

[5] Kawashima, T. (1975). Reverse ordering of services in tandem queues, *Memoirs of the Defense Academy*, **15**, 151-159.

[6] Miyazawa, M. (1976). Stochastic order relations among $GI/G/1$ queues with a common traffic intensity, *J. Operat. Res. Soc. Japan*, **19**, 193-208.

[7] Sakasegawa, H. and Yamazaki, G. (1977). Inequalities and approximation formula for the mean delay time in tandem queueing systems, *Ann. Inst. Statist. Math.*, **29**, A, 445-466.

[8] Stidham, S., Jr. (1970). On the optimality of single-server queuing systems, *Operat. Res.*, **18**, 708-732.

[9] Stoyan, H. and Stoyan, D. (1969). Monotonieeigenschaften der Kundewartezeiten im Model $GI/G/1$, *Zeit. angew. Math. Mech.*, **49**, 729-734.

[10] Suzuki, T. (1964). On a tandem queue with blocking, *J. Operat. Res. Soc. Japan*, **6**, 137-157.

[11] Tumura, Y. and Ishikawa, A. (1978). Numerical calculation of the tandem queueing systems, *TRU Math.*, **14**, 57-70.