Ann. Inst. Statist. Math.
33 (1981), Part A, 101-113

ASYMPTOTIC BEHAVIOR OF DIFFERENCE BETWEEN A FINITE PREDICTOR AND AN INFINITE PREDICTOR FOR A WEAKLY STATIONARY STOCHASTIC PROCESS

AKIO ARIMOTO

(Received June 18, 1980; revised Nov. 10, 1980)

Let σ_T^2 be the variance of the difference between the finite predictor made from the finite history of a weakly stationary stochastic process with the observation time T and the infinite predictor made from the infinite history of the same process. In this paper, we shall give conditions under which we can evaluate the decreasing speed of σ_T^2 as T tends to infinity. These conditions are related to the analytic properties of the outer factor of the spectral density function.

1. Preliminaries

Let $X(t)$ be a complex valued weakly stationary stochastic process of $t \in R$ (real line) with mean value zero and finite variance and $f(x)$ be the spectral density for the covariance function of $X(t)$, which satisfies conditions

$$f(x) \in L^1(R, dx)\,, \qquad f(x)>0, \text{ a.e.}$$

and

$$\int_{-\infty}^{\infty} \frac{\log f(x)}{1+x^2} dx > -\infty\,. \tag{1.1}$$

We shall denote by $\|h\|=\left(\int_{-\infty}^{\infty} |h(x)|^2 dx\right)^{1/2}$ and $\|k\|_f=\left(\int_{-\infty}^{\infty} |k(x)|^2 f(x) \cdot dx\right)^{1/2}$ the norms of functions $h \in L^2(R, dx)$ and $k \in L^2(R, f(x)dx)$, respectively. $e_s=e_s(x)=e^{isx}$ and $H_{(a,b)}$ denotes the closed linear span of the functions e_t : $a \leqq t \leqq b$ in $L^2(R, f(x)dx)$ for a, b : $-\infty \leqq a \leqq b \leqq \infty$. We shall denote by $\hat{h}=\mathcal{F}h$ the Fourier transform of $h \in L^2(R, dx)$ which is defined by

$$\hat{h}=\lim_{A\to\infty} \psi_A \qquad \text{in } L^2(R, dx) \tag{1.2}$$

with

$$\psi_A(x)=\frac{1}{\sqrt{2\pi}}\int_{-A}^{A} h(t)e^{-itx}dt\,. \tag{1.3}$$

We shall also write this as

$$\hat{h}(x)=\underset{A\to\infty}{\text{l.i.m.}}\ \psi_A(x)\,. \tag{1.4}$$

Let $H^{2^+}(H^{2^-})$ be Hardy space over the upper (lower) half plane in the complex domain. Under the condition (1.1), there exists an outer function $g\in H^{2^+}$ such that

$$f(x)=|g(x)|^2\,,\ \text{a.e.},\qquad x\in R\,, \tag{1.5}$$

$$gH_{(0,\infty)}=H^{2^+} \tag{1.6}$$

and its analytic extension into the upper half plane is

$$g(z)=\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1+\lambda z}{\lambda-z}\,\frac{\log f(\lambda)}{1+\lambda^2}d\lambda\right\},\qquad z=x+iy \tag{1.7}$$

(Ibragimov-Rozanov [6] p. 34). $\bar{g}(z)=\overline{g(\bar{z})}$ is analytic extension which is the outer factor of $f(x)$ in the lower half plane whose value on the real line is $\bar{g}(x)$, the complex conjugate of $g(x)$. $H^{2^+}(H^{2^-})$ may be also defined by

$$H^{2^+}=\{h\in L^2(R, dx)\,;\ \hat{h}(u)=0\ \text{a.e.},\ u\leqq 0\} \tag{1.8}$$

$$H^{2^-}=\{h\in L^2(R, dx)\,;\ \hat{h}(v)=0\ \text{a.e.},\ v\geqq 0\} \tag{1.9}$$

and we have $L^2(R, dx)=H^{2^+}\oplus H^{2^-}$. We shall denote by $P\,(Q)$ the orthogonal projection from $L^2(R, dx)$ onto $H^{2^+}(H^{2^-})$ and by U the orthogonal projection from $L^2(R, f(x)dx)$ onto $H_{(-\infty,0)}$.

Let the modulus of continuity of an a.e. bounded function $\phi(\lambda)$ be

$$\omega(\delta, \phi)=\sup_{0<|h|\leqq\delta}\|\phi(\lambda+h)-\phi(\lambda)\|_\infty\,, \tag{1.10}$$

where

$$\|\phi\|_\infty=\underset{-\infty<\lambda<\infty}{\text{ess sup}}\,|\phi(\lambda)|\,. \tag{1.11}$$

Write $\phi\in\Lambda_\alpha$, when $\omega(\delta, \phi)=O(\delta^\alpha)$ as δ tends to zero, $0<\alpha\leqq 1$. The mean modulus of continuity of a function $\phi\in L^p(R, dx)$ is defined by

$$\omega_p(\delta, \phi)=\sup_{0<|h|\leqq\delta}\|\phi(\lambda+h)-\phi(\lambda)\|_p\,, \tag{1.12}$$

where

$$\|\phi\|_p=\left(\int_{-\infty}^{\infty}|\phi(\lambda)|^p d\lambda\right)^{1/p}.$$

2. Finite prediction and infinite prediction

The classical Szegö's theorem can be formulated in the following way using the projection operator defined in the preceding section.

THEOREM A (Seghier [7], p. 391). *For $s>0$, an infinite predictor (or the projection of e_s onto $H_{(-\infty,0)}$) is given by the formula:*

$$U(e_s)=\bar{g}^{-1}Q(e_s\bar{g})=e_s-\bar{g}^{-1}P(e_s\bar{g}) , \tag{2.1}$$

where $\bar{g}(x)$ is the complex conjugate of an outer factor $g(x)\in H^{2^+}$ such that $|g(x)|^2=f(x)$ is the spectral density of the covariance for a weakly stationary process $X(t)$.

In order to give a finite predictor, we need some other notations. Let M_T be the linear operator from H^{2^+} into H^{2^-} defined by

$$M_T\theta=Q(e_Tg/\bar{g}\theta) , \qquad \text{for } \theta\in H^{2^+} \tag{2.2}$$

and M_T^* be the linear operator from H^{2^-} into H^{2^+} defined by

$$M_T^*\theta=P(e_{-T}\bar{g}/g\theta) , \qquad \text{for } \theta\in H^{2^-} . \tag{2.3}$$

Using the stationarity of our stochastic process and shifting the time scale, we can modify the Seghier's Proposition 3 ([7]) in the following way.

THEOREM B (Modification of Seghier's Proposition 3, [7]). *Let P_T be the orthogonal projection from $L^2(R, f(x)dx)$ onto $H_{(-\infty,0)}\cap H_{(-T,\infty)}$ and assume the operator $I-M_T^*M_T$ can be inversible. Then for $s>0$,*

$$P_T(e_s)=e_s-e_{-T}g^{-1}M_T(I-M_T^*M_T)^{-1}b_s+\bar{g}^{-1}(I-M_T^*M_T)^{-1}b_s , \tag{2.4}$$

where $b_s=-P(\bar{g}e_s)$.

According to Seghier's result, the projection of e_s onto $H_{(-\infty,a)}\cap H_{(-a,\infty)}$, $a>0$ is given by

$$e_s-e_{-a}g^{-1}M(I-M^*M)^{-1}b_s+e_a\bar{g}^{-1}(I-M^*M)^{-1}b_s , \tag{2.5}$$

where $s>a$, $b_s=-P(\bar{g}e_{s-a})$, $M(\theta)=Q(e_{2a}g/\bar{g}\theta)$ for $\theta\in H^{2+}$ and $M^*(\theta)=P(e_{-2a}\bar{g}/g\theta)$ for $\theta\in H^{2-}$. We shall have (2.4) when we multiply (2.5) by e_{-a} and replace $s-a$ by s and $2a$ by T.

$\|\cdot\|$ also denotes the operator norm without ambiguity.

LEMMA 2.1. *Operator norms of M_T and M_T^* satisfy conditions:*

$$\text{(i)} \quad \|M_T\|\leqq 1 , \qquad \|M_T^*\|\leqq 1 \tag{2.6}$$

and

(ii) $\|M_T\|$, $\|M_T^*\|$ *are nonincreasing in* T.

PROOF. We have

$$\begin{aligned}(2.7)\qquad \|M_T\theta\| &= \|Q(e_T g/\bar{g}\theta)\| \\ &\leqq \|Q\|\cdot\|e_T g/\bar{g}\theta\| = \|Q\|\cdot\|\theta\|,\end{aligned}$$

while Q is a projection operator and hence $\|Q\|\leqq 1$ and $\|M_T\|\leqq 1$. For the proof of the second part of the lemma, we use the fact that the Fourier transform $\mathcal{F}$ and its inverse transform $\mathcal{F}^{-1}$ are isometric operators. We have for $\theta\in H^{2+}$,

$$\begin{aligned}(2.8)\qquad \|M_T\theta\| &= \|Q(e_T g/\bar{g}\theta)\| \\ &= \|\mathcal{F}^{-1}((\mathcal{F}(e_T g/\bar{g}\theta))1_{R^-})\| \\ &= \|(\mathcal{F}(e_T g/\bar{g}\theta))1_{R^-}\| \\ &= \left(\int_{-\infty}^{0}\left|\underset{A\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A} e^{iTx}g(x)/\bar{g}(x)\theta(x)e^{-itx}dx\right|^2 dt\right)^{1/2} \\ &= \lim_{A\to\infty}\left(\int_{T}^{\infty}\left|\frac{1}{\sqrt{2\pi}}\int_{-A}^{A} e^{itx}g(x)/\bar{g}(x)\theta(x)dx\right|^2 dt\right)^{1/2},\end{aligned}$$

where 1_{R^-} denotes the indicator function of $(-\infty, 0)$. Thus for each $\theta\in H^{2+}$ it is evident from (2.8) that $\|M_T\theta\|$ is nonincreasing in T.

The following theorem plays an important role for giving the finite predictor.

THEOREM C (Dym [3], p. 402). *If* f^{-1} *is locally integrable, then*

$$(2.9)\qquad H_{(-\infty,0)}\cap H_{(-T,\infty)} = \bigcap_{\varepsilon>0} H_{(-T-\varepsilon,\varepsilon)}\,.$$

If also $\|M_c\|<1$ *for some* $c\geqq 0$, *then*

$$(2.10)\qquad H_{(-\infty,0)}\cap H_{(-T,\infty)} = H_{(-T,0)}$$

for every $T>c$.

It will be assumed throughout in this section and the followings that $\|M_c\|<1$ for some $c\geqq 0$ and consider only T's which are larger than c since we shall deal with the degree of the asymptotic smallness of $\|M_T\|$ as T goes to infinity. When it is also assumed that f^{-1} is locally integrable, the finite predictor of $X(s)$, $s>0$ knowing the history of $X(t)$, $-T\leqq t\leqq 0$ is given from the projection of e_s into $H_{(-T,\infty)}\cap H_{(-\infty,0)}$ and its isometric transformation $e_u\to X(u)$ in view of Theorem C.

We shall give conditions in theorems of the next sections, under which we can evaluate the asymptotic order of the variance of its isometric transform of the difference between the finite predictor (2.4) and the infinite predictor (2.1) when the length T of an observation time interval goes to infinity. We might have the better approximation of the infinite predictor by using the finite predictor when the variance of the difference between them decreases faster.

The difference between the infinite predictor and the finite predictor is given from (2.1) and (2.4).

$$\begin{aligned}(2.11)\quad & U(e_s)-P_T(e_s)\\ &=e_s+\bar{g}^{-1}b_s-(e_s-e_{-T}g^{-1}M_T(I-M_T^*M_T)^{-1}b_s+\bar{g}^{-1}(I-M_T^*M_T)^{-1}b_s)\\ &=e_{-T}g^{-1}(I-M_TM_T^*)^{-1}M_Tb_s-\bar{g}^{-1}M_T^*(I-M_TM_T^*)^{-1}M_Tb_s\,,\end{aligned}$$

where $b_s=-P(\bar{g}e_s)$. Here we used the fact that if $\|M_T\|<1$, then

$$(2.12)\qquad (I-M_T^*M_T)^{-1}=I+M_T^*M_T+(M_T^*M_T)^2+(M_T^*M_T)^3+\cdots\,.$$

The variance σ_T^2 of the random variable transformed isometrically from (2.11) can be evaluated in the following way.

$$\begin{aligned}(2.13)\quad \sigma_T &=\|U(e_s)-P_T(e_s)\|_f\\ &=\|e_{-T}g^{-1}(I-M_TM_T^*)^{-1}M_Tb_s-\bar{g}^{-1}M_T^*(I-M_TM_T^*)^{-1}M_Tb_s\|_f\\ &\leqq\|(I-M_TM_T^*)^{-1}\|(1+\|M_T^*\|)\|M_Tb_s\|\,,\end{aligned}$$

while we have

$$(2.14)\qquad \|(I-M_TM_T^*)^{-1}\|\leqq\frac{1}{1-\|M_T^*\|\,\|M_T\|}\leqq\frac{1}{1-\varepsilon}\,,$$

as far as $\|M_T\|\leqq\varepsilon<1$, $T\geqq c$ for some nonnegative number c and we have assumed this. Hence we have seen from (2.13) and (2.14) that the speed of decrease for σ_T as T tends to infinity is less than the constant multiple of the one for $\|M_Tb_s\|$.

In the following section, using the relation (2.8) we shall give theorems containing conditions under which we can evaluate the speed approaching zero for the variance σ_T^2 defined by (2.13) as T tends to infinity and we shall prove them in Sections 4 and 5.

3. Theorems

σ_T^2 was defined in (2.13) as the variance of the difference between the finite predictor made from the finite history with the observation interval T and the infinite predictor made from the infinite history. We shall deal with the decreasing speed of σ_T as T tends to infinity.

Let $g(x)$ be an outer function for the spectral density $f(x)$ of the stationary stochastic process $X(t)$ ((1.5)-(1.7)).

THEOREM 1. *If $e^{ix\alpha}g(x)$, $\alpha>0$ agrees a.e. on the real axis with the reciprocal of an entire function of exponential type$\leqq\alpha$* (Kawata [5], p. 461), *then we have*

(3.1) $$\sigma_T=0 \qquad \text{for } T\geqq 2\alpha\,.$$

THEOREM 2. *Suppose the function $\bar{g}(x)^{-1}$ an outer function in the lower half plane can be extended analytically into the strip, $z=x+iy$, $0\leqq y<\beta$ in the upper half plane. Moreover we assume this analytic extension $g(z)/\bar{g}(z)$ is bounded in the closed strip $0\leqq y\leqq\gamma$ for some non-negative number $\gamma<\beta$:*

(3.2) $$|g(z)/\bar{g}(z)|\leqq K_\gamma\,,$$

where K_γ is a finite constant only depending on γ. Then we have

(3.3) $$\sigma_T=O(K_\gamma e^{-\gamma T})$$

for large T.

Proofs of these theorems are given in Section 4.

Example 1. If the spectral density is

(3.4) $$f(x)=\frac{1}{1+x^2}\,,$$

then the outer function is

(3.5) $$g(x)=\frac{1}{x+i}$$

(This is an outer function. See Devinatz [1], p. 86.) and its reciprocal function is $\bar{g}(x)^{-1}=x-i$. In this case $\bar{g}(z)^{-1}$ is an entire function. By Theorem 1, $\sigma_T=0$ for any $T\geqq 0$.

Example 2. If the spectral density is

(3.6) $$f(x)=\frac{x^2+4}{(x^2+1)^2}\,,$$

then the outer function is

(3.7) $$g(x)=\frac{x+2i}{(x+i)^2}$$

and

(3.8) $$\frac{g(z)}{\bar{g}(z)}=\frac{(z-i)^2(z+2i)}{(z+i)^2(z-2i)}$$

which is analytic in $z=x+iy$, $-1<y<2$ and bounded in z, $0\leqq y\leqq\gamma<2$. By Theorem 2, we have

(3.9) $$\sigma_T=O(e^{-\gamma T})\,, \qquad \text{for any } \gamma<2\,.$$

THEOREM 3. *Let* $\|M_T\|<1$ *for* $T \geqq c$, *some positive constant. Then*

$$\sigma_T^2 = O\left(\sum_{k=0}^{\infty} \omega^2(1/2^k T, g/\bar{g})\right), \tag{3.10}$$

for $T \geqq c$.

COROLLARY. *Suppose* $\|M_T\|<1$ *for* $T \geqq c$, *for some positive constant and*

$$\operatorname{Arg} g(x) \in \Lambda_\beta \tag{3.11}$$

for some $\beta>0$, *where for any complex number* z, $\operatorname{Arg} z$ *is the principal argument of* z, $-\pi<\operatorname{Arg} z \leqq \pi$. *Then we have for* $T \geqq c$

$$\sigma_T = O(1/T^\beta)\,. \tag{3.12}$$

Proofs of Theorem 3 and this corollary are given in Section 5.

4. Proofs of Theorems 1 and 2

(I) *Proof of Theorem* 1

We have

$$\|M_T b_s\| \leqq \|M_T\| \, \|b_s\| \tag{4.1}$$

and

$$\begin{aligned} \|b_s\| &= \|P(e_s \bar{g})\| \\ &\leqq \|g\| = \left(\int_{-\infty}^{\infty} f(x)dx\right)^{1/2}, \end{aligned} \tag{4.2}$$

which is the square root of the variance of the stochastic process and is finite. Hence we have only to consider the speed of decrease for $\|M_T\|$ instead of σ_T in view of (2.13). Under the condition of Theorem 1, we can find from Lemma 2.1 of Dym [2], p. 25 that $e^{2i\alpha x}\cdot g(x)\bar{g}^{-1}(x)$ agree a.e. on the real line with an inner function $j(x)$ in the upper half plane. In this case, from (2.8) we have for $T \geqq 2\alpha$ that

$$\begin{aligned} \|M_T\theta\|^2 &= \int_T^\infty \underset{A\to\infty}{\text{l.i.m.}} \left| \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{itx} g(x)\bar{g}^{-1}(x)\theta(x)dx \right|^2 dt \\ &= \int_T^\infty \underset{A\to\infty}{\text{l.i.m.}} \left| \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{i(t-2\alpha)x} j(x)\theta(x)dx \right|^2 dt \\ &= 0 \end{aligned} \tag{4.3}$$

because $\theta \in H^{2^+}$ and $j\theta \in H^{2^+}$.

$$\|M_T\|^2 = \sup_{\theta \in H^{2^+}} \|M_T\theta\|^2/\|\theta\|^2 \tag{4.4}$$

$$=0\,,$$

for $T\geqq 2\alpha$, which completes the proof of Theorem 1.

(II) *Proof of Theorem* 2

By assumptions of our theorem, we can find using the Cauchy's integral formula that

$$\int_{\Gamma_A} e^{itz}g(z)\bar{g}^{-1}(z)\theta(z)dz=0\,, \tag{4.5}$$

where Γ_A is the contour of a rectangle in the upper half plane, $z=x+iy$, $0<\varepsilon\leqq y\leqq\gamma$, $-A\leqq x\leqq A$ for positive A (See Fig. 1). Hence we can write from (4.5)

$$\begin{aligned}(4.6)\qquad &\int_{-A}^{A} e^{itx-\varepsilon t}g(x+i\varepsilon)\bar{g}^{-1}(x+i\varepsilon)\theta(x+i\varepsilon)dx\\ &+i\int_{\varepsilon}^{\gamma} e^{iAt-yt}g(A+iy)\bar{g}^{-1}(A+iy)\theta(A+iy)dy\\ &=\int_{-A}^{A} e^{itx-\gamma t}g(x+i\gamma)\bar{g}^{-1}(x+i\gamma)\theta(x+i\gamma)dx\\ &+i\int_{\varepsilon}^{\gamma} e^{-iAt-yt}g(-A+iy)\bar{g}^{-1}(-A+iy)\theta(-A+iy)dy\,.\end{aligned}$$

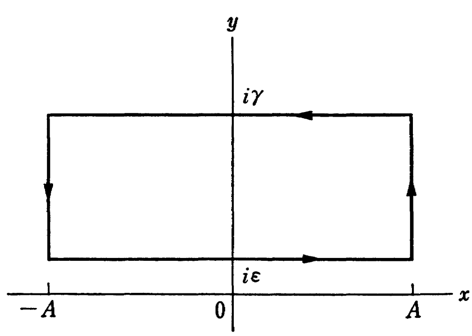

Fig. 1

Let

$$[\hat{G}(t,y)]_A=\frac{1}{\sqrt{2\pi}}\int_{-A}^{A} G(x+iy)e^{itx}dx\,, \tag{4.7}$$

where $G(z)=g(z)\bar{g}^{-1}(z)\theta(z)$, $z=x+iy$.

Since from our assumption, $G(x+iy)\in L^2(R,dx)$, $y>0$, there exists for each $y>0$,

$$\hat{G}(t,y)=\underset{A\to\infty}{\text{l.i.m.}}\,[\hat{G}(t,y)]_A\,. \tag{4.8}$$

Since $\theta(z)$ belongs to H^{2+}, we have for some constant $C>0$

$$(4.9) \qquad |\theta(z)| \leqq C/\sqrt{\varepsilon}\,, \qquad \text{for } z=x+iy\,,\ y \geqq \varepsilon\,,$$

and $\theta(z)$ converges to zero as $|z| \to \infty$ uniformly in y, $y \geqq \varepsilon$ (Kawata [5], p. 219). Thus from (3.2) $G(\pm A+iy)$ tends to zero boundedly in y, $\varepsilon \leqq y \leqq \gamma$ as A tends to infinity and hence from (4.6) and (4.8) we have

$$(4.10) \qquad e^{-\varepsilon t}\hat{G}(t, \varepsilon) = e^{-\gamma t}\hat{G}(t, \gamma)\,.$$

If we denote by $\hat{G}(t) = \underset{A\to\infty}{\text{l.i.m.}} \int_{-A}^{A} G(x)e^{itx}dx$ the Fourier transform of $G(x) = G(x+0i) \in L^2(R, dx)$, then we have by the Parseval's identity and the Minkowski's inequality

$$(4.11) \qquad \begin{aligned} \|\hat{G}(t, \varepsilon) - \hat{G}(t)\| &= \|G(x+i\varepsilon) - G(x)\| \\ &= \left(\int_{-\infty}^{\infty} \left| \frac{g(x+i\varepsilon)}{\bar{g}(x+i\varepsilon)}\theta(x+i\varepsilon) - \frac{g(x)}{\bar{g}(x)}\theta(x) \right|^2 dx\right)^{1/2} \\ &\leqq \left(\int_{-\infty}^{\infty} \left| \frac{g(x+i\varepsilon)}{\bar{g}(x+i\varepsilon)} - \frac{g(x)}{\bar{g}(x)} \right|^2 |\theta(x+i\varepsilon)|^2 dx\right)^{1/2} \\ &\quad + \left(\int_{-\infty}^{\infty} \left| \frac{g(x+i\varepsilon)}{\bar{g}(x+i\varepsilon)} \right|^2 |\theta(x+i\varepsilon) - \theta(x)|^2 dx\right)^{1/2}. \end{aligned}$$

Since

$$(4.12) \qquad \lim_{\varepsilon\to 0} \frac{g(x+i\varepsilon)}{\bar{g}(x+i\varepsilon)} = \frac{g(x)}{\bar{g}(x)}\,, \qquad \text{a.e.},$$

the first term in (4.11) tends to zero by the dominated convergence and the second term tends to zero by the property of H^{2+} function $\theta(x)$.

From (2.8), (4.10) and (4.11)

$$(4.13) \qquad \begin{aligned} &\|M_T\theta\|^2 \\ &= \lim_{A\to\infty} \int_T^{\infty} \frac{1}{\sqrt{2\pi}} \left| \int_{-A}^{A} e^{itx} g(x)\bar{g}^{-1}(x)\theta(x)dx \right|^2 dt \\ &= \int_T^{\infty} \left| \underset{\varepsilon\to 0}{\text{l.i.m.}}\ \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{itx-\varepsilon t} g(x+i\varepsilon)\bar{g}^{-1}(x+i\varepsilon)\theta(x+i\varepsilon)dx \right|^2 dt \\ &= \lim_{A\to\infty} \int_T^{\infty} \left| \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{ity-\gamma t} g(x+i\gamma)\bar{g}^{-1}(x+i\gamma)\theta(x+i\gamma)dx \right|^2 dt \\ &\leqq e^{-2\gamma T} K_\gamma^2 \int_{-\infty}^{\infty} |\theta(x)|^2 dx \end{aligned}$$

which is the desired result.

5. Proofs of Theorem 3 and Corollary

Proofs of Theorem 3 and its corollary will be given as a series of certain lemmas.

LEMMA 1. *Let $\|M_T\|<1$ for $T\geqq c$, some positive c. Then*

$$\sigma_T^2=O\Big(\sum_{k=0}^{\infty}\omega_2^2(1/2^kT,\, g\bar{g}^{-1}b_s)\Big) \tag{5.1}$$

holds.

Denote by $F(t)$ the inverse Fourier transform of $g\bar{g}^{-1}b_s(x)$, then we have

$$F(t)=\underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{\sqrt{2\pi}}\int_{-A}^{A} g(x)\bar{g}^{-1}(x)b_s(x)e^{itx}dx \tag{5.2}$$

and

$$\begin{aligned}&\frac{g(x+h/2)}{\bar{g}(x+h/2)}b_s(x+h/2)-\frac{g(x-h/2)}{\bar{g}(x-h/2)}b_s(x-h/2)\\ &\quad=\underset{A\to\infty}{\text{l.i.m.}}\left\{-\frac{2i}{\sqrt{2\pi}}\int_{-A}^{A}F(t)\sin\frac{th}{2}e^{-itx}dt\right\}.\end{aligned} \tag{5.3}$$

By the Parseval's equality, it follows that

$$\begin{aligned}&\int_{-\infty}^{\infty}\left|\frac{g(x+h/2)}{\bar{g}(x+h/2)}b_s(x+h/2)-\frac{g(x-h/2)}{\bar{g}(x-h/2)}b_s(x-h/2)\right|^2dx\\ &\quad=4\int_{-\infty}^{\infty}|F(t)|^2\sin^2\frac{th}{2}dt\end{aligned} \tag{5.4}$$

which is at most equal to $\omega_2^2(h,\, g\bar{g}^{-1}b_s)$. Thus we have from (5.4)

$$\omega_2^2(h,\, g\bar{g}^{-1}b_s)\geqq\int_{\pi/2h}^{\pi/h}|F(t)|^2dt \tag{5.5}$$

because of $\sin x\geqq\dfrac{2}{\pi}x$ for $0\leqq x\leqq\pi/2$. Accordingly

$$\omega_2^2(h/2^k,\, g\bar{g}^{-1}b_s)\geqq\int_{\pi 2^k/2h}^{\pi 2^{k+1}/2h}|F(t)|^2dt\,,\qquad k=0,1,2,\cdots. \tag{5.6}$$

By adding these inequalities, we find

$$\int_{1/h}^{\infty}|F(t)|^2dt\leqq C\sum_{k=0}^{\infty}\omega_2^2(h/2^k,\, g\bar{g}^{-1}b_s)\,, \tag{5.7}$$

for some constant $C>0$.

On the other hand, from (2.8)

$$\|M_Tb_s\|^2=\int_T^{\infty}|F(t)|^2dt \tag{5.8}$$

and thus it follows from (2.14), (5.7) and (5.8)

(5.9) $$\sigma_T^2 \leqq C \sum_{k=0}^{\infty} \omega_2(1/2^k T, g\bar{g}^{-1}b_s)^2$$

which completes the proof.

LEMMA 2. *We have*

(5.10) $$\omega_2(\delta, g\bar{g}^{-1}b_s) \leqq C\omega(\delta, g\bar{g}^{-1})$$

for some constant $C>0$.

From the definition (1.12) of the mean modulus of continuity, we have

(5.11) $$\omega_2(\delta, g\bar{g}^{-1}b_s) = \sup_{|h|\leqq\delta}\left(\int_{-\infty}^{\infty}\left|\frac{g(x+h)}{\bar{g}(x+h)}b_s(x+h)-\frac{g(x)}{\bar{g}(x)}b_s(x)\right|^2 dx\right)^{1/2}.$$

The last integral in (5.11) is less than

(5.12) $$\left(\int_{-\infty}^{\infty}|b_s(x+h)-b_s(x)|^2 dx\right)^{1/2}+\left(\int_{-\infty}^{\infty}|b_s(x)|^2\left|\frac{g(x+h)}{\bar{g}(x+h)}-\frac{g(x)}{\bar{g}(x)}\right|^2 dx\right)^{1/2}$$

by the Minkowski's inequality.

On the other hand, we have

(5.13) $$\hat{\bar{g}}(t)=\mathcal{F}(\bar{g})(t)=0 \qquad \text{for } t>0\,,$$

because $\bar{g}(t)\in H^{2^-}$. Thus we have from (2.4)

(5.14) $$\begin{aligned} b_s(x) &= -P(\bar{g}e_s) = -\mathcal{F}^{-1}(\mathcal{F}(\bar{g}e_s)1_{R^+}) \\ &= -\frac{1}{\sqrt{2\pi}}\int_0^s e^{itx}\hat{\bar{g}}(t-s)dt\,. \end{aligned}$$

Since $\hat{\bar{g}}(u)=\hat{g}(-u)$, we have

(5.15) $$\bar{b}_s(x)=\frac{-1}{\sqrt{2\pi}}\int_0^s e^{i(u-s)x}\hat{g}(u)du$$

and

(5.16) $$\overline{b_s(x+h)}-\overline{b_s(x)}=\frac{1}{\sqrt{2\pi}}\int_0^s e^{i(u-s)x}(e^{i(u-s)h}-1)\hat{g}(u)du\,.$$

By the Parseval's equality

(5.17) $$\begin{aligned}\int_{-\infty}^{\infty}|b_s(x+h)-b_s(x)|^2dx &= \int_0^s|e^{i(u-s)h}-1|^2|\hat{g}(u)|^2du \\ &\leqq h^2s^2\int_0^s|\hat{g}(u)|^2du \leqq h^2s^2\int_{-\infty}^{\infty}f(x)dx\,.\end{aligned}$$

Since $\lim_{h\to 0}\omega(h,\phi)/h>0$ holds for a function $\phi(x)$ which is not identically a constant (Timan [8], p. 104), we have from (5.12), (5.17) and (4.2) the desired result (5.10).

LEMMA 3. *Suppose*

$$\mathrm{Arg}\, g(x) \in \Lambda_\beta\,, \tag{5.18}$$

then we have

$$\omega(\delta, g\bar{g}^{-1}) \leqq C\delta^\beta \tag{5.19}$$

for some constant $C>0$.

Since we have

$$\frac{g(x)}{\bar{g}(x)} = e^{2i\,\mathrm{Arg}\, g(x)}\,, \tag{5.20}$$

it follows that

$$\begin{aligned}\left|\frac{g(x+h)}{\bar{g}(x+h)} - \frac{g(x)}{\bar{g}(x)}\right| &\leqq \left|e^{2i\,\mathrm{Arg}\, g(x+h)} - e^{2i\,\mathrm{Arg}\, g(x)}\right| \\ &\leqq 2\left|\mathrm{Arg}\, g(x+h) - \mathrm{Arg}\, g(x)\right| \\ &\leqq Ch^\beta\end{aligned} \tag{5.21}$$

for some constant $C>0$, and thus we have (5.19).

Now combining the Lemmas 1 and 2, we have Theorem 3. Furthermore using the Lemma 3, we have the corollary.

Acknowledgement

The author would like to express his thanks to Professor Tatsuo Kawata for his valuable guidance. He is also grateful to the referee for his useful comments.

MUSASHI INSTITUTE OF TECHNOLOGY

REFERENCES

[1] Devinatz, A. (1967). *On Wiener-Hopf Operators*, Proc. Irvine Conf. on Functional Analysis, Washington D.C., 81-118.
[2] Dym, H. (1977). Trace formulas for a class of Toeplitz-like operators, *Israel J. Math.*, **27**, 21-48.
[3] Dym, H. (1978). A problem in trigonometric approximation theory, *Illinois J. Math.*, **22**, 402-403.
[4] Grenander, U. and G. Szegö (1958). *Toeplitz Forms and Their Applications*, Univ. of California Press, Berkeley.
[5] Kawata, T. (1972). *Fourier Analysis in Probability Theory*, Academic Press, New York.
[6] Ibragimov, I. A. and Rozanov, Yu. A. (1978). *Gaussian Random Processes*, Springer-Ver.

[7] Seghier, A. (1978). Prediction d'un processus stationnaire du second ordre de covariance connue sur un intervale fini, *Illinois J. Math.*, **22**, 389-401.

[8] Timan, A. F. (1966). *Theory of Approximation of Functions of a Real Variable*, Hindustan Pub. Cor., India.