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BAYES ESTIMATION WITH SPHERICALLY SYMMETRIC, CONVEX LOSS

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Summary

It is desired to estimate a parameter $\theta \in \mathcal{R}^n$ with the loss function of the form $L(\theta, a)=W(\|\theta-a\|)$, where $W: \mathcal{R}^+ \to \mathcal{R}^+$ is convex, differentiable, and non-decreasing. With this structure a characterization of Bayes estimators is given. Also it is noted that if the sample space, $\mathcal{X}$, for the observation, X, is a complete separable metric space then a Bayes estimator exists.

1. Introduction

In DeGroot and Rao [2] a characterization of a Bayes estimator for θ when the loss is of the form $L(\theta, a)=W(|\theta-a|)$ is given. They show that if $W: \mathcal{R}^+ \to \mathcal{R}^+$ is non-decreasing, differentiable, and convex, a measurable function δ is a Bayes estimator for θ if and only if the following inequalities hold a.e. (Π): [Π being the marginal distribution for X.]

$$\int_{\theta \geqq \delta(x)} W'(\theta-\delta(x))dF(\theta|x) \geqq \int_{\theta<\delta(x)} W'(\delta(x)-\theta)dF(\theta|x)$$

and

$$\int_{\theta>\delta(x)} W'(\theta-\delta(x))dF(\theta|x) \leqq \int_{\theta \leqq \delta(x)} W'(\delta(x)-\theta)dF(\theta|x) .$$

Here $F(\cdot|x)$ is the posterior distribution function when the observed value of X is x.

In this work a similar characterization is given for θ being a vector valued parameter and $L(\theta, a)=W(\|\theta-a\|)$, where W is as above.

The case when $L(\theta, a)=\sum_{i=1}^{n} W_i(|\theta_i-a_i|)$ can be handled using the DeGroot and Rao results if each W_i is as above. In this case each

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coordinate, $\delta_i(x)$, of $\delta(x)$ satisfies the DeGroot and Rao characterization with W_i, δ_i, $F_i(\theta_i|x)$ substituted for W, δ, and $F(\theta|x)$ respectively for each $i=1, 2, \cdots, n$. Here $F_i(\theta_i|x)$ is the posterior distribution function for θ_i (the ith coordinate of θ) given $X=x$.

In DeGroot and Rao [3] similar results are given to those in this paper. DeGroot and Rao, however, have more restrictions on the loss function, W, yet work in a more general parameter space, Banach space. The results here and in DeGroot and Rao [3] are closely related to the very abstract results in Strasser [6]. Contained herein is a direct and easily accessible proof for the important case when the parameter space is $\mathcal{R}^n$.

Recall, a Bayes estimate for the given value of x is a number, δ_x^*, such that

$$\int_{\mathcal{R}^n} W(\|\theta-\delta_x^*\|)dF(\theta|x)=\inf \int_{\mathcal{R}^n} W(\|\theta-a\|)dF(\theta|x) .$$

Thus the problem of finding a Bayes estimate is a problem of finding a minimizer of

$$\int_{\mathcal{R}^n} W(\|\theta-a\|)dF(\theta)$$

for a specified distribution function F.

In what follows, all vectors are written as column vectors with T denoting the transpose operation.

In Section 2 the solutions of this minimization problem are characterized and some properties of the minimizing values are discussed. In Section 3 this minimization is discussed as it relates to Bayes estimators. In addition, it is shown that if the sample space, $\mathcal{X}$, is a separable, complete, metric space then a Bayes estimator will always exist.

2. The minimization problem

Suppose W is as in the introduction; convex, non-decreasing, and differentiable on $(0, \infty)$. Define $W'(0)=\lim_{\varepsilon\to 0^+} W'(\varepsilon)$. To avoid a triviality assume that W is not identically constant. Also, throughout we assume

$$\int_{\mathcal{R}^n} W(\|\theta-a\|)dF(\theta)<\infty .$$

With this assumption it follows that

$$\int_{\mathcal{R}^n} W'(\|\theta-a\|)dF(\theta)<\infty .$$

Because, since for each $\varepsilon>0$, from the convexity of W, we have

$$\begin{aligned}\int_{\mathcal{R}^n} W'(\|\theta-a\|)dF(\theta) &\leqq \int_{\mathcal{R}^n} \frac{W(\|\theta-a\|+\varepsilon)-W(\|\theta-a\|)}{\varepsilon}dF(\theta) \\ &= \int_{\mathcal{R}^n} \frac{W(\|\theta-a\|+\varepsilon)}{\varepsilon}dF(\theta) \\ &\quad -\int_{\mathcal{R}^n} \frac{W(\|\theta-a\|)}{\varepsilon}dF(\theta)<\infty .\end{aligned}$$

A proof of the finiteness is not difficult, but is too tedious to include.

Now, let

$$U(a)=\int_{\mathcal{R}^n} W(\|\theta-a\|)dF(\theta) \qquad \text{for } \|a\|<\infty . \tag{2.1}$$

LEMMA 1. *U: $\mathcal{R}^n \to \mathcal{R}^+$ is convex.*

PROOF. Define $Q_\theta(a)=W(\|\theta-a\|)$. Then if $b, c \in \mathcal{R}^n$ it follows that

$$\begin{aligned}Q_\theta\Big(\frac{1}{2}(b+c)\Big) &= W\Big(\Big\|\theta-\frac{1}{2}(b+c)\Big\|\Big) \\ &= W\Big(\Big\|\frac{1}{2}(\theta-b)+\frac{1}{2}(\theta-c)\Big\|\Big) \\ &\leqq W\Big(\frac{1}{2}\|\theta-b\|+\frac{1}{2}\|\theta-c\|\Big) \\ &\leqq \frac{1}{2}W(\|\theta-b\|)+\frac{1}{2}W(\|\theta-c\|) \\ &= \frac{1}{2}Q_\theta(b)+\frac{1}{2}Q_\theta(c) .\end{aligned}$$

Thus each Q_θ : $\mathcal{R}^n \to \mathcal{R}^+$ is convex. Thus, if $b, c \in \mathcal{R}^n$ and $b\neq c$ we have

$$\begin{aligned}U\Big(\frac{1}{2}(b+c)\Big) &= \int_{\mathcal{R}^n} W\Big(\Big\|\theta-\frac{1}{2}(b+c)\Big\|\Big)dF(\theta) \\ &\leqq \int_{\mathcal{R}^n}\Big(\frac{1}{2}Q_\theta(b)+\frac{1}{2}Q_\theta(c)\Big)dF(\theta) \\ &= \frac{1}{2}U(b)+\frac{1}{2}U(c) .\end{aligned}$$

Thus U is convex.

LEMMA 2. $\operatorname*{Lim}_{\|a\|\to\infty} U(a)=\infty$.

PROOF. Fatou's lemma implies

$$\lim_{\|a\|\to\infty} U(a) \geqq \int_{\mathcal{R}^n} \lim_{\|a\|\to\infty} W(\|\theta-a\|)dF(\theta) .$$

But for each θ we have $\lim_{\|a\|\to\infty} W(\|\theta-a\|)=\infty$ since W is non-decreasing, convex, and not identically constant. Thus the lemma is establised.

Lemmas 1 and 2 imply that the set of values which minimize U is a non-empty, closed, bounded, convex set in $\mathcal{R}^n$. Call this set M. Now suppose $\xi \in \mathcal{R}^n$ with $\|\xi\|=1$. Define

$$U_\xi(a)=\lim_{t\to 0^+} \frac{U(a+t\xi)-U(a)}{t} . \tag{2.2}$$

By Theorem A.1 of the appendix, we see that $a \in M$ if and only if $U_\xi(a)\geqq 0$ for all ξ with $\|\xi\|=1$.

Now, define

$$Q_{\theta,\xi}(a)=\begin{cases} W'(\|\theta-a\|)\dfrac{(a-\theta)^T\xi}{\|a-\theta\|} & \text{for } a\neq\theta , \\ W'(0) & \text{for } a=\theta . \end{cases}$$

LEMMA 3. *If* $\|a\|<\infty$ *then*

$$U_\xi(a)=\int_{\mathcal{R}^n} Q_{\theta,\xi}(a)dF(\theta)<\infty .$$

PROOF. Let $\{t_k\}_{k=1}^\infty$ be a sequence of positive reals converging to zero. For any fixed a consider the sequence $\{G_{\xi,k}(\cdot)\}_{k=1}^\infty$ defined by

$$G_{\xi,k}(\theta)=\frac{W(\|\theta-a-t_k\xi\|)-W(\|\theta-a\|)}{t_k} .$$

Note that $\lim_{k\to\infty} G_{\xi,k}(\theta)=Q_{\theta,\xi}(a)$ and that

$$\lim_{k\to\infty}\int_{\mathcal{R}^n} G_{\xi,k}(\theta)dF(\theta)=\lim_{k\to\infty}\frac{U(a+t_k\xi)-U(a)}{t_k}=U_\xi(a) .$$

Thus, the lemma is established once we show that the limit may be moved inside the integral. Thus consider $|G_{\xi,k}(\theta)|$. We first want to show that

$$|G_{\xi,k}(\theta)|\leqq\frac{W(\|\theta-a\|+t_k)-W(\|\theta-a\|)}{t_k}=H_k(\theta) .$$

Now, note that

$$W(\|\theta-a-t_k\xi\|)\leqq W(\|\theta-a\|+t_k)$$

since W is non-decreasing. Also note that

$$W(\|\theta-a-t_k\xi\|)\geqq W(\|\theta-a\|-t_k)$$

for $t_k<\|\theta-a\|$ for the same reason. Thus if $t_k<\|\theta-a\|$ then

$$
\begin{aligned}
&|W(\|\theta-a-t_k\xi\|)-W(\|\theta-a\|)| \\
&\leqq \begin{cases} W(\|\theta-a\|+t_k)-W(\|\theta-a\|) & \text{for } \|\theta-a-t_k\xi\|>\|\theta-a\| \\ W(\|\theta-a\|)-W(\|\theta-a\|-t_k) & \text{for } \|\theta-a-t_k\xi\|\leqq\|\theta-a\| . \end{cases}
\end{aligned}
$$

However, if $t_k<\|\theta-a\|$ then

$$W(\|\theta-a\|)-W(\|\theta-a\|-t_k)\leqq W(\|\theta-a\|+t_k)-W(\|\theta-a\|) .$$

This follows since W is convex. To see this, suppose $0<y<x$, then

$$\frac{W(x+y)+W(x-y)}{2}\geqq W\Big(\frac{1}{2}(x+y)+\frac{1}{2}(x-y)\Big)=W(x) .$$

Thus

$$W(x+y)+W(x-y)\geqq 2W(x) ,$$

which implies

$$W(x+y)-W(x)\geqq W(x)-W(x-y) .$$

Thus if $t_k<\|\theta-a\|$ then $|G_{\xi,k}(\theta)|\leqq H_k(\theta)$. Now consider $t_k\geqq\|\theta-a\|$. For this case we have

$$W(0)\leqq W(\|\theta-a-t_k\xi\|)\leqq W(\|\theta-a\|+t_k) .$$

Thus,

$$
\begin{aligned}
W(0)-W(\|\theta-a\|)&\leqq W(\|\theta-a-t_k\xi\|)-W(\|\theta-a\|) \\
&\leqq W(\|\theta-a\|+t_k)-W(\|\theta-a\|) .
\end{aligned}
$$

But

$$W(\|\theta-a\|)-W(0)\leqq W(\|\theta-a\|+t_k)-W(\|\theta-a\|) . \tag{2.3}$$

Now (2.3) holds since $t_k\geqq\|\theta-a\|$ implies $\frac{1}{2}\|\theta-a\|+\frac{1}{2}t_k\geqq\|\theta-a\|$, which implies

$$W\Big(\frac{1}{2}\|\theta-a\|+\frac{1}{2}t_k\Big)\geqq W(\|\theta-a\|) .$$

However, the convexity of W implies

$$\frac{W(\|\theta-a\|+t_k)+W(0)}{2}\geqq W\Big(\frac{1}{2}\|\theta-a\|+\frac{1}{2}t_k\Big)$$

which implies inequality (2.3). Thus we see $|G_{\xi,k}(\theta)|\leqq H_k(\theta)$. But note, $H_k(\theta)$ is a monotonically decreasing, non-negative sequence with $\int_{\mathfrak{R}^n} H_k(\theta)dF(\theta)<\infty$ for each k, so

$$\lim_{k\to\infty}\int_{\mathcal{R}^n} H_k(\theta)dF(\theta)=\int_{\mathcal{R}^n}\lim_{k\to\infty} H_k(\theta)dF(\theta)\,,$$
$$=\int_{\mathcal{R}^n} W'(\|\theta-a\|)dF(\theta)<\infty\,.$$

Thus the Lebesgue generalized convergence theorem implies that

$$\lim_{k\to\infty}\int_{\mathcal{R}^n} G_{\xi,k}(\theta)dF(\theta)=\int_{\mathcal{R}^n}\lim_{k\to\infty} G_{\xi,k}(\theta)dF(\theta)=\int_{\mathcal{R}^n} Q_{\theta,\xi}(a)dF(\theta)\,.$$

Thus, the lemma is established.

THEOREM 1. *The set of values that minimize U is a non-empty, bounded, closed, convex set $M\subset\mathcal{R}^n$ such that $a\in M$ if and only if*

$$\sum_{i=1}^{n}\left[\int_{\mathcal{R}^n-\{a\}} W'(\|\theta-a\|)\frac{a_i-\theta_i}{\|\theta-a\|}dF(\theta)\right]^2\leqq[W'(0)\mu_F(a)]^2\,, \tag{2.4}$$

where μ_F is the measure induced by F.

PROOF. The only property of M that is not obvious from Lemmas 1, 2, and 3 is that $a\in M$ if and only if (2.4) holds. Now, $a\in M$ if and only if $U_\xi(a)\geqq 0$ for all ξ with $\|\xi\|=1$. But

$$U_\xi(a)=\int_{\mathcal{R}^n} Q_{\theta,\xi}(a)dF(\theta)$$
$$=W'(0)\mu_F(a)+\int_{\mathcal{R}^n-\{a\}} W'(\|\theta-a\|)\frac{(a-\theta)^T\xi}{\|a-\theta\|}dF(\theta)\,.$$

Thus $a\in M$ if and only if

$$\int_{\mathcal{R}^n-\{a\}} W'(\|\theta-a\|)\frac{(a-\theta)^T\xi}{\|a-\theta\|}dF(\theta)\geqq -W'(0)\mu_F(a)\,.$$

However,

$$\int_{\mathcal{R}^n-\{a\}} W'(\|\theta-a\|)\frac{(a-\theta)^T\xi}{\|a-\theta\|}dF(\theta) \tag{2.5}$$
$$=\sum_{i=1}^{n}\xi_i\int_{\mathcal{R}^n-\{a\}} W'(\|\theta-a\|)\frac{(a_i-\theta_i)}{\|\theta-a\|}dF(\theta)\,.$$

Now the right hand side of (2.5)$\geqq -W'(0)\mu_F(a)$ for all ξ with $\|\xi\|=1$ if, and only if, (2.4) holds. Thus the theorem is proved.

LEMMA 4. *Suppose W is strictly convex. For $y, z\in\mathcal{R}^n$ define $B(y, z)=\{x\in\mathcal{R}^n|z^T(x-y)=0\}$. Suppose $\mu_F(B(y, z))<1$ for each $y, z\in\mathcal{R}^n$, i.e. no $n-1$ dimensional hyperplane has probability one under F. Then U is strictly convex.*

PROOF. The proof follows as in the proof of Lemma 1. We now conclude

$$W\left(\frac{1}{2}\|\theta-b\|+\frac{1}{2}\|\theta-c\|\right)\leqq\frac{1}{2}Q_\theta(b)+\frac{1}{2}Q_\theta(c)$$

with strict inequality when $\|\theta-b\|\neq\|\theta-c\|$. But $\{\theta\in\mathscr{R}^n\,|\,\|\theta-b\|=\|\theta-c\|\}$ $=\left\{\theta\in\mathscr{R}^n\,\middle|\,(c-b)^T\left[\theta-\frac{1}{2}(b+c)\right]=0\right\}=B\left(\frac{1}{2}(b+c),\ b-c\right)=B^*$. Thus

$$\begin{aligned}U\left(\frac{1}{2}(b+c)\right)&=\int_{\mathscr{R}^n}W\left(\left\|\theta-\frac{1}{2}(b+c)\right\|\right)dF(\theta)\\&=\int_{B^*}W\left(\left\|\theta-\frac{1}{2}(b+c)\right\|\right)dF(\theta)+\int_{\mathscr{R}^n-B^*}W\left(\left\|\theta-\frac{1}{2}(b+c)\right\|\right)dF(\theta)\\&<\int_{B^*}\left(\frac{1}{2}Q_\theta(b)+\frac{1}{2}Q_\theta(c)\right)dF(\theta)+\int_{\mathscr{R}^n-B^*}\left(\frac{1}{2}Q_\theta(b)+\frac{1}{2}Q_\theta(c)\right)dF(\theta)\\&<\frac{1}{2}U(b)+\frac{1}{2}U(c)\,.\end{aligned}$$

Thus U is strictly convex.

COROLLARY 1. *If W is strictly convex and $\mu_F(B(x,y))<1$ for each $x, y\in\mathscr{R}^n$ then M contains only one point, the unique point such that* (2.4) *holds.*

This corollary is a direct consequence of Theorem 1 and Lemma 4 and the observation that a strictly convex function has at most one minimizer.

3. Bayes Estimators

For a given prior distribution let Π be the marginal distribution for X. For $x\in\mathscr{X}$, let $F(\cdot|x)$ be the posterior distribution when the observed value of X is x. Let D be the class of all measurable from $\mathscr{X}$ to $\mathscr{R}^n$. Suppose all previous assumptions about W and F are still in force when F is replaced with $F(\cdot|x)$ for each $x\in\mathscr{X}$.

Now, a Bayes estimator for θ is a function $\delta\in D$ that minimizes the Bayes risk

$$\int_{\mathscr{X}}\int_{\mathscr{R}^n}W(\|\theta-\delta(x)\|)dF(\theta|x)d\Pi(x)\,.$$

We will use the results of Section 2 to characterize this class of estimators.

For each $x\in\mathscr{X}$ and $\|a\|<\infty$, let $U(a|x)$ be defined as in (2.4) with $F(\theta|x)$ in place of $F(\theta)$. Let $M(x)$ be the set of values of a which minimize $U(a|x)$. For each a we can choose $U(a|\cdot)$ as a measurable function of x, as in Doob ([4], p. 27), upon the assumption of the existence of the condition distribution function. With these preliminaries we see that we have established the following theorem.

THEOREM 2. *$\delta \in D$ is a Bayes estimator for θ if and only if δ satisfies the following*

$$\sum_{i=1}^{n}\left[\int_{\mathcal{R}^n-\{\delta(x)\}} W'(\|\theta-\delta(x)\|)\frac{\delta_i(x)-\theta_i}{\|\theta-\delta(x)\|}dF(\theta|x)\right]^2 \leqq [W'(0)\mu_{F(\cdot|x)}(\delta(x))]^2 .$$

The next theorem present sufficient conditions for a Bayes estimator to exist.

THEOREM 3. *If $\mathcal{X}$ is a separable, complete metric space then there exists a measurable function δ such that $\delta(x) \in M(x)$ for each $x \in \mathcal{X}$.*

The proof is a direct application of Corollary 1 in Brown and Purves [1].

The following corollary is a direct application of these results.

COROLLARY 2. *If W is strictly convex, and $\mu_{F(\cdot|x)}(B(y, z))<1$ for each $x \in \mathcal{X}$ and $y, z \in \mathcal{R}^n$, and $\mathcal{X}$ is a separable, complete, metric space, then δ defined by $\delta(x)=M(x)$ is the unique (a.e. (Π)) Bayes estimator.*

4. An application

These results can be applied when the posterior distribution is elliptically symmetric. This is shown for the case when the distribution has a density in the following theorem.

THEOREM 4. *Suppose X is a random variable whose density at x is given by $g((x-u)^T A(x-u))$ for some vector u and some positive definite* (p.d.) *matrix A and some $g: \mathcal{R}^+ \to \mathcal{R}^+$. Then $\min_a \mathrm{E}\, W(\|X-a\|) = \mathrm{E}\, W(\|X-u\|)$ provided $\mathrm{E}\, W(\|X\|)<\infty$.*

PROOF. By Theorem 1 it suffices to show that

$$\mathrm{E}\left[\frac{W'(\|X-u\|)}{\|X-u\|}(X-u)\right]=0 . \tag{4.1}$$

But since A is p.d. there exists an orthogonal matrix, B, such that B^TAB is a diagonal matrix with strictly positive diagonal elements, say $d_1, d_2, \cdots, d_n$. Let $Y=B^T(X-u)$. The density for Y at y is given by $g((By)^T ABy)|\det B|=g\left(\sum_{j=1}^{n} d_j y_j^2\right)$, where y_j is the jth coordinate of y. Thus the left hand side of (4.1) equals

$$\text{(4.2)} \qquad \mathrm{E}\left[\frac{W'(\|BY\|)}{\|BY\|}BY\right]=B\,\mathrm{E}\left[\frac{W'(\|Y\|)}{\|Y\|}Y\right].$$

But the ith coordinate of $\mathrm{E}\left[\frac{W'(\|Y\|)}{\|Y\|}\right]$ is given by

$$\text{(4.3)} \qquad \int_{\mathcal{R}^n}\frac{W'(\|y\|)}{\|y\|}y_i g\Big(\sum_{j=1}^n d_j y_j^2\Big)dy\,.$$

But by symmetry

$$\int_{\mathcal{R}}\frac{W'(\|y\|)}{\|y\|}y_i g\Big(\sum_{j=1}^n d_j y_j^2\Big)dy_i=0\,.$$

So (4.3)=0. So (4.2)=0, which implies that (4.1)=0. Thus the theorem is established.

For example, if the posterior for data x is normal with mean u_x and variance-covariance matrix Σ_x, p.d., then for any loss function of the form in this work a Bayes rule is given by $\delta(x)=u_x$, provided that the risk is finite. This includes, of course, the loss function $L(\theta, a)=\|\theta-a\|$.

Appendix

In this appendix we prove a result used in the body of the paper.

THEOREM A.1. *$a\in M$ if, and only if, $U_\xi(a)\geqq 0$ for all ξ with $\|\xi\|=1$, where $U_\xi(a)$ is defined in* (2.2).

PROOF. Roberts and Varberg ([5], p. 62) define

$$\text{(A.1)} \qquad U'_+(a;v)=\lim_{t\downarrow 0}\frac{U(a+tv)-U(a)}{t}\,,$$

and note that $U'_+(a;v)$ always exists if U is convex. Note that if $\|v\|=1$, we can write $U'_+(a;v)=U_v(a)$.

(If) By (A.1) we note that for $x\neq a$,

$$\begin{aligned} U'_+(a, x-a)&=\lim_{t\downarrow 0}\frac{U\Big(a+t\|x-a\|\cdot\dfrac{x-a}{\|x-a\|}\Big)-U(a)}{t}\\ &=\|x-a\|\lim_{t\downarrow 0}\frac{U\Big(a+t\|x-a\|\cdot\dfrac{x-a}{\|x-a\|}\Big)-U(a)}{\|x-a\|t}\\ &=\|x-a\|U_{(x-a)/\|x-a\|}(a)\,. \end{aligned}$$

However, Roberts and Varberg ([5], p. 117) state that $U(x)-U(a)\geqq$

$U'_+(a; x-a)$. So we have $U(x)-U(a) \geqq \|x-a\| U_{(x-a)/\|x-a\|}(a) \geqq 0$. Thus U is minimum at a.

(Only if) Suppose U is minimum at a. This implies $U(x)-U(a) \geqq 0$. We see that the numerator inside the limit in (2.2) is never negative. Thus we must have $U_\xi(a) \geqq 0$.

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