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DISTRIBUTION OF THE CANONICAL CORRELATION MATRIX

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Summary

Generalized canonical correlation matrix is associated with canonical correlation analysis, multivariate analysis of variance, a large variety of statistical tests and regression problems. In this paper two methods of deriving the distribution are given and the exact distribution is given in an elegant form. The techniques of derivation are applicable to all versions of the generalized canonical correlation matrices, nonnull distributions in generalized analysis of variance problems and also they give rise to a simpler derivation of the distribution of the multiple correlation coefficient.

1. Introduction

Let (X', Y') be a $1\times(p+q)$ multinormal vector with covariance matrix Σ where X is $p\times 1$ and Y is $q\times 1$ and Σ is $(p+q)\times(p+q)$ and positive definite. Consider a simple random sample of size N from this population and let S be the corrected sample sum of products matrix, that is, $S=(s_{ij})$, $s_{ij}=\sum_{k=1}^{N}(x_{ik}-\bar{x}_i)(x_{jk}-\bar{x}_j)$, $\bar{x}_i=\sum_{k=1}^{N} x_{ik}/N$ where x_{ik} denotes the kth observation on the ith component. Consider the following partitioning of S and Σ.

$$S=\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad \Sigma=\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}, \quad S'_{21}=S_{12}, \quad \Sigma'_{21}=\Sigma_{12}$$

where S_{11} and Σ_{11} are $p\times p$ and S_{22} and Σ_{22} are $q\times q$. It is well-known that S is Wishart distributed with $n=N-1$ degrees of freedom and parameter matrix Σ. Let

$$R=S_{11}^{-1/2}S_{12}S_{22}^{-1}S_{21}S_{11}^{-1/2} \tag{1.1}$$

where $S_{11}^{1/2}$ denotes the square root of S_{11}. In our discussion the following notations will be used. A symmetric positive definite matrix A is denoted by $A>0$, $0<A<I$ implies that $A>0$ and $I-A>0$, $A^{1/2}$ denotes the square root of A when A is positive definite. But in the transformations that are used in the following discussions we need only a representation of A in the form $A=BB'$ where B' is the transpose of the nonsingular matrix B. In other words $A^{1/2}$ need not be symmetric. Hence $A^{1/2}BA^{1/2}$ will mean that the post-multiplier of B is the transpose of the pre-multiplier of B.

When $p=1$ the matrix R in (1.1) is a scalar quantity and it is the square of the multiple correlation coefficient of X on Y and in the general case when $p>1$ it denotes the generalized canonical correlation matrix. Several problems such as multivariate analysis of variance and canonical correlation analysis are associated with R of (1.1). An enumeration of such problems may be seen from Khatri [3]. A large variety of test statistics are associated with $\operatorname{tr} R$, $\operatorname{tr} R(I-R)^{-1}$, $(\operatorname{tr} R^{-1})^{-1}$ and $(\operatorname{tr}(I-R)R^{-1})^{-1}$. A particular case of the distribution of R in (1.1) is done in Khatri [3] by using a rather lengthy procedure. Using more or less the same approach an extension of Khatri's result is given in Srivastava [7]. In both of these papers the representations are not elegant and compact. Srivastava's representation involves zonal polynomials and dimensions of the representations of certain symmetric groups. From the following derivations one can see that the problem is rather a simple one. Two methods of derivations of the exact density will be given in this article. The exact distribution of R is easily available through the method of M-transforms introduced in Mathai [5]. The M-transform or the generalized Mellin transform of a scalar function $f(A)$ of the $p\times p$ symmetric positive definite matrix argument A is defined by

$$M_f(s)=\int_{A>0}|A|^{s-(p+1)/2}f(A)dA \tag{1.2}$$

whenever the integral exists, where $|A|$ is the determinant of A and dA is the differential element $da_{11}da_{12}\cdots da_{1p}da_{22}\cdots da_{2p}\cdots da_{pp}$. When $p=1$, (1.2) corresponds to the Mellin transform in the scalar case but when $p>1$ it does not correspond to the Mellin transform in the multivariable case. An independent theory of functions of matrix argument and a new definition for hypergeometric function of matrix argument are developed in Mathai [5]. With the help of the M-transform we will derive the distribution of R in (1.1) and we will also give an alternate derivation with the help of generalized Laplace transforms. We start with a simpler derivation of the distribution of the multiple correlation coefficient.

The square of the multiple correlation is R in (1.1) with $p=1$. Let $U=1-R$. Then since S_{11} in this case is scalar,

$$U=1-(S_{12}S_{22}^{-1}S_{21})/S_{11}=(S_{11}-S_{12}S_{22}^{-1}S_{21})/S_{11}=|S|/\{|S_{22}|S_{11}\}\,. \tag{1.3}$$

The hth moment of U is available by integrating out over the Wishart density. That is,

$$\text{(1.4)}\qquad \mathrm{E}\,(U^h)=\{|2\Sigma|^{n/2}\Gamma_{p+q}(n/2)\}^{-1}\int_{S>0}\{|S|/S_{11}|S_{22}|\}^h \times|S|^{n/2-(p+q+1)/2}e^{-(1/2)\,\mathrm{tr}\,\Sigma^{-1}S}dS$$

where, for example,

$$\text{(1.5)}\qquad \Gamma_k(\alpha)=\pi^{k(k-1)/4}\Gamma(\alpha)\Gamma(\alpha-1/2)\Gamma(\alpha-1)\cdots\Gamma(\alpha-(k-1)/2)\,.$$

We may replace S_{11}^{-h} and $|S_{22}|^{-h}$ by their equivalent integral representations, namely,

$$\text{(1.6)}\qquad S_{11}^{-h}=\{\Gamma(h)\}^{-1}\int_0^\infty x^{h-1}e^{-S_{11}x}dx\,,\qquad R(h)>0$$

and

$$\text{(1.7)}\qquad |S_{22}|^{-h}=\{\Gamma_q(h)\}^{-1}\int_{\Lambda>0}|\Lambda|^{h-(q+1)/2}e^{-\mathrm{tr}\,\Lambda S_{22}}d\Lambda\,,\qquad R(h)>(q-1)/2$$

where $R(\cdot)$ denotes the real part of $(\cdot)$. Under these substitutions and replacing $S/2$ by S we have

$$\text{(1.8)}\qquad \mathrm{E}\,(U^h)=\{\Gamma_q(h)\Gamma(h)\Gamma_{p+q}(n/2)|\Sigma|^{n/2}\}^{-1} \times\int_{x>0}\int_{\Lambda>0}\int_{S>0}x^{h-1}|\Lambda|^{h-(q+1)/2}|S|^{n/2+h-(p+q+1)/2}e^{-\mathrm{tr}\,AS}dS\,d\Lambda\,dx$$

where

$$A=\begin{bmatrix}\sigma^{11}+x & \Sigma^{12}\\ \Sigma^{21} & \Sigma^{22}+\Lambda\end{bmatrix},\qquad \Sigma^{-1}=\begin{bmatrix}\Sigma^{11} & \Sigma^{12}\\ \Sigma^{21} & \Sigma^{22}\end{bmatrix}.$$

Integrating out S we get

$$\text{(1.9)}\qquad \mathrm{E}\,(U^h)=\Gamma_{p+q}(n/2+h)\{|\Sigma|^{n/2}\Gamma_{p+q}(n/2)\Gamma(h)\Gamma_q(h)\}^{-1} \times\int_{x>0}\int_{\Lambda>0}x^{h-1}|\Lambda|^{h-(q+1)/2}|A|^{-(n/2+h)}d\Lambda\,dx\,.$$

When the population multiple correlation coefficient

$$\text{(1.10)}\qquad \rho^2=\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}/\sigma_{11}$$

is zero then $|A|$ reduces to a simple form and the integrations of x and Λ are very simple. We consider the general case when $\rho^2>0$. In this case,

$$\text{(1.11)}\qquad |A|=B(1+B^{-1}x)|\Lambda+\Sigma^{22}|$$

where

(1.12) $$B=\sigma^{11}-\Sigma^{12}(\Lambda+\Sigma^{22})^{-1}\Sigma^{21}=\sigma^{11}|\Lambda+\Sigma_{22}^{-1}|/|\Lambda+\Sigma^{22}|\,.$$

Collecting the factors containing x and integrating one gets

(1.13) $$\{\Gamma(h)\}^{-1}\int_0^\infty x^{h-1}(1+B^{-1}x)^{-(n/2+h)}dx=B^h\Gamma(n/2)/\Gamma(n/2+h)\,.$$

Now collecting factors containing Λ we get

(1.14) $$\begin{aligned}&\{(\sigma^{11})^{-n/2}/\Gamma_q(h)\}\int_{\Lambda>0}|\Lambda|^{h-(q+1)/2}|\Lambda+\Sigma^{22}|^{-h}|\Lambda+\Sigma_{22}^{-1}|^{-n/2}d\Lambda\\&=(\sigma^{11})^{-n/2}|\Sigma_{22}|^{n/2}\{\Gamma_q(n/2)/\Gamma_q(n/2+h)\}\\&\quad\times{}_2F_1(n/2,h\,;\,n/2+h\,;\,I-\Sigma_{22}^{1/2}\Sigma^{22}\Sigma_{22}^{1/2})\,.\end{aligned}$$

The integral in (1.14) is available by transforming the variable Λ to $U^{-1}-I$ so that $0<U<I$ and then the differential element is $d\Lambda=|U|^{-(q+1)}dU$. Now one can use Euler's representation of a ${}_2F_1$, for example, see Herz ([2], (2.12)) or the results in Mathai [5] to evaluate (1.14). Now substituting back in (1.8) and simplifying the gammas one gets

(1.15) $$\begin{aligned}\mathrm{E}\,(U^h)=&\{\Gamma(n/2-q/2+h)\Gamma(n/2)/\Gamma(n/2-q/2)\Gamma(n/2+h)\}\\&\times{}_2F_1(n/2,h\,;\,n/2+h\,;\,I-\Sigma_{22}^{1/2}\Sigma^{22}\Sigma_{22}^{1/2})\,,\\&\qquad\text{for } R(h)>(q-1)/2\,.\end{aligned}$$

This is one representation. But from the symmetry of simplifications in (1.11) to (1.14) we can see that $\mathrm{E}\,(U^h)$ is also given by

(1.16) $$\begin{aligned}\mathrm{E}\,(U^h)=&\{\Gamma(n/2-q/2+h)\Gamma(n/2)/\Gamma(n/2-q/2)\Gamma(n/2+h)\}\\&\times{}_2F_1(n/2,h\,;\,n/2+h\,;\,1-\sigma_{11}\sigma^{11})\,.\end{aligned}$$

The density of U can be represented in a number of ways. One representation is available in terms of a ${}_2F_1$ from the uniqueness properties of Mellin transforms. Since (1.15) is valid for all complex h such that $R(h)>(q-1)/2$, (1.15) uniquely determines the densities of U and R. The density of R can be directly written down from (1.15) but for the sake of simplicity we will show it explicitly by using the following known result.

(1.17) $$\begin{aligned}&\int_0^1 x^{a-1}(1-x)^{b-1}{}_2F_1(\alpha,\beta\,;\,\gamma\,;\,\delta x)dx\\&=\{\Gamma(a)\Gamma(b)/\Gamma(a+b)\}{}_3F_2(a,\alpha,\beta\,;\,a+b,\gamma\,;\,\delta)\\&\qquad\text{for } R(a),R(b)>0\,.\end{aligned}$$

Put $a=q/2$, $b=n/2-q/2+h$, $\alpha=n/2$, $\beta=n/2$, $\gamma=q/2$ then we get

(1.18) $$\int_0^1(1-x)^h x^{q/2-1}(1-x)^{n/2-q/2-1}{}_2F_1(n/2,n/2\,;\,q/2\,;\,\delta x)dx$$

$$=\{\Gamma(q/2)\Gamma(n/2-q/2+h)/\Gamma(n/2+h)\}$$
$$\times {}_3F_2(n/2,\, n/2,\, q/2\,;\; n/2+h,\, q/2\,;\; \delta)$$
$$=\{\Gamma(q/2)\Gamma(n/2-q/2+h)/\Gamma(n/2+h)\}(1-\delta)^{-n/2}$$
$$\times {}_2F_1(h,\, n/2\,;\; n/2+h\,;\; \delta/(\delta-1))\,.$$

The last step is done with the help of the known formula

(1.19) $${}_2F_1(a,\, b\,;\; c\,;\; z)=(1-z)^{-b}{}_2F_1(c-a,\, b\,;\; c\,;\; z/(z-1))\,.$$

Comparing (1.18) with (1.16) for $\delta=\rho^2$ we get the density of R as,

(1.20) $$f(R)=\{(1-\rho^2)^{n/2}\Gamma(n/2)/\Gamma(n/2-q/2)\Gamma(q/2)\}$$
$$\times R^{q/2-1}(1-R)^{n/2-q/2-1}{}_2F_1(n/2,\, n/2\,;\; q/2\,;\; \rho^2R)\,, \qquad 0\leqq R\leqq 1$$

where R is the square of the multiple correlation coefficient of X on Y when X is scalar. It may be noticed that this method of derivation is easier than the other methods available in the literature, see for example Kshirsagar ([4], pp. 94–99), Anderson ([1], pp. 93–96).

One advantage of proceeding as above is that all the steps that we have used in the derivation of (1.20) are directly extensible to matrix cases. In the general case when $p>1$ we have

(1.21) $$R=S_{11}^{-1/2}S_{12}S_{22}^{-1}S_{11}^{-1/2} \quad \text{and} \quad P=\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1/2}$$

where P is the population matrix corresponding to R. Let

$$U=|I-R|=|S_{11}-S_{12}S_{22}^{-1}S_{21}|/|S_{11}|=|S|/\{|S_{11}||S_{22}|\}\,.$$

From (1.2) it is seen that the M-transform of the density of R is $\mathrm{E}\,(U^h)$ with $h=s-(p+1)/2$. Following the same steps as in (1.3) to (1.8) we get

(1.22) $$\mathrm{E}\,(U^h)=\Gamma_{p+q}(n/2+h)\{\Gamma_{p+q}(n/2)|\Sigma|^{n/2}\Gamma_p(h)\Gamma_q(h)\}^{-1}$$
$$\times\int_{\Lambda_1>0}\int_{\Lambda_2>0}|\Lambda_1|^{h-(p+1)/2}|\Lambda_2|^{h-(q+1)/2}|A|^{-(n/2+h)}d\Lambda_1\, d\Lambda_2$$

where

$$A=\begin{bmatrix} \Sigma^{11}+\Lambda_1 & \Sigma^{12} \\ \Sigma^{21} & \Sigma^{22}+\Lambda_2 \end{bmatrix}.$$

Now proceeding as in (1.11) to (1.20) we get for $q\geqq p$,

(1.23) $$\mathrm{E}\,(U^h)=\{\Gamma_p(n/2-q/2+h)\Gamma_p(n/2)/\Gamma_p(n/2-q/2)\Gamma_p(n/2+h)\}$$
$$\times {}_2F_1(n/2,\, h\,;\; n/2+h\,;\; I-\Sigma_{11}^{1/2}\Sigma^{11}\Sigma_{11}^{1/2})$$

and

(1.24) $$f(R)=\{|I-P|^{n/2}\Gamma_p(n/2)/\Gamma_p(n/2-q/2)\Gamma_p(q/2)\}|R|^{q/2-(p+1)/2}$$

$$\times|I-R|^{n/2-(p+q+1)/2}{}_2F_1(n/2, n/2\,;\; q/2\,;\; P^{1/2}RP^{1/2})\,,\qquad 0<R<I,\quad 0<P<I\,.$$

In Mathai [5] the uniqueness of $f(A)$ from $M_f(s)$ of (1.2) is not established. Hence as it stands we have one function in (1.24) for which the M-transform is (1.23). In the next section we will show that $f(R)$ in (1.24) is in fact the unique density of R. This will be established with the help of the uniqueness of the Laplace and inverse Laplace transforms in the case of functions of matrix argument. This also gives another method of deriving the density of R. This method is more lengthy but it involves several interesting steps which are also applicable in other problems. It is not difficult to see that all the integrations and interchanges carried out in the following section are valid. Hence we won't list the conditions at each step.

2. Derivation of the density through Laplace transforms

The Laplace transform of the density of R in (1.1), denoted by $L_R(B)$, is the expected value of $e^{-\mathrm{tr}\,BR}$, that is,

$$\text{(2.1)}\qquad L_R(B)=\mathrm{E}\,(e^{-\mathrm{tr}\,BR})\,,\qquad R=S_{11}^{-1/2}S_{12}S_{22}^{-1}S_{21}S_{11}^{-1/2}$$

where the variables are the distinct elements of R and B is a symmetric complex matrix with diagonal elements b_{ii} and nondiagonal elements $b_{ij}/2$, $i\neq j$, $b_{ij}=b_{ji}$. A discussion of the uniqueness of the inverse Laplace transform may be seen from Herz ([2], pp. 478–480). Since R is a function of S and S is Wishart distributed we get $L_R(B)$ by integrating out over the Wishart density. That is,

$$\text{(2.2)}\qquad L_R(B)=\{\Gamma_{p+q}(n/2)|\Sigma|^{n/2}\}^{-1}\int_{S>0}e^{-\mathrm{tr}\,BR}|S|^{n/2-(p+q+1)/2}e^{-\mathrm{tr}\,\Sigma^{-1}S}dS$$

where $S/2$ is replaced by S for convenience. But

$$\text{(2.3)}\qquad |S|=|S_{22}||S_{11}||I-S_{11}^{-1/2}S_{12}S_{22}^{-1}S_{21}S_{11}^{-1/2}|\,.$$

Make the transformations $S_{22}^{-1/2}S_{21}=U_{21}$, $S_{11}^{-1/2}U_{12}=V_{12}$ then $dS_{21}=|S_{22}|^{p/2}dU_{21}$, $dU_{12}=|S_{11}|^{q/2}dV_{12}$ for fixed S_{11} and S_{22} and $R=V_{12}V_{21}$. Under these transformations the Laplace transform becomes

$$\text{(2.4)}\quad L_R(B)=\{\Gamma_{p+q}(n/2)|\Sigma|^{n/2}\}^{-1}$$
$$\times\int_{S_{11}}\int_{S_{22}}\int_{V_{12}}|S_{11}|^{n/2-(p+1)/2}|S_{22}|^{n/2-(q+1)/2}|I-V_{12}V_{21}|^{n/2-(p+q+1)/2}$$
$$\times\exp\{-\mathrm{tr}\,BV_{12}V_{21}-\mathrm{tr}\,\Sigma^{11}S_{11}-\mathrm{tr}\,S_{11}^{1/2}\Sigma^{12}S_{22}^{1/2}V_{21}$$
$$-\mathrm{tr}\,S_{22}^{1/2}\Sigma^{21}S_{11}^{1/2}V_{12}-\mathrm{tr}\,\Sigma^{22}S_{22}\}dS_{11}dS_{22}dV_{12}\,.$$

Let $q\geqq p$. Consider the class of $q\times p$ matrices of the type V_{21}. Let

$Z=V_{12}V_{21}>0$. Then we have a unique representation of the form $V_{21}=W_{21}Z^{1/2}$ where W_{21} is an element of the Stieffel manifold, that is, $W_{12}W_{21}=I$. Then $dV_{21}=2^{-p}|Z|^{q/2-(p+1)/2}dW_{21}dZ$ (see for example Herz ([2], p. 482)). Also

$$(2.5)\quad L_R(B)=\{2^p|\Sigma|^{n/2}\Gamma_{p+q}(n/2)\}^{-1}\int_{I>Z>0}|Z|^{q/2-(p+1)/2}e^{-\operatorname{tr}BZ}|I-Z|^{n/2-(p+q+1)/2}$$
$$\times\int_{S_{11}}\int_{S_{22}}\int_{W_{21}}|S_{11}|^{n/2-(p+1)/2}|S_{22}|^{n/2-(q+1)/2}$$
$$\times\exp\{-\operatorname{tr}\Sigma^{11}S_{11}-\operatorname{tr}\Sigma^{22}S_{22}-2\operatorname{tr}Z^{1/2}S_{11}^{1/2}\Sigma^{12}S_{22}^{1/2}W_{21}\}$$
$$\times dS_{11}dS_{22}dW_{21}dZ\,.$$

Now consider the integration of W_{21}. From Herz ([2], p. 494 (3.5)′) we have

$$(2.6)\qquad \int_{V'V=I}e^{-2\operatorname{tr}T'V}dV=2^p\pi^{pq/2}A_\delta(-T'T)\,,\qquad \delta=(q-p-1)/2$$

where the Bessel function is given by the following representation:

$$(2.7)\qquad A_\delta(M)=(2\pi i)^{-p(p+1)/2}\int_X e^{\operatorname{tr}X}e^{-\operatorname{tr}MX^{-1}}|X|^{-q/2}dX\,,\qquad i=(-1)^{1/2}\,.$$

Hence

$$(2.8)\qquad \int_{W_{21}}\exp\{-2\operatorname{tr}Z^{1/2}S_{11}^{1/2}\Sigma^{12}S_{22}^{1/2}W_{21}\}dW_{21}$$
$$=2^p\pi^{pq/2}(2\pi i)^{-p(p+1)/2}\int_U e^{\operatorname{tr}U+\operatorname{tr}MU^{-1}}|U|^{-q/2}dU$$

where

$$M=Z^{1/2}S_{11}^{1/2}\Sigma^{12}S_{22}\Sigma^{21}S_{11}^{1/2}Z^{1/2}\,.$$

Now collecting the factors containing S_{22} we get

$$(2.9)\quad \int_{S_{22}>0}|S_{22}|^{n/2-(q+1)/2}\exp\{-\operatorname{tr}\Sigma^{22}S_{22}+\operatorname{tr}S_{22}\Sigma^{21}S_{11}^{1/2}Z^{1/2}U^{-1}S_{11}^{1/2}\Sigma^{12}\}dS_{22}$$
$$=\Gamma_q(n/2)|\Sigma^{22}-\Sigma^{21}S_{11}^{1/2}Z^{1/2}U^{-1}Z^{1/2}S_{11}^{1/2}\Sigma^{12}|^{-n/2}\,.$$

The factors containing U can be integrated out directly or one can use the following known property to write down the result.

$$(2.10)\qquad A_\delta(M)=\{\Gamma_p(q/2)\}^{-1}{}_0F_1(q/2\,;\,-M)\,,\qquad \delta=(q-p-1)/2\,.$$

If direct integration is used then one may need the property

$$(2.11)\qquad \begin{vmatrix}\Sigma^{22} & \Sigma^{21}S_{11}^{1/2}Z^{1/2}\\ Z^{1/2}S_{11}^{1/2}\Sigma^{12} & U\end{vmatrix}=|U||\Sigma^{22}-\Sigma^{21}S_{11}^{1/2}Z^{1/2}U^{-1}Z^{1/2}S_{11}^{1/2}\Sigma^{12}|$$
$$=|\Sigma^{22}||U||I-U^{-1}Z^{1/2}S_{11}^{1/2}\Sigma^{12}(\Sigma^{22})^{-1}\Sigma^{21}S_{11}^{1/2}Z^{1/2}|$$

and the inverse Laplace representation for ${}_1F_1$, see for example Herz ([2], p. 487). Now

$$\Sigma^{11}=(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1}=\Sigma_{11}^{-1/2}(I-P)^{-1}\Sigma_{11}^{-1/2} \tag{2.12}$$

where

$$P=\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1/2} .$$

Put

$$(I-P)^{-1/2}\Sigma_{11}^{-1/2}S_{11}\Sigma_{11}^{-1/2}(I-P)^{-1/2}=X$$

then

$$dS_{11}=|\Sigma_{11}|^{(p+1)/2}|I-P|^{(p+1)/2}dX$$

and writing

$$(I-P)^{-1/2}\Sigma_{11}^{-1/2}S_{11}^{1/2}=X^{1/2}$$

we have

$$\begin{aligned} Z^{1/2}S_{11}^{1/2}\Sigma^{12}(\Sigma^{22})^{-1}\Sigma^{21}S_{11}^{1/2}Z^{1/2} &= Z^{1/2}S_{11}^{1/2}\Sigma_{11}^{-1/2}(I-P)^{-1}P\Sigma_{11}^{-1/2}S_{11}^{1/2}\Sigma^{1/2} \\ &= Z^{1/2}(X^{1/2})'PX^{1/2}Z^{1/2} . \end{aligned}$$

This can always be put in the form $Y^{1/2}P^{1/2}Z(P^{1/2})'(Y^{1/2})'$ since Z, P, X are all symmetric and positive definite. The differential element does not change and also $\operatorname{tr} X=\operatorname{tr} Y$. This can be seen by substituting the symmetric square roots and then using the property that $\operatorname{tr} ABC=\operatorname{tr} ACB$ when A, B, C are square symmetric matrices of the same order. Now collecting the factors containing S_{11} one gets

$$\begin{aligned} &\int_{S_{11}>0}|S_{11}|^{n/2-(p+1)/2}e^{-\operatorname{tr}\Sigma^{11}S_{11}} \\ &\quad\times{}_1F_1(n/2\,;\ q/2\,;\ Z^{1/2}S_{11}^{1/2}\Sigma^{12}(\Sigma^{22})^{-1}\Sigma^{21}S_{11}^{1/2}Z^{1/2})dS_{11} \\ &=|\Sigma_{11}|^{n/2}|I-P|^{n/2}\int_{Y>0}|Y|^{n/2-(p+1)/2}e^{-\operatorname{tr} Y} \\ &\quad\times{}_1F_1(n/2\,;\ q/2\,;\ Y^{1/2}P^{1/2}Z(P^{1/2})'(Y^{1/2})')dY \\ &=|\Sigma_{11}|^{n/2}|I-P|^{n/2}\Gamma_p(n/2){}_2F_1(n/2,\ n/2\,;\ q/2\,;\ P^{1/2}ZP^{1/2}) , \\ &\qquad 0<Z<I . \end{aligned} \tag{2.13}$$

The second integral in (2.13) is available from standard results, see for example, Herz ([2], (2.1)'). Collecting the constants we have

$$\begin{aligned} &\{2^p\pi^{pq/2}\Gamma_q(n/2)\Gamma_p(n/2)|\Sigma^{22}|^{-n/2}|\Sigma_{11}|^{n/2}|I-P|^{n/2}\}/ \\ &\qquad\{\Gamma_{p+q}(n/2)|\Sigma|^{n/2}2^p\Gamma_p(q/2)\} \\ &=|I-P|^{n/2}\Gamma_p(n/2)/\{\Gamma_p(n/2-q/2)\Gamma_p(q/2) . \end{aligned} \tag{2.14}$$

The simplification is done by using the facts that

(2.15) $|\Sigma|=|\Sigma_{11}||\Sigma^{22}|^{-1}$ and $\Gamma_{p+q}(n/2)/\Gamma_q(n/2)=\pi^{pq/2}\Gamma_p(n/2-q/2)$.

Thus we have

$$L_R(B)=|I-P|^{n/2}\Gamma_p(n/2)\{\Gamma_p(n/2-q/2)\Gamma_p(q/2)\}^{-1} \times\int_{I>Z>0} e^{-\mathrm{tr}\,BZ}|Z|^{q/2-(p+1)/2}|I-Z|^{n/2-(p+q+1)/2} \times{}_2F_1(n/2, n/2\,;\ q/2\,;\ P^{1/2}ZP^{1/2})dZ\,. \tag{2.16}$$

From the uniqueness of the inverse Laplace transform it follows that the density of R is uniquely determined by

$$f(R)=\Gamma_p(n/2)|I-P|^{n/2}\{\Gamma_p(q/2)\Gamma_p(n/2-q/2)\}^{-1} \times|R|^{q/2-(p+1)/2}|I-R|^{n/2-(p+q+1)/2} \times{}_2F_1(n/2, n/2\,;\ q/2\,;\ P^{1/2}RP^{1/2})\,,\quad 0<R<I,\quad 0<P<I. \tag{2.17}$$

The distributions in other cases can be written from the symmetry of the partitioning of S and Σ, with interchanges of rows and columns if necessary.

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