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ASYMPTOTIC DISTRIBUTION OF A GENERALIZED HOTELLING'S T_0^2 IN THE DOUBLY NONCENTRAL CASE

TAKESI HAYAKAWA

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1. Introduction

Let S_1 be a non-central Wishart matrix with n_1 degrees of freedom and with a non-centrality parameter matrix Ω_1, and S_2 a non-central Wishart matrix with n_2 degrees of freedom and with a non-centrality parameter matrix Ω_2, respectively. Hotelling's generalized T_0^2 statistics is defined as

$$(1) \qquad T=T_0^2/n_2=\mathrm{tr}\, S_1S_2^{-1} .$$

When n_2 becomes large under $\Omega_2=0$, the distribution of T_0^2 approaches that of χ^2 based on mn_1 degrees of freedom. The asymptotic expansion of the distribution of T_0^2 under $\Omega_2=0$ is obtained by various methods, Ito [8], Siotani [10], Davis [2], Fujikoshi [6], Hayakawa [7], Muirhead [9] and Yoong-Sin Lee [12].

Recently Davis [3], [4] defined an invariant polynomial of two matrix arguments and gave some formulas useful in multivariate statistical analysis. The exact probability density function of T under $\Omega_2\neq 0$ has been obtained by Davis [3], using generalized Laguerre polynomials of two matrix arguments. Davis's expression is the form

$$(2) \qquad f(T)=\frac{\Gamma_m\left(\frac{n_1+n_2}{2}\right)}{\Gamma\left(\frac{mn_1}{2}\right)\Gamma_m\left(\frac{n_2}{2}\right)}\,\mathrm{etr}\,(-\Omega_1-\Omega_2)T^{mn_1/2-1}\sum_{k=0}^{\infty}\frac{(-T)^k}{k!\left(\frac{mn_1}{2}\right)_k}$$
$$\cdot\sum_{l=0}^{\infty}\sum_{\kappa}\sum_{\lambda}\sum_{\phi\in\kappa\cdot\lambda}\frac{\left(\frac{n_1+n_2}{2}\right)_\phi}{l!\left(\frac{n_2}{2}\right)_\lambda}\theta_\phi^{\kappa,\lambda}L_{\kappa,\lambda;\phi}^{n_1/2-p}(\Omega_1,\Omega_2)\ ,$$

where

$$\Gamma_m(a)=\pi^{m(m-1)/4}\prod_{\alpha=1}^{m}\Gamma\left(a-\frac{\alpha-1}{2}\right)\ ,\qquad p=\frac{1}{2}(m+1)$$

$$(a)_\phi=\prod_{\alpha=1}^{m}\left(a-\frac{\alpha-1}{2}\right)_{f_\alpha}, \qquad (a)_n=a(a+1)\cdots(a+n-1),$$

and $L_{\kappa,\lambda;\phi}^{n_1/2-p}(\Omega_1, \Omega_2)$ is a generalized Laguerre polynomial of matrix arguments Ω_1 and Ω_2 corresponding to a partition $\kappa=\{k_1, k_2, \cdots, k_m\}$ of k, $\lambda=\{l_1, l_2, \cdots, l_m\}$ of l, and $\phi=\{f_1, f_2, \cdots, f_m\}$ of $f=k+l$ into not more than m parts. In this paper we give the asymptotic expansion of the probability density function of T_0^2 under $\Omega_2\neq 0$ up to order $1/n_2$.

2. Some useful formulas for invariant polynomials and generalized Laguerre polynomials of two matrix arguments

Davis [3], [4] gave the tables of coefficients of invariant polynomials of two matrix arguments up to order 5. Inverting the relation between $\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(X, Y)$ and functions of traces, we have following relations

$$(3)\qquad \begin{bmatrix}(XY)\\(X)(Y)\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2 & -1\\ 2 & 2\end{bmatrix}\begin{bmatrix}(1,1;2)\\(1,1;1^2)\end{bmatrix}$$

$$(4)\qquad \begin{bmatrix}(X^2Y)\\(X^2)(Y)\\(XY)(X)\\(X)^2(Y)\end{bmatrix}=\frac{1}{4}\begin{bmatrix}4 & -1 & -1 & 1\\ 4 & 4 & -2 & -2\\ 4 & -1 & 2 & -2\\ 4 & 4 & 4 & 4\end{bmatrix}\begin{bmatrix}(2,1;3)\\(2,1;21)\\(1^2,1;21)\\(1^2,1;1^3)\end{bmatrix}$$

$$(5)\qquad \begin{bmatrix}(X^3Y)\\(X^3)(Y)\\(X^2Y)(X)\\(X^2)(XY)\\(X^2)(X)(Y)\\(XY)(X)^2\\(X^3)(Y)\end{bmatrix}=\frac{1}{24}\begin{bmatrix}24 & -4 & -4 & -1 & 2 & 2 & -3\\ 24 & 24 & -6 & -6 & -6 & 6 & 6\\ 24 & -4 & 6 & -6 & 0 & -4 & 6\\ 24 & -4 & -4 & 14 & -4 & -4 & 6\\ 24 & 24 & 4 & 4 & 4 & -12 & -12\\ 24 & -4 & 16 & 4 & -8 & 8 & -12\\ 24 & 24 & 24 & 24 & 24 & 24 & 24\end{bmatrix}\begin{bmatrix}(3,1;4)\\(3,1;31)\\(21,1;31)\\(21,1;2^2)\\(21,1;21^2)\\(1^3,1;21^2)\\(1^3,1;1^4)\end{bmatrix}$$

$$(6)\qquad \begin{bmatrix}(X^2Y^2)\\(XYXY)\\(X^2Y)(Y)\\(XY^2)(X)\\(X^2)(Y^2)\\(XY)^2\\(X^2)(Y)^2\\(XY)(X)(Y)\\(Y^2)(X)^2\\(X)^2(Y)^2\end{bmatrix}=\frac{1}{32}\begin{bmatrix}32 & 4 & -8 & -8 & 4 & -8 & 4 & 4\\ 32 & -24 & -12 & 0 & 0 & 0 & 0 & -12\\ 32 & 4 & -8 & 16 & -8 & -8 & 4 & -8\\ 32 & 4 & -8 & -8 & 4 & 16 & -8 & -8\\ 32 & 32 & 32 & -16 & -16 & -16 & -16 & 8\\ 32 & -24 & 12 & 0 & 0 & 0 & 0 & 24\\ 32 & 32 & 32 & 32 & 32 & -16 & -16 & -16\\ 32 & 4 & -8 & 16 & -8 & 16 & -8 & 16\\ 32 & 32 & 32 & -16 & -16 & 32 & 32 & -16\\ 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32\end{bmatrix}$$

$$\begin{matrix} 1 & -4 \\ 6 & -4 \\ -2 & 8 \\ -2 & 8 \\ 8 & 8 \\ -12 & 8 \\ -16 & -16 \\ 4 & -16 \\ -16 & -16 \\ 32 & 32 \end{matrix} \left[\begin{matrix} (2, 2; 4) \\ (2, 2; 31) \\ (2, 2; 2^2) \\ (2, 1^2; 31) \\ (2, 1^2; 21^2) \\ (1^2, 2; 31) \\ (1^2, 2; 21^2) \\ (1^2, 1^2; 2^2) \\ (1^2, 1^2; 21^2) \\ (1^2, 1^2; 1^4) \end{matrix}\right],$$

where (X) stands for trace X and $(\kappa, \lambda; \phi)$ stands for $\theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(X, Y)$, $\theta_\phi^{\kappa,\lambda} = C_\phi^{\kappa,\lambda}(I, I)/C_\phi(I)$, respectively. In this paper $\sum_{\kappa,\lambda;\phi}$ stands for $\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\kappa}\sum_{\lambda}\sum_{\phi\in\kappa\cdot\lambda}$ for the simplicity of summation notation. The fundamental formulas for $C_\phi^{\kappa,\lambda}(X, Y)$ are

$$\text{etr}\,(xX+yY) = \sum_{\kappa,\lambda;\phi} \theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(xX, yY)/k!l! \tag{7}$$

$$|I-(xX+yY)|^{-a} = \sum_{\kappa,\lambda;\phi} (a)_\phi \theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(xX, yY)/k!l!\ . \tag{8}$$

The following lemmas are important in our argument.

LEMMA 1. *Let $C_\phi^{\kappa,\lambda}(X, Y)$ be the invariant polynomial corresponding to a partition κ of k, λ of l and ϕ of $f=k+l$ for $m\times m$ symmetric matrices X and Y.*

Put

$$a_1(\phi) = \sum_{\alpha=1}^{m} f_\alpha(f_\alpha-\alpha) \tag{9}$$

$$a_2(\phi) = \sum_{\alpha=1}^{m} f_\alpha(4f_\alpha^2 - 6\alpha f_\alpha + 3\alpha^2)\ . \tag{10}$$

Then the following equalities hold.

$$\sum_{\kappa,\lambda;\phi} \theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(xX, yY)/(k-r)!(l-s)! = (\text{tr}\, xX)^r (\text{tr}\, yY)^s\, \text{etr}\,(W) \tag{11}$$

$$\sum_{\kappa,\lambda;\phi} a_1(\phi)\theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(xX, yY)/k!l! = \text{tr}\, W^2\, \text{etr}\,(W) \tag{12}$$

$$\begin{aligned}\sum_{\kappa,\lambda;\phi} & a_2(\phi)\theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(xX, yY)/k!l! \\ &= [4\,\text{tr}\, W^3 + 3\,\text{tr}\, W^2 + 3(\text{tr}\, W)^2 + \text{tr}\, W]\, \text{etr}\,(W)\end{aligned} \tag{13}$$

$$\begin{aligned}\sum_{\kappa,\lambda;\phi} & a_1^2(\phi)\theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(xX, yY)/k!l! \\ &= [(\text{tr}\, W^2)^2 + 4\,\text{tr}\, W^3 + \text{tr}\, W^2 + (\text{tr}\, W)^2]\, \text{etr}\,(W)\end{aligned} \tag{14}$$

(15) $$\sum_{\kappa,\lambda;\phi} a_1(\kappa)\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(xX, yY)/k!l!=x^2 \operatorname{tr} X^2 \operatorname{etr}(W) ,$$

where $W=xX+yY$.

PROOF. Differentiating both sides of (7) r times with respect to x and s times with respect to y, respectively and multiplying $x^r y^s$, we have (11).

Sugiura and Fujikoshi [11] gave for a symmetric matrix W the following equalities.

(16) $$\sum_{f=0}^{\infty}\sum_{\phi} a_1(\phi)C_\phi(W)/f!=\operatorname{tr} W^2 \operatorname{etr}(W)$$

(17) $$\sum_{f=0}^{\infty}\sum_{\phi} a_2(\phi)C_\phi(W)/f!=[4 \operatorname{tr} W^3+3 \operatorname{tr} W^2+3(\operatorname{tr} W)^2+\operatorname{tr} W] \operatorname{etr}(W)$$

(18) $$\sum_{f=0}^{\infty}\sum_{\phi} a_1^2(\phi)C_\phi(W)/f!=[(\operatorname{tr} W^2)^2+4 \operatorname{tr} W^3+\operatorname{tr} W^2+(\operatorname{tr} W)^2] \operatorname{etr}(W) .$$

Setting $W=xX+yY$ on the left-hand sides of these equalities and using the binomial expansion due to Davis [3],

(19) $$C_\phi(xX+yY)=\sum_{\kappa,\lambda(\phi\in\kappa\cdot\lambda)}\binom{f}{k}\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(xX, yY)$$

where $f=k+l$, we have (12), (13) and (14).

The LHS of (15) is expressed with the help of (16) as follows

(20) $$\begin{aligned}\text{LHS of (15)}&=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\kappa}\sum_{\lambda} a_1(\kappa)\sum_{\phi\in\kappa\cdot\lambda}\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(xX, yY)/k!l!\\&=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\kappa}\sum_{\lambda} a_1(\kappa)C_\kappa(xX)C_\lambda(yY)/k!l!\\&=x^2 \operatorname{tr} X^2 \operatorname{etr}(W) .\end{aligned}$$

LEMMA 2. *Using same notations as Lemma* 1, *we have following equalities.*

(21) $$\sum_{\kappa,\lambda;\phi} a_1(\phi)(a)_\phi\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(xX, yY)/k!l!$$
$$=\frac{a}{2}\{(\operatorname{tr} V)^2+(2a+1) \operatorname{tr} V^2\}|I-(xX+yY)|^{-a}$$

(22) $$\sum_{\kappa,\lambda;\phi} a_1(\phi)^2(a)_\phi\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(xX, yY)/k!l!$$
$$\begin{aligned}=\frac{a}{4}\{&2(2a+1)(\operatorname{tr} V)^2+2(2a+3) \operatorname{tr} V^2+4(\operatorname{tr} V)^3\\&+12(2a+1) \operatorname{tr} V \operatorname{tr} V^2+8(2a^2+3a+2) \operatorname{tr} V^3\\&+a(\operatorname{tr} V)^4+2(2a^2+a+2)(\operatorname{tr} V)^2 \operatorname{tr} V^2\end{aligned}$$

$$+(2a+1)(2a^2+a+2)(\mathrm{tr}\,V^2)^2+8(2a+1)\,\mathrm{tr}\,V\,\mathrm{tr}\,V^3$$
$$+2(8a^2+10a+5)\,\mathrm{tr}\,V^4\}|I-(xX+yY)|^{-a}$$

$$(23)\qquad \sum_{\kappa,\lambda;\phi} a_2(\phi)(a)_\phi\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(xX,\, yY)/k!l!$$
$$=\frac{a}{2}\{2\,\mathrm{tr}\,V+3(2a+1)(\mathrm{tr}\,V)^2+3(2a+3)\,\mathrm{tr}\,V^2+2(\mathrm{tr}\,V)^3$$
$$+6(2a+1)\,\mathrm{tr}\,V\,\mathrm{tr}\,V^2+4(2a^2+3a+2)\,\mathrm{tr}\,V^3\}$$
$$\cdot\,|I-(xX+yY)|^{-a}\,,$$

where $V=(xX+yY)(I-(xX+yY))^{-1}$.

PROOF. Take $Z=xX+yY$ in formulas of Fujikoshi ([6], Lemma 3) and use the binomial expansion (19) on the left-hand side, then we have (21), (22) and (23).

Davis [3] defined a generalized Laguerre polynomial $L_{\kappa,\lambda;\phi}^{n/2-p}(X, Y)$ as follows

$$(24)\qquad L_{\kappa,\lambda;\phi}^{n/2-p}(X, Y)=\mathrm{etr}\,(X)\int_{R>0}\mathrm{etr}\,(-R)|R|^{n/2-p}C_\phi^{\kappa,\lambda}(R,\, Y)A_{n/2-p}(RX)dR$$

where $A_{n/2-p}(RX)$ is a Bessel function of a matrix argument, and $L_{\kappa,\lambda;\phi}^{n/2-p}\cdot(X, Y)$ is expressed as

$$(25)\qquad L_{\kappa,\lambda;\phi}^{n/2-p}(X, Y)=\left(\frac{n}{2}\right)_\kappa C_\phi(I)\sum_{r=0}^{k}\sum_{\phi;\tau\in\rho\cdot\lambda}(-1)^r b_{\rho,\lambda;\tau}^{\kappa,\lambda;\phi}C_\tau^{\rho,\lambda}(X,\, Y)\Big/\left(\frac{n}{2}\right)_\rho C_\tau(I)$$

where $b_{\rho,\lambda;\tau}^{\kappa,\lambda;\phi}$ is the coefficient of an expansion of the form

$$(26)\qquad C_\phi^{\kappa,\lambda}(I+X,\, Y)/C_\phi(I)=\sum_{r=0}^{k}\sum_{\phi;\tau\in\rho\cdot\lambda}b_{\rho,\lambda;\tau}^{\kappa,\lambda;\phi}C_\tau^{\rho,\lambda}(X,\, Y)/C_\tau(I)\,.$$

The coefficients $b_{\rho,\lambda;\tau}^{\kappa,\lambda;\phi}$ are tabulated in Appendix up to $f=3$.

LEMMA 3. *Let* $L_{\kappa,\lambda;\phi}^{n/2-p}(X, Y)$ *be a generalized Laguerre polynomial defined by* (24), *and* $a_1(\phi)$ *is given by* (9). *Then the following equalities hold.*

$$(27)\qquad \sum_\kappa\sum_\lambda\sum_{\phi\in\kappa\cdot\lambda}\theta_\phi^{\kappa,\lambda}L_{\kappa,\lambda;\phi}^{n/2-p}(X, Y)=(\mathrm{tr}\,Y)^l L_k^{mn/2-1}(\mathrm{tr}\,X)$$

$$(28)\qquad \sum_\kappa\sum_\lambda\sum_{\phi\in\kappa\cdot\lambda}a_1(\phi)\theta_\phi^{\kappa,\lambda}L_{\kappa,\lambda;\phi}^{n/2-p}(X, Y)$$
$$=l(l-1)\,\mathrm{tr}\,Y^2(\mathrm{tr}\,Y)^{l-2}L_k^{mn/2-1}(\mathrm{tr}\,X)$$
$$+kl(\mathrm{tr}\,Y)^{l-1}\{n\,\mathrm{tr}\,Y\,L_{k-1}^{mn/2}(\mathrm{tr}\,X)-2\,\mathrm{tr}\,XY\,L_{k-1}^{mn/2+1}(\mathrm{tr}\,X)\}$$
$$+k(k-1)(\mathrm{tr}\,Y)^l\left\{\frac{mn(m+n+1)}{4}L_{k-2}^{mn/2+1}(\mathrm{tr}\,X)\right.$$

$$-(m+n+1)\operatorname{tr} X L_{k-2}^{mn/2+2}(\operatorname{tr} X)+\operatorname{tr} X^2 L_{k-2}^{mn/2+3}(\operatorname{tr} X)\Big\}$$

(29) $$\sum_{\kappa}\sum_{\lambda}\sum_{\phi\in\kappa\cdot\lambda} a_1(\kappa)\theta_\phi^{\kappa,\lambda} L_{\kappa,\lambda;\phi}^{n/2-p}(X, Y)$$

$$=k(k-1)(\operatorname{tr} Y)^l\Big\{\frac{mn(m+n+1)}{4} L_{k-2}^{mn/2+1}(\operatorname{tr} X)$$

$$-(m+n+1)\operatorname{tr} X L_{k-2}^{mn/2+2}(\operatorname{tr} X)+\operatorname{tr} X^2 L_{k-2}^{mn/2+3}(\operatorname{tr} X)\Big\}$$

(30) $$\sum_{\kappa}\sum_{\lambda}\sum_{\phi\in\kappa\cdot\lambda} a_1(\lambda)\theta_\phi^{\kappa,\lambda} L_{\kappa,\lambda;\phi}^{n/2-p}(X, Y)=l(l-1)\operatorname{tr} Y^2(\operatorname{tr} Y)^{l-2} L_k^{mn/2-1}(\operatorname{tr} X)\ ,$$

where $L_k^\alpha(z)$ *is a univariate Laguerre polynomial defined by*

(31) $$k!L_k^\alpha(z)=e^z z^{-\alpha}\frac{d^k}{dz^k}(e^{-z}z^{k+\alpha})\ ,$$

Erdelyi A. et al. [5].

PROOF. As all formulas are proved by a similar way, we give the proof of (28). From the definition of $L_{\kappa,\lambda;\phi}^{n/2-p}(X, Y)$ and Lemma 1, (12), the generating function of $\sum_{\kappa}\sum_{\lambda}\sum_{\phi\in\kappa\cdot\lambda} a_1(\phi)\theta_\phi^{\kappa,\lambda} L_{\kappa,\lambda;\phi}^{n/2-p}(X, Y)$ is given by

(32) $$\operatorname{etr}(-X)\sum_{\kappa,\lambda;\phi} x^k y^l a_1(\phi)\theta_\phi^{\kappa,\lambda} L_{\kappa,\lambda;\phi}^{n/2-p}(X, Y)/k!l!$$

$$=\int_{R>0} A_{n/2-p}(XR)|R|^{n/2-p}\operatorname{etr}(-R)\sum_{\kappa,\lambda;\phi} x^k y^l a_1(\phi)\theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(R, Y)dR$$

$$=\int_{R>0} A_{n/2-p}(XR)|R|^{n/2-p}\operatorname{etr}(-(1-x)R)$$

$$\cdot\operatorname{etr}(yY)\operatorname{tr}(xR+yY)^2 dR$$

$$=\operatorname{etr}(yY)(1-x)^{-mn/2}\int_{R>0} A_{n/2-p}\Big(\frac{X}{1-x}R\Big)|R|^{n/2-p}\operatorname{etr}(-R)$$

$$\cdot\Big\{\frac{x^2}{(1-x)^2}\operatorname{tr} R^2+\frac{2xy}{1-x}\operatorname{tr} RY+y^2\operatorname{tr} Y^2\Big\}dR\ .$$

Using $\operatorname{tr} R^2=C_{(2)}(R)-(1/2)C_{(1^2)}(R)$ and $\operatorname{tr} RY=C_{(2)}^{1,1}(R, Y)-(1/2)C_{(1^2)}^{1,1}(R, Y)$ and the definition of a generalized Laguerre polynomials of a matrix argument, Constantine ([1], (14)) and (24), we have

(33) $$\sum_{\kappa,\lambda;\phi} x^k y^l a_1(\phi)\theta_\phi^{\kappa,\lambda} L_{\kappa,\lambda;\phi}^{n/2-p}(X, Y)/k!l!$$

$$=\operatorname{etr}(yY)\operatorname{etr}\Big(-\frac{x}{1-x}X\Big)(1-x)^{-mn/2}$$

$$\cdot\Big[\frac{x^2}{(1-x)^2}\Big\{L_{(2)}^{n/2-p}\Big(\frac{X}{1-x}\Big)-\frac{1}{2}L_{(1^2)}^{n/2-p}\Big(\frac{X}{1-x}\Big)\Big\}$$

$$+\frac{2xy}{1-x}\Big\{L_{1,1;(2)}^{n/2-p}\Big(\frac{X}{1-x}, Y\Big)-\frac{1}{2}L_{1,1;(1^2)}^{n/2-p}\Big(\frac{X}{1-x}, Y\Big)\Big\}$$

$$
\begin{aligned}
&+y^2 \operatorname{tr} Y^2\Big] \\
&= \operatorname{etr}(yY) \operatorname{etr}\left(-\frac{x}{1-x}X\right)(1-x)^{-mn/2} \\
&\cdot \left[\frac{x^2}{(1-x)^2}\left\{\frac{mn(m+n+1)}{4}-(m+n+1)\frac{\operatorname{tr} X}{1-x}+\frac{\operatorname{tr} X^2}{(1-x)^2}\right\}\right. \\
&\left.+\frac{xy}{1-x}\left\{n \operatorname{tr} Y-\frac{2}{1-x}\operatorname{tr} XY\right\}+y^2 \operatorname{tr} Y^2\right].
\end{aligned}
$$

Hence by the use of the generating function of a univariate Laguerre polynomials, i.e.,

$$
(34) \qquad (1-x)^{-\alpha-t} \exp\left\{-\frac{xz}{1-x}\right\}=\sum_{k=0}^{\infty}\frac{x^k}{k!}L_k^{\alpha}(z), \qquad (|x|<1)
$$

comparing the coefficients of $x^k y^l$ on both side of (33), we have (28).

LEMMA 4. *Let $g_{2\alpha+2j}(x, z)$ be a probability density function of a non-central chi-square random variable with $2\alpha+2j$ degrees of freedom and non-centrality parameter z.*

Define

$$
g_{2\alpha+2j}(x, z)=\exp(-z)h_{2\alpha+2j}, \qquad (\alpha)(x)=x^{\alpha-1}/2^{\alpha}\Gamma(\alpha),
$$

then the following equalities hold.

$$
(35) \qquad (\alpha)(x)\sum_{k=0}^{\infty}\frac{(-x/2)^k L_k^{\alpha-1}(z)}{k!(\alpha)_k}=h_{2\alpha}
$$

$$
(36) \qquad (\alpha)(x)\sum_{k=0}^{\infty}\frac{(-x/2)^k L_k^{\alpha}(z)}{k!(\alpha)_k}=-h_{2\alpha+2}+h_{2\alpha}
$$

$$
(37) \qquad (\alpha)(x)\sum_{k=0}^{\infty}\frac{(-x/2)^k L_k^{\alpha+1}(z)}{k!(\alpha)_k}=h_{2\alpha+4}-2h_{2\alpha+2}+h_{2\alpha}
$$

$$
(38) \qquad (\alpha)(x)\sum_{k=0}^{\infty}\frac{(-x/2)^k L_k^{\alpha-1}(z)}{(k-1)!(\alpha)_k}=-zh_{2\alpha+4}+(z-\alpha)h_{2\alpha+2}.
$$

PROOF. Hayakawa [7].

3. Derivation of the asymptotic probability density function of T_0^2

In this section we denote n_1 as n. Let us write

$$
(39) \qquad x=n_2T=T_0^2.
$$

Then the p.d.f. of x is represented by (40)

$$(40)\qquad \frac{n_2^{-mn/2}\Gamma_m\left(\frac{n+n_2}{2}\right)}{\Gamma_m\left(\frac{n_2}{2}\right)\Gamma\left(\frac{mn}{2}\right)}\operatorname{etr}(-\Omega_1-\Omega_2)x^{mn/2-1}$$

$$\cdot\sum_{k=0}^{\infty}\frac{(-x)^k}{k!\left(\frac{mn}{2}\right)_k}\sum_{l=0}^{\infty}\frac{1}{l!}\sum_{\kappa}\sum_{\lambda}\sum_{\phi\in\kappa\cdot\lambda}\frac{\left(\frac{n+n_2}{2}\right)_\phi}{\left(\frac{n_2}{2}\right)_\lambda n_2^k}\theta_\phi^{\kappa,\lambda}L_{\kappa,\lambda;\phi}^{n/2-p}(\Omega_1,\Omega_2)\ .$$

Noting

$$\frac{\Gamma_m\left(\frac{n+n_2}{2}\right)n_2^{-mn/2}}{\Gamma_m\left(\frac{n_2}{2}\right)\Gamma\left(\frac{mn}{2}\right)}=\frac{1}{2^{mn/2}\Gamma\left(\frac{mn}{2}\right)}\left\{1+\frac{1}{4n_2}mn(n-m-1)+O\left(\frac{1}{n_2^2}\right)\right\}$$

and

$$\left(\frac{n+n_2}{2}\right)_\phi\Big/\left(\frac{n_2}{2}\right)_\lambda n_2^k=\frac{1}{2^k}\left\{1+\frac{1}{n_2}\{a_1(\phi)+nf-a_1(\lambda)\}+O\left(\frac{1}{n_2^2}\right)\right\}\ ,$$

we have

$$(41)\qquad \frac{1}{2^{mn/2}\Gamma\left(\frac{mn}{2}\right)}\operatorname{etr}(-\Omega_1-\Omega_2)x^{mn/2-1}\sum_{k=0}^{\infty}\frac{\left(-\frac{x}{2}\right)^k}{k!\left(\frac{mn}{2}\right)_k}$$

$$\cdot\sum_{l=0}\frac{1}{l!}\sum_{\kappa}\sum_{\lambda}\sum_{\phi\in\kappa\cdot\lambda}\theta_\phi^{\kappa,\lambda}L_{\kappa,\lambda;\phi}^{n/2-p}(\Omega_1,\Omega_2)$$

$$\cdot\left[1+\frac{1}{n_2}\left\{a_1(\phi)+nf-a_1(\lambda)+\frac{mn(n-m-1)}{4}\right\}+O\left(\frac{1}{n_2^2}\right)\right]\ .$$

The first term is obvious from (27) and (35)

$$(42)\qquad \frac{\operatorname{etr}(-\Omega_1-\Omega_2)}{2^{mn/2}\Gamma\left(\frac{mn}{2}\right)}x^{mn/2-1}\sum_{k=0}^{\infty}\frac{\left(-\frac{x}{2}\right)^k}{k!\left(\frac{mn}{2}\right)_k}\sum_{l=0}^{\infty}\frac{(\operatorname{tr}\Omega_2)^l}{l!}L_k^{mn/2-1}(\operatorname{tr}\Omega_1)$$

$$=g_{mn}(x,\operatorname{tr}\Omega_1)\ .$$

The term of order $1/n_2$ becomes by the use of (27), (28) and (29).

$$(43)\qquad \frac{\operatorname{etr}(-\Omega_1-\Omega_2)}{2^{mn/2}\Gamma\left(\frac{mn}{2}\right)}x^{mn/2-1}\sum_{k=0}^{\infty}\frac{\left(-\frac{x}{2}\right)^k}{k!\left(\frac{mn}{2}\right)_k}$$

$$\cdot \sum_{l=0}^{\infty} \frac{1}{l!}\Big[k(k-1)(\mathrm{tr}\,\Omega_2)^l\Big\{\frac{mn(m+n+1)}{4}L_{k-2}^{mn/2+1}(\mathrm{tr}\,\Omega_1)$$
$$-(n+m+1)\,\mathrm{tr}\,\Omega_1 L_{k-2}^{mn/2+2}(\mathrm{tr}\,\Omega_1)+\mathrm{tr}\,\Omega_1^2 L_{k-2}^{mn/2+2}(\mathrm{tr}\,\Omega_2)\Big\}$$
$$+kl(\mathrm{tr}\,\Omega_2)^{l-1}\{n\,\mathrm{tr}\,\Omega_2 L_{k-1}^{mn/2}(\mathrm{tr}\,\Omega_1)-2\,\mathrm{tr}\,\Omega_1\Omega_2 L_{k-1}^{mn/2+1}(\mathrm{tr}\,\Omega_2)\}$$
$$+nf(\mathrm{tr}\,\Omega_2)^l L_k^{mn/2-1}(\mathrm{tr}\,\Omega_1)+\frac{mn(n-m-1)}{4}$$
$$\cdot(\mathrm{tr}\,\Omega_2)^l L_k^{mn/2-1}(\mathrm{tr}\,\Omega_1)\Big]\,.$$

Applying (35), (36), (37) and (38) to (43), we have

$$A\equiv\Big\{n\,\mathrm{tr}\,\Omega_2+\frac{1}{4}mn(n-m-1)\Big\}g_{mn}(x,\,\mathrm{tr}\,\Omega_1) \tag{44}$$
$$+\Big\{n\,\mathrm{tr}\,\Omega_1+2\,\mathrm{tr}\,\Omega_1\Omega_2-n\,\mathrm{tr}\,\Omega_2-\frac{1}{2}mn^2\Big\}g_{mn+2}(x,\,\mathrm{tr}\,\Omega_1)$$
$$+\Big\{\frac{mn(m+n+1)}{4}-(m+n+1)\,\mathrm{tr}\,\Omega_1+\mathrm{tr}\,\Omega_1^2$$
$$-2\,\mathrm{tr}\,\Omega_1\Omega_2-n\,\mathrm{tr}\,\Omega_1\Big\}g_{mn+4}(x,\,\mathrm{tr}\,\Omega_1)$$
$$+\{(m+n+1)\,\mathrm{tr}\,\Omega_1-2\,\mathrm{tr}\,\Omega_1^2\}g_{mn+6}(x,\,\mathrm{tr}\,\Omega_1)$$
$$+\mathrm{tr}\,\Omega_1^2\,g_{mn+8}(x,\,\mathrm{tr}\,\Omega_1)\,.$$

We summarize our results in

THEOREM. *Let S_1 and S_2 be independent non-central Wishart matrix with n and n_2 degrees of freedom, and non-centrality parameter matrices Ω_1 and Ω_2, respectively. Then the asymptotic expansion of the probability density function of a generalized Hotelling's $x=T_0^2=n_2\,\mathrm{tr}\,S_1S_2^{-1}$ is given by*

$$f(x)=g_{mn}(x,\,\mathrm{tr}\,\Omega_1)+\frac{A}{n_2}+O\Big(\frac{1}{n_2^2}\Big) \tag{45}$$

where $g_{mn}(x,\mathrm{tr}\,\Omega_1)$ is a p.d.f. of a non-central chi-square random variable with mn degrees of freedom and non-centrality parameter $\mathrm{tr}\,\Omega_1$ and A is given by (44), *respectively.*

COROLLARY. *Putting $\Omega_2=0$ in* (45), *we have same result in Hayakawa* [7].

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Appendix

$\kappa, \lambda; \phi$ \ $\rho, \lambda; \phi$	0,0;0	1,0;1	0,1;1	1,1;2	1,1;1^2	2,1;3	2,1;21	1^2,1;21	1^2,1;1^3
0, 0; 0	1								
1, 0; 1	1	1							
0, 1; 1			1						
1, 1; 2			1	1					
1, 1; 1^2			1		1				
2, 1; 3			1	2		1			
2, 1; 21			$\frac{2}{3}$	$\frac{2}{9}$	$\frac{10}{9}$		1		
1^2, 1; 21			$\frac{\sqrt{5}}{3}$	$\frac{4\sqrt{5}}{9}$	$\frac{2\sqrt{5}}{9}$			1	
1^2, 1; 1^3			1		2				1

The values of $b_{\rho,\lambda;\tau}^{\kappa,\lambda;\phi}$ up to order $f=3$ in the expansion

$$C_{\phi}^{\kappa,\lambda}(I+X, Y)/C_{\phi}(I)=\sum_{r=0}^{k} \sum_{\phi;\tau\in\rho\cdot\lambda} b_{\rho,\lambda;\tau}^{\kappa,\lambda;\phi} C_{\tau}^{\rho,\lambda}(X, Y)/C_{\tau}(I) .$$

Entries not shown in the table are zero.

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