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# ASYMPTOTIC DISTRIBUTION OF A CRAMÉR-VON MISES TYPE STATISTIC FOR TESTING SYMMETRY WHEN THE CENTER IS ESTIMATED

SIGEO AKI

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## Summary

In this paper we investigate the effect of estimating the center of symmetry on a Cramér-von Mises type statistic for testing the symmetry of a distribution function. The test statistic is defined by

$$nT_0[F_n]=n\int_{-\infty}^{\infty}\{F_n(x)+F_n(2S[F_n]-x)-1\}^2dF_n(x)\ ,$$

where $F_n$ is the empirical distribution function and $S[F_n]$ is an estimator of the center of $F$ which is consistent with the order $n^{1/2}$ and has von Mises derivative. The asymptotic distribution of $nT_0[F_n]$ under the null hypothesis is obtained. The distribution depends on the distribution $F$ and on the estimator $S[F_n]$.


## 1. Introduction

Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of i.i.d. random variables defined on a single probability space $(\Omega, \mathcal{B}, P)$ with a continuous distribution function $F$. $F_n$ denotes the empirical distribution function of the variables $X_1, X_2, \cdots, X_n$. There are many statistics for testing the null hypothesis $(H_0)$ that $F$ is symmetric about a specified value. For example we may mention a (weighted) sign test statistic and a Cramér-von Mises type statistic for testing symmetry. The latter is investigated by Filippova [1] (Example 9) and Rothman and Woodroofe [5].

Now we consider a problem of testing the null hypothesis $(H)$ that $F$ is symmetric about an unknown center. Since it is very rare in practice that one knows the center of symmetry, we can say that the hypothesis $H$ is more practical than the hypothesis $H_0$. Under the hypothesis $H$ we must estimate the center of $F$ using some estimator. The problem of finding the best estimator is very difficult because we do not know the distribution shape of $F$. But at any rate we can use any estimator satisfying the assumption (A2) in Section 2, for example Huber's $M$-estimator and $L$-estimator. Then the statistic $nT_0[F_n]$ defined in (1.1) below can be regarded as a test statistic of the hypothesis $H$,

$$nT_0[F_n]=n\int_{-\infty}^{\infty}\{F_n(x)+F_n(2S[F_n]-x)-1\}^2dF_n(x)\ , \tag{1.1}$$

where $S[F_n]$ is an estimator of the center of $F$. The statistic is a Cramér-von Mises type statistic for testing symmetry with the center estimated. It is very important, I think, for practical purposes investigating its distribution.

In Section 2 we give the asymptotic distribution of $nT_0[F_n]$ under the null hypothesis $H$. Let $nT[F_n]$ be defined by

$$nT[F_n]=n\int_{-\infty}^{\infty}\{F_n(x)+F_n(2S[F_n]-x)-1\}^2dF(x)\ , \tag{1.2}$$

which is replaced the measure $F_n$ of the integral of $nT_0[F_n]$ by $F$. In Theorem 1, $nT_0[F_n]$ and $nT[F_n]$ are shown to be asymptotically equivalent. In Theorem 2 we investigate the asymptotic distribution of $nT[F_n]$. The asymptotic distribution depends on the distribution $F$ and the estimator $S$. After all we arrive at the conclusion that estimating the unknown center has a very severe effect on the asymptotic distribution of the Cramér-von Mises type statistic for testing symmetry.

## 2. Results

In this section we study the asymptotic distribution of $nT_0[F_n]$ under the null hypothesis $H$. Suppose that $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of i.i.d. random variables with a continuous distribution function $F$. Let $m$ be the median of $F$. In order to state assumptions on $F$ and $S$, we need two definitions. Suppose a real valued functional $R$ is defined on a set $\sigma_R$ of real functions of a real argument.

DEFINITION 1. The functional $R$ is called $m$ times differentiable at the point $V\in\sigma_R$ with respect to the set $\tau\subset\sigma_R$ which is assumed to be star-shaped at the point $V$ if the following conditions are satisfied:

(1) For any $t\in[0, 1]$, $p=1, 2,\cdots, m$, and any function $W\in\tau$

$$\frac{d^p}{dt^p}R[(1-t)V+tW]$$

exists.

(2) There exist functions $R^{(p)}[V: y_1,\cdots, y_p]$ of $p$ arguments, $p=$

$1, \cdots, m$, which depend on $V$, such that for any function $W \in \tau$, the relation

$$\frac{d^p}{dt^p} R[(1-t)V+tW]\Big|_{t=0}$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} R^{(p)}[V: y_1, \cdots, y_p] \prod_{i=1}^{p} d[W(y_i)-V(y_i)] ,$$
$$p=1, \cdots, m ,$$

holds.

DEFINITION 2 (Filippova). The functional $R$ is called a Mises functional of order $m$ at the point $F$ ($F$ is a distribution function) if the following conditions are satisfied:

(1) There exists a star-shaped set $\tau_R \subset \sigma_R$ at the point $F$ such that

$$\lim_{n\to\infty} \mathrm{P}\,\{F_n \in \tau_R\} = 1 .$$

(2) The functional $R$ is $m$ times differentiable at the point $F$ with respect to the set $\tau_R$.

(3) For any $\varepsilon > 0$, $\delta > 0$ and $p=1, \cdots, m$, it holds that

$$\lim_{n\to\infty} \mathrm{P}\left\{ n^{(p/2)-\delta} \sup_{0\leq t\leq 1} \left| \frac{d^p}{dt^p} R[(1-t)F+tF_n] \right| > \varepsilon \right\} = 0 .$$

ASSUMPTIONS

(A1) $F$ is three times differentiable except for a set of Lebesgue measure zero.

(A2) $S[F_n]$ is a consistent estimator of $m$ with the order $n^{1/2}$ and is a Mises functional of order 3 at $F$.

THEOREM 1. *If Assumptions* (A1) *and* (A2) *are satisfied,*

$$\int_{-\infty}^{\infty} n\{F_n(x)+F_n(2S[F_n]-x)-1\}^2 d[F_n(x)-F(x)] \tag{2.1}$$

*converges to zero in probability as* $n \to \infty$ *under the null hypothesis* $H$.

Before the proof of Theorem 1 we state two lemmas.

LEMMA 1 (Pyke and Shorack [4]). *There exists a sequence of random processes* $\{\Gamma_n(t)\colon 0\leq t\leq 1\}$, $n\geq 1$ *which have the same distributions as empirical distribution functions of independent random variables with uniform distribution on* $[0, 1]$ *and there exists a Brownian bridge* $\{\beta(t)\colon 0\leq t\leq 1\}$ *such that* $\{\Gamma_n(t)\colon 0\leq t\leq 1\}$, $n\geq 1$ *and* $\{\beta(t)\colon 0\leq t\leq 1\}$ *are defined on a single probability space and they satisfy the following relation*

$$\sup_t |n^{1/2}(\Gamma_n(t)-t)-\beta(t)| \to 0 \qquad \textit{almost surely as } n\to\infty .$$

We define $h_n(x)$ and $h(x)$ by

$$h_n(x)=n\{\Gamma_n(F(x))+\Gamma_n(F(2S[\Gamma_n(F)]-x))-1\}^2 ,$$

$$h(x)=\Big\{\beta(F(x))+\beta(F(2m-x)) + 2f(2m-x)\int_0^1 S^{(1)}[F\colon F^{-1}(s)]d\beta(s)\Big\}^2 ,$$

where $f$ is the derivative of $F$ and $S^{(1)}[F\colon \cdot]$ is the real valued function of a real argument which exists from the assumption (A2) (see Definition 1 and Definition 2).

LEMMA 2. *If Assumptions* (A1) *and* (A2) *are satisfied,*

$$\sup_x |h_n(x)-h(x)| \to 0 \qquad \textit{in probability as } n\to\infty$$

*under the null hypothesis H.*

PROOF. Note that

$$h_n(x)=n\{\Gamma_n(F(x))-F(x)+\Gamma_n(F(2S[\Gamma_n(F)]-x))-F(2S[\Gamma_n(F)]-x) + F(2S[\Gamma_n(F)]-x)-F(2m-x)\}^2 .$$

Then we have

$$\begin{aligned}
&\sup_x |h_n(x)-h(x)| \\
&\leqq \Big(\sup_x |n^{1/2}(\Gamma_n(F(x))-F(x))-\beta(F(x))| \\
&\quad +\sup_x |n^{1/2}(\Gamma_n(F(2S[\Gamma_n(F)]-x))-F(2S[\Gamma_n(F)]-x)) \\
&\quad -\beta(F(2S[\Gamma_n(F)]-x))| \\
&\quad +\sup_x |\beta(F(2S[\Gamma_n(F)]-x))-\beta(F(2m-x))| \\
&\quad +\sup_x \Big| n^{1/2}(F(2S[\Gamma_n(F)]-x)-F(2m-x)) \\
&\quad -2f(2m-x)\int_0^1 S^{(1)}[F\colon F^{-1}(s)]d\beta(s)\Big|\Big) \\
&\quad \times(\sup_x ((h_n(x))^{1/2}+(h(x))^{1/2})) .
\end{aligned}$$

By Lemma 1 we can see the first and the second terms of the right-hand side converge to zero almost surely. The fourth term converges to zero in probability by the property of Mises functional $S$. Furthermore it holds that

$$\sup_x ((h_n(x))^{1/2}+(h(x))^{1/2})=O_p(1) .$$

Consequently we have the result of Lemma 2 if the third term is shown to converge to zero in probability. But it is shown by the theorem of Lévy on the sample path of Brownian motion (see Lévy [3] or Hida [2]), i.e. for any constant $c>1$ and for almost all $\omega \in \Omega$, there exists $\delta=\delta(\omega)>0$ and if $|t-t'|<\delta$ then

$$|\beta(t', \omega)-\beta(t, \omega)| \leqq c\{2|t-t'| \log (1/|t'-t|)\}^{1/2}$$

holds.

For any $\eta>0$, we will determine $d>0$ as

$$\mathrm{P}\{\omega;\ \delta(\omega)>d\} \geqq 1-\eta/2 .$$

Since $S[F_n]$ converges to $m$ in probability, $F$ is differentiable and the derivative $f$ is bounded, there exists an integer $n_0$ and

$$\mathrm{P}\{\sup_x |F(2S[\Gamma_n(F)]-x)-F(2m-x)|<d\} \geqq 1-\eta/2 \qquad \text{for all } n \geqq n_0 .$$

By Lévy's theorem it holds that

$$\begin{aligned}\mathrm{P}\{\sup_x &|\beta(F(2S[\Gamma_n(F)]-x))-\beta(F(2m-x))| \\ &\leqq c\{2\mu|S[\Gamma_n(F)]-m| \log (1/(\mu|2(S[\Gamma_n(F)]-m)|))\}^{1/2}\} \\ &\geqq 1-\eta \qquad \text{for all } n \geqq n_0,\ \text{where } \mu=\sup_x f(x) .\end{aligned}$$

Since $S[\Gamma_n(F)]-m$ converges to zero in probability,

$$2\mu|S[\Gamma_n(F)]-m| \log (1/(\mu|2(S[\Gamma_n(F)]-m)|)$$

also converges to zero in probability. Then we have

$$\sup_x |\beta(F(2S[\Gamma_n(F)]-x))-\beta(F(2m-x))| \to 0 \quad \text{in probability as } n \to \infty .$$

PROOF OF THEOREM 1. (2.1) is equal to

$$\int_0^1 h_n(F^{-1}(t))d[\Gamma_n(t)-t] .$$

We set $h_n(F^{-1}(t))=g_n(t)$ and $h(F^{-1}(t))=g(t)$.

For any function $\psi(t)$ on $[0, 1]$ and for any positive integer $k$, we define a step function $(\psi)_k(t)=\psi(i/k)$ on $[(i-1)/k, i/k)$, $i=1,\cdots,k$. Then we can easily see the following inequality.

$$\begin{aligned}&\left|\int_0^1 g_n(t)d[\Gamma_n(t)-t]\right| \\ &\quad \leqq \left|\int_0^1 (g_n(t)-(g_n)_k(t))d\Gamma_n(t)\right| + \left|\int_0^1 (g_n(t)-(g_n)_k(t))dt\right|\end{aligned}$$

$$+\left|\int_0^1 (g_n)_k(t)d[\Gamma_n(t)-t]\right| .$$

The first term of the right-hand side is estimated as

$$\begin{aligned}&\left|\int_0^1 (g_n(t)-(g_n)_k(t))d\Gamma_n(t)\right|\\ &\quad\leqq \sup_t |g_n(t)-g(t)|+\sup_t |g(t)-(g)_k(t)|+\sup_t |(g)_k(t)-(g_n)_k(t)|\\ &\quad\leqq 2\sup_t |g_n(t)-g(t)|+\sup_t |g(t)-(g)_k(t)| .\end{aligned}$$

Similarly the second term is estimated as

$$\left|\int_0^1 (g_n(t)-(g_n)_k(t))dt\right|\leqq 2\sup_t |g_n(t)-g(t)|+\sup_t |g(t)-(g)_k(t)| .$$

Note that $\int_a^b d[\Gamma_n(t)-t]=(\Gamma_n(b)-b)-(\Gamma_n(a)-a)$.

Then we have

$$\begin{aligned}&\left|\int_0^1 (g_n)_k(t)d[\Gamma_n(t)-t]\right|\\ &\quad\leqq 2k\sup_t |g_n(t)|\sup_t |\Gamma_n(t)-t|\\ &\quad\leqq 2k(\sup_t |g_n(t)-g(t)|+\sup_t |g(t)|)\sup_t |\Gamma_n(t)-t| ,\end{aligned}$$

where $\sup_t |g(t)|=\sup_x |h(x)|=O_p(1)$.

Fix $k$ sufficiently large so that $\sup_t |g(t)-g_k(t)|$ is sufficiently small. And let $n$ go to infinity.

Then (2.2) and (2.3) hold:

(2.2) $\sup_t |g_n(t)-g(t)|\to 0$ in probability as $n\to\infty$ (by Lemma 2).

(2.3) $\sup |\Gamma_n(t)-t|\to 0$ almost surely as $n\to\infty$ (by the Glivenko-Cantelli theorem).

Therefore we have the desired result.

THEOREM 2. *If Assumptions* (A1) *and* (A2) *are satisfied, the asymptotic distribution of* (1.2) *under the hypothesis $H$ is represented by the following double stochastic integral of a Brownian bridge,*

(2.4) $$\int_0^1\int_0^1 \psi[F: F^{-1}(u), F^{-1}(v)]d\beta(u)d\beta(v) ,$$ *where*

(2.5) $$\psi[F: y, z]=2\int_{-\infty}^{\infty} \chi_{(-\infty,x]}(y)\chi_{(-\infty,x]}(z)dF(x)$$

$$+2\int_{-\infty}^{\infty}\chi_{(-\infty,2m-x]}(y)\chi_{(-\infty,x]}(z)dF(x)$$

$$+2S^{(1)}[F: y]\int_{-\infty}^{\infty}f(x)\{\chi_{(-\infty,2m-x]}(z)+\chi_{(-\infty,x]}(z)\}dF(x)$$

$$+2S^{(1)}[F: z]\int_{-\infty}^{\infty}f(x)\{\chi_{(-\infty,2m-x]}(y)+\chi_{(-\infty,x]}(y)\}dF(x)$$

$$+4\left\{\int_{-\infty}^{\infty}(f(2m-x))^2dF(x)\right\}S^{(1)}[F: y]S^{(1)}[F: z]\ ,$$

*in* (2.5) $\chi_A(\cdot)$ *denotes the indicator function of* $A$.

*Remark.* In the first and the second term of the right-hand side of (2.5) we use the change of variables $y=F^{-1}(u)$ and $z=F^{-1}(v)$. Then we can see

$$2\left\{\int_{-\infty}^{\infty}\chi_{(-\infty,x]}(y)\chi_{(-\infty,x]}(z)dF(x)+\int_{-\infty}^{\infty}\chi_{(-\infty,2m-x]}(y)\chi_{(-\infty,x]}(z)dF(x)\right\}$$

$$=\begin{cases} 2\{1-\max(u,v)\} & \text{if } u+v>1 \\ 2\{2-\max(u,v)-u-v\} & \text{if } u+v\leqq 1\ . \end{cases}$$

The double stochastic integral of this kernel represents the asymptotic distribution of the test statistic when the center is known (see Filippova [1]). Consequently the other part of the right-hand side of (2.5) is added by estimating the center. And we can see that this part depends on the distribution $F$ and the estimator $S$.

We can easily prove Theorem 2 by using the following Proposition 1 and Proposition 2.

We define

$$\bar{T}[F_n]=\int_{-\infty}^{\infty}\Big\{F_n(x)-F(x)+F_n(2m-x)-F(2m-x)+2f(2m-x)\int_{-\infty}^{\infty}S^{(1)}[F: u]d[F_n(u)-F(u)]\Big\}^2 dF(x)\ .$$

PROPOSITION 1. If Assumptions (A1) and (A2) are satisfied, $nT[F_n]$ and $n\bar{T}[F_n]$ are asymptotically equivalent, i.e.

$$n(T[F_n]-\bar{T}[F_n])\to 0 \qquad \text{in probability as } n\to\infty\ .$$

PROPOSITION 2. If Assumptions (A1) and (A2) are satisfied, it holds that

$$\bar{T}[F_n]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi[F: y, z]d[F_n(y)-F(y)]d[F_n(z)-F(z)]\ .$$

PROOF OF PROPOSITION 1. We set $F_n^{(t)}=(1-t)F+tF_n$ for all $t\in[0,1]$. In this case

$$T[F_n^{(t)}]=\int_{-\infty}^{\infty}\{(1-t)F(x)+tF_n(x)+(1-t)F(2S[F_n^{(t)}]-x)+tF_n(S[F_n^{(t)}]-x)-1\}^2dF(x)$$

is not differentiable with respect to $t$.

Therefore we define

$$T_1[F_n^{(t)}]=\int_{-\infty}^{\infty}\{(1-t)F(x)+tF_n(x)+(1-t)F(2S[F_n^{(t)}]-x)+ts_n(2S[F_n^{(t)}]-x)-1\}^2dF(x)\ ,$$

which is differentiable with respect to $t$, where $s_n(x)$ is the following random function defined for every $\omega\in\Omega$ by

$$s_n(x)=\begin{cases}\dfrac{i+1}{n}+\dfrac{\int_{X_{(i)}-1/n^3}^{x}\exp[1/\{(u-X_{(i)}+1/n^3)(u-X_{(i)})\}]du}{n\int_{X_{(i)}-1/n^3}^{X_{(i)}}\exp[1/\{(u-X_{(i)}+1/n^3)(u-X_{(i)})\}]du} & (\text{if } X_{(i)}-1/n^3\leqq x\leqq X_{(i)},\ i=1,2,\cdots,n)\\ F_n(x) & (\text{otherwise})\ .\end{cases}$$

In the above formula $X_{(1)}\leqq\cdots\leqq X_{(n)}$ denote the order statistics of $X_1,\cdots,X_n$. We define $A_n(x,t)$ by

$$A_n(x,t)=(1-t)F(x)+tF_n(x)+(1-t)F(2S[F_n^{(t)}]-x)-1\ .$$

Then it holds that for any real $x$, for any $t\in[0,1]$, and for any integer $n$,

$$|A_n(x,t)|\leqq 2\ . \tag{2.6}$$

And we have by the definition of $T_1[F_n^{(t)}]$, Schwarz's inequality, and (2.6),

$$\begin{aligned}|T[F_n^{(t)}]-T_1[F_n^{(t)}]|\\ &\leqq\int_{-\infty}^{\infty}|\{F_n(2S[F_n^{(t)}]-x)\}^2-\{s_n(2S[F_n^{(t)}]-x)\}^2|dF(x)\\ &\quad+\int_{-\infty}^{\infty}2|A_n(x,t)|\cdot|F_n(2S[F_n^{(t)}]-x)-s_n(2S[F_n^{(t)}]-x)|dF(x)\\ &\leqq\left[\left\{\int_{-\infty}^{\infty}(F_n(2S[F_n^{(t)}]-x)+s_n(2S[F_n^{(t)}]-x))^2dF(x)\right\}^{1/2}\right.\\ &\quad\left.+2\left\{\int_{-\infty}^{\infty}|A_n(x,t)|^2dF(x)\right\}^{1/2}\right]\end{aligned}$$

$$\times\left[\int_{-\infty}^{\infty}\{F_n(2S[F_n^{(t)}]-x)-s_n(2S[F_n^{(t)}]-x)\}^2dF(x)\right]^{1/2}.$$

By the definition of $s_n$, it holds that $\sup_u|F_n(u)-s_n(u)|\leqq 1/n$ and Lebesgue measure of $\{x;\ s_n(x)\neq F_n(x)\}=1/n^2$. The latter immediately implies $F$-measure of $\{x;\ s_n(x)\neq F_n(x)\}\leqq\mu/n^2$, where $\mu=\sup_x f(x)$. Then we have

$$|T[F_n^{(t)}]-T_1[F_n^{(t)}]|\leqq 6\mu^{1/2}/n^2 \qquad \text{for all } t\in[0, 1].$$

Consequently,

$$n(T[F_n^{(t)}]-T_1[F_n^{(t)}])\to 0$$
$$\text{as } n\to\infty \text{ for all } \omega\in\Omega \text{ and for all } t\in[0, 1].$$

Hence it suffices to investigate the asymptotic distribution of $T_1[F_n^{(t)}]$. We have (2.7)–(2.11) by differentiating $T_1[F_n^{(t)}]$ with respect to $t$.

$$(2.7)\qquad \frac{dT_1[F_n^{(t)}]}{dt}=\int_{-\infty}^{\infty}2\{(1-t)F(x)+tF_n(x)+(1-t)F(2S[F_n^{(t)}]-x)$$
$$+ts_n(2S[F_n^{(t)}]-x)-1\}g(t, x)dF(x),$$

where

$$g(t, x)=F_n(x)-F(x)-F(2S[F_n^{(t)}]-x)$$
$$+2(1-t)f(2S[F_n^{(t)}]-x)\frac{dS[F_n^{(t)}]}{dt}+s_n(2S[F_n^{(t)}]-x)$$
$$+2ts_n'(2S[F_n^{(t)}]-x)\frac{dS[F_n^{(t)}]}{dt}.$$

$$(2.8)\qquad \left.\frac{dT_1[F_n^{(t)}]}{dt}\right|_{t=0}=0.$$

$$(2.9)\qquad \frac{d^2T_1[F_n^{(t)}]}{dt^2}=\int_{-\infty}^{\infty}2\Big[\{g(t, x)\}^2+\{(1-t)F(x)+tF_n(x)$$
$$+(1-t)F(2S[F_n^{(t)}]-x)+ts_n(2S[F_n^{(t)}]-x)-1\}$$
$$\times\frac{dg(t, x)}{dt}\Big]dF(x),$$

where

$$\frac{dg(t, x)}{dt}=-4f(2S[F_n^{(t)}]-x)\frac{dS[F_n^{(t)}]}{dt}+4(1-t)f'(2S[F_n^{(t)}]-x)$$
$$\times\left\{\frac{dS[F_n^{(t)}]}{dt}\right\}^2+2(1-t)f(2S[F_n^{(t)}]-x)\frac{d^2S[F_n^{(t)}]}{dt^2}$$
$$+4s_n'(2S[F_n^{(t)}]-x)\frac{dS[F_n^{(t)}]}{dt}+4ts_n''(2S[F_n^{(t)}]-x)$$

$$\times\left\{\frac{dS[F_n^{(t)}]}{dt}\right\}^2+2s_n'(2S[F_n^{(t)}]-x)\frac{d^2S[F_n^{(t)}]}{dt^2}\ .$$

$$\text{(2.10)}\quad \left.\frac{d^2T_1[F_n^{(t)}]}{dt^2}\right|_{t=0}=2\int_{-\infty}^{\infty}\{g(0,x)\}^2dF(x)\ .$$

$$\text{(2.11)}\quad \frac{d^3T_1[F_n^{(t)}]}{dt^3}=\int_{-\infty}^{\infty}\left[6g(t,x)\frac{dg(t,x)}{dt}+2\{(1-t)F(x)+tF_n(x)\right.$$
$$+(1-t)F(2S[F_n^{(t)}]-x)+ts_n(2S[F_n^{(t)}]-x)-1\}$$
$$\left.\times\frac{d^2g(t,x)}{dt^2}\right]dF(x)\ ,$$

where

$$\frac{d^2g(t,x)}{dt^2}=-12f'(2S[F_n^{(t)}]-x)\left\{\frac{dS[F_n^{(t)}]}{dt}\right\}^2-6f(2S[F_n^{(t)}]-x)$$
$$\times\frac{d^2S[F_n^{(t)}]}{dt^2}+8(1-t)f''(2S[F_n^{(t)}]-x)\left\{\frac{dS[F_n^{(t)}]}{dt}\right\}^3$$
$$+12(1-t)f'(2S[F_n^{(t)}]-x)\frac{dS[F_n^{(t)}]}{dt}\ \frac{d^2S[F_n^{(t)}]}{dt^2}$$
$$+2(1-t)f(2S[F_n^{(t)}]-x)\frac{d^3S[F_n^{(t)}]}{dt^3}$$
$$+12s_n''(2S[F_n^{(t)}]-x)\left\{\frac{dS[F_n^{(t)}]}{dt}\right\}^2$$
$$+6s_n'(2S[F_n^{(t)}]-x)\frac{d^2S[F_n^{(t)}]}{dt^2}$$
$$+8ts_n^{(3)}(2S[F_n^{(t)}]-x)\left\{\frac{dS[F_n^{(t)}]}{dt}\right\}^3$$
$$+12ts_n''(2S[F_n^{(t)}]-x)\frac{dS[F_n^{(t)}]}{dt}\ \frac{d^2S[F_n^{(t)}]}{dt^2}$$
$$+2ts_n'(2S[F_n^{(t)}]-x)\frac{d^3S[F_n^{(t)}]}{dt^3}\ .$$

By Assumption (A2) and from (2.7), (2.9) and (2.11) we have

$$\text{(2.12)}\qquad n^{p/2-\delta}\sup_t\left|\frac{d^pT_1[F_n^{(t)}]}{dt^p}\right|\to 0\qquad\text{in probability}$$

for any $\delta>0$ and $p=1, 2, 3$. Set

$$B_n(x)=F_n(x)-F(x)-F(2m-x)+2f(2m-x)$$
$$\times\int_{-\infty}^{\infty}S^{(1)}[F: u]d[F_n(u)-F(u)]\ ,$$

then we have

(2.13) $$|B_n(x)|\leqq 4 \quad \text{for sufficiently large } n\,.$$

From the definition of $\bar{T}[F_n]$ and by Schwarz's inequality and (2.13), we have

$$\begin{aligned}
&\left|\left(\frac{d^2T_1[F_n]}{dt^2}\Big|_{t=0}\right)-2\bar{T}[F_n]\right| \\
&\leqq\int_{-\infty}^{\infty} 2|\{s_n(2m-x)\}^2-\{F_n(2m-x)\}^2|dF(x) \\
&\quad +4\int_{-\infty}^{\infty} |B_n(x)||s_n(2m-x)-F_n(2m-x)|dF(x) \\
&\leqq(4+16)\left\{\int_{-\infty}^{\infty} (s_n(2m-x)-F_n(2m-x))^2dF(x)\right\}^{1/2} \\
&\leqq 20\mu^{1/2}/n^2 \qquad \text{for sufficiently large } n\,.
\end{aligned}$$

Consequently we have

(2.14) $$n\left(\frac{d^2T_1[F_n^{(t)}]}{dt^2}\Big|_{t=0}-2\bar{T}[F_n^{(t)}]\right)\to 0 \qquad \text{as } n\to\infty \text{ for all } \omega\in\Omega\,.$$

Let's assume that a functional $R[F_n^{(t)}]$ is $(m+1)$ times differentiable with respect to $t$, and

(2.15) $$\frac{d^pR[F_n^{(t)}]}{dt^p}\Big|_{t=0}=0\,, \qquad p=1,\cdots,m-1\,,$$

hold. Suppose moreover that

(2.16) $$n^{q/2-\delta}\sup_t\left|\frac{d^qR[F_n^{(t)}]}{dt^q}\right|\to 0 \qquad \text{in probability as } n\to\infty$$

for any $\delta>0$ and $q=1,\cdots,m+1$.

If we set $t=1$ on the Taylor expansion for $R[F_n^{(t)}]$ of order $(m+1)$ at the point $t=0$,

$$\begin{aligned}
R[F_n^{(t)}]=R[F]+\frac{t}{1!}\frac{dR[F_n^{(t)}]}{dt}\Big|_{t=0}+\cdots+\frac{t^m}{m!}\frac{d^mR[F_n^{(t)}]}{dt^m}\Big|_{t=0} \\
+\frac{t^{m+1}}{(m+1)!}\frac{d^{m+1}R[F_n^{(t)}]}{dt^{m+1}}\Big|_{t=\theta}\,,
\end{aligned}$$

then we have by (2.15)

(2.17) $$R[F_n]-R[F]-\frac{1}{m!}\frac{d^mR[F_n^{(t)}]}{dt^m}\Big|_{t=0}=\frac{1}{(m+1)!}\frac{d^{m+1}R[F_n^{(t)}]}{dt^{m+1}}\Big|_{t=\theta'}\,.$$

Then (2.16) implies

(2.18) $$n^{m/2}(R[F_n]-R[F])-\frac{n^{m/2}}{m!}\frac{d^mR[F_n^{(t)}]}{dt^m}\Big|_{t=0}\to 0 \qquad \text{in probability}\,.$$

By (2.7)–(2.12), $T_1[F_n^{(t)}]$ satisfies (2.15) and (2.16) with $m=2$. Therefore we have

$$(2.19) \quad nT_1[F_n]-\frac{n}{2!}\frac{d^2T_1[F_n^{(t)}]}{dt^2}\Big|_{t=0}\to 0 \quad \text{in probability as } n\to\infty .$$

Proposition 1 now follows from (2.14) and (2.19).

PROOF OF PROPOSITION 2. By the definition of $\bar{T}[F_n]$ we have

$$\begin{aligned}\bar{T}[F_n]=&\int_{-\infty}^{\infty}\{F_n(x)-F(x)\}^2dF(x)+\int_{-\infty}^{\infty}\{F_n(2m-x)-F(2m-x)\}^2dF(x)\\&+4\left\{\int_{-\infty}^{\infty}S^{(1)}[F:u]d[F_n(u)-F(u)]\right\}^2\int_{-\infty}^{\infty}f^2(2m-x)dF(x)\\&+2\int_{-\infty}^{\infty}\{F_n(x)-F(x)\}\{F_n(2m-x)-F(2m-x)\}dF(x)\\&+4\int_{-\infty}^{\infty}S^{(1)}[F:u]d[F_n(u)-F(u)]\\&\times\int_{-\infty}^{\infty}\{F_n(x)-F(x)\}f(2m-x)dF(x)\\&+4\int_{-\infty}^{\infty}S^{(1)}[F:u]d[F_n(u)-F(u)]\\&\times\int_{-\infty}^{\infty}\{F_n(2m-x)-F(2m-x)\}f(2m-x)dF(x) .\end{aligned}$$

The result of Proposition 2 is shown by the following calculations (2.20)–(2.24).

$$\begin{aligned}(2.20)\quad &\int_{-\infty}^{\infty}\{F_n(x)-F(x)\}^2dF(x)\\&=\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}(F_n(x)-F(x))\chi_{(-\infty,x]}(z)dF(x)\right\}d[F_n(z)-F(z)]\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\chi_{(-\infty,x]}(y)\chi_{(-\infty,x]}(z)dF(x)\right\}\\&\quad\times d[F_n(y)-F(y)]d[F_n(z)-F(z)] .\end{aligned}$$

$$\begin{aligned}(2.21)\quad &\int_{-\infty}^{\infty}\{F_n(2m-x)-F(2m-x)\}^2dF(x)\\&=\int_{-\infty}^{\infty}\{F_n(y)-F(y)\}^2dF(y)\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\chi_{(-\infty,x]}(y)\chi_{(-\infty,x]}(z)dF(x)\right\}\\&\quad\times d[F_n(y)-F(y)]d[F_n(z)-F(z)] .\end{aligned}$$

$$(2.22)\quad \int_{-\infty}^{\infty}\{F_n(2m-x)-F(2m-x)\}\{F_n(x)-F(x)\}dF(x)$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\chi_{(-\infty,2m-x]}(z)\chi_{(-\infty,x]}(y)dF(x)\right\}$$
$$\times d[F_n(y)-F(y)]d[F_n(z)-F(z)] .$$

(2.23) $$\int_{-\infty}^{\infty} S^{(1)}[F: u]d[F_n(u)-F(u)]\int_{-\infty}^{\infty}\{F_n(x)-F(x)\}f(2m-x)dF(x)$$
$$=\int_{-\infty}^{\infty} S^{(1)}[F: u]d[F_n(u)-F(u)]$$
$$\times\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(2m-x)\chi_{(-\infty,x]}(z)dF(x)\right)d[F_n(z)-F(z)]$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left\{\int_{-\infty}^{\infty}\chi_{(-\infty,x]}(z)f(2m-x)dF(x)\right\}S^{(1)}[F: y]\right]$$
$$\times d[F_n(y)-F(y)]d[F_n(z)-F(z)] .$$

(2.24) $$\int_{-\infty}^{\infty} S^{(1)}[F: u]d[F_n(u)-F(u)]$$
$$\times\int_{-\infty}^{\infty}\{F_n(2m-x)-F(2m-x)\}f(2m-x)dF(x)$$
$$=\int_{-\infty}^{\infty} S^{(1)}[F: u]d[F_n(u)-F(u)]\int_{-\infty}^{\infty}\{F_n(x)-F(x)\}f(x)dF(x)$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left\{\int_{-\infty}^{\infty} f(x)\chi_{(-\infty,x]}(z)dF(x)\right\}S^{(1)}[F: y]\right]$$
$$\times d[F_n(y)-F(y)]d[F_n(z)-F(z)] .$$

By Theorem 1 and Theorem 2 we can conclude that the asymptotic distribution of $nT_0[F_n]$ under the hypothesis $H$ is equal to the distribution of the double stochastic integral defined by (2.4).

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