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LINEAR PREDICTION OF RECORD VALUES FOR THE TWO PARAMETER EXPONENTIAL DISTRIBUTION

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Abstract

This paper deals with the problem of predicting the sth record value based on the first m record values ($s>m$) when the observations are from the exponential distribution. Various estimates for the sth record value are obtained and their mean square errors are compared.

1. Introduction

Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables (rv) with cumulative distribution function (CDF) $F(x)$ and the corresponding probability density function (pdf) $f(x)$. Let $Y_n = \max(X_1, \cdots, X_n)$ for $n \geqq 1$. We say X_j is a record value of $\{X_n\}$ if $Y_j > Y_{j-1}$. Define $L(0)=1$, $L(n)=\min\{j \mid j>L(n-1),\ X_j > X_{L(n-1)}\}$. Then $X_{L(n)}$, $n \geqq 0$ is the sequence of record values. We will denote $X \in \mathrm{E}(x, \mu, \sigma)$, if the pdf $f(x)$ of X is of the form

$$(1.1)\qquad f(x)=\begin{cases} \sigma^{-1}\exp(-\sigma^{-1}(x-\mu))\ , & \text{for } x>\mu,\ \sigma>0\ , \\ 0\ , & \text{otherwise}\ ; \end{cases}$$

and $X \in G_n(x, \mu, \sigma)$, if the pdf $f(x)$ of X is of the type

$$(1.2)\qquad f(x)=\begin{cases} (\Gamma(n))^{-1}\sigma^{-n}(x-\mu)^{n-1}\exp(-\sigma^{-1}(x-\mu))\ , & \text{for } x>\mu,\ \sigma>0\ , \\ 0\ , & \text{otherwise}\ . \end{cases}$$

In this paper we will consider various linear estimates of the sth record value based on the first m record values ($s>m$) when $X \in \mathrm{E}(x, \mu, \sigma)$.

2. Preliminaries

LEMMA 1. *If* $X_k \in \mathrm{E}(x, \mu, \sigma)$, $k \geqq 1$, *then*

Key words: Exponential distribution, record values, prediction, linear estimation, invariance estimation, least square estimation.

(i) $X_{L(m)} \in G_{m+1}(x, \mu, \sigma)$ *and*
(ii) $Z_{n,m} | X_{L(m)} = u \in G_{n-m}(x, 0, \sigma)$
where $Z_{n,m} = X_{L(n)} - X_{L(m)}$.

PROOF. For (i) see Ahsanullah [1]. The joint pdf f_1 of $X_{L(0)}$, $X_{L(1)}, \cdots, X_{L(n)}$ is known, Resnick [5] p. 69, as

$$(2.1)\qquad f_1(x_0, x_1, \cdots, x_n) = \begin{cases} r(x_0)r(x_1)\cdots r(x_{n-1})f(x_n) \\ \qquad \text{for } \mu < x_0 < x_1 < \cdots < x_{n-1} < x_n < \infty , \\ 0 , \qquad \text{otherwise} , \end{cases}$$

where $r(x) = f(x)/(1-F(x))$, $F(x) < 1$. Integrating out $x_0, x_1, \cdots, x_{m-1}$, $x_{m+1}, \cdots, x_{n-1}$, we get the pdf f_2 of $X_{L(m)}$ and $X_{L(n)}$ as

$$(2.2)\qquad f_2(x_m, x_n) = \begin{cases} (\Gamma(n-m)\Gamma(m+1))^{-1} R^m(x_m) r(x_m)(R(x_n) - R(x_m))^{n-m-1} \\ \qquad \cdot f(x_n) , \qquad \text{for } \mu < x_m < x_n < \infty , \\ 0 , \qquad \text{otherwise} , \end{cases}$$

where $R(x) = -\log_e \bar{F}(x)$, for $\bar{F}(x) > 0$, $\bar{F}(x) = 1 - F(x)$ and $r(x) = d/dx \cdot R(x)$. Similarly it can be shown that the pdf of $X_{L(m)}$ can be written as

$$(2.3)\qquad f_{X_{L(m)}}(x_m) = \begin{cases} (\Gamma(m+1))^{-1} R^m(x_m) f(x_m) , & \text{for } \mu < x_m < \infty , \\ 0 , & \text{otherwise} . \end{cases}$$

Hence the pdf f_3 of $X_{L(n)} | X_{L(m)} = x_m$ is

$$f_3(x_n | X_{L(m)} = x_m) = \begin{cases} (\Gamma(n-m))^{-1}(R(x_n) - R(x_m))^{n-m-1} f(x_n)(\bar{F}(x_m))^{-1} , \\ \qquad \text{for } \mu < x_m < x_n < \infty , \\ 0 , \qquad \text{otherwise} . \end{cases}$$

Substituting $R(x) = \sigma^{-1}(x-\mu)$, for $x > \mu$, $f(x) = \sigma^{-1} \exp(-\sigma^{-1}(x-\mu))$ and $Z_{n,m} = X_{L(n)} - X_{L(m)}$, we get the pdf f_4 of $Z_{n,m} | X_{L(m)} = x_m$ as

$$(2.4)\qquad f_4(z | X_{L(m)} = x_m) = \begin{cases} (\Gamma(n-m))^{-1} \sigma^{m-n} z^{n-m-1} \exp(-\sigma^{-1} z) \\ \qquad \text{for } 0 < z < \infty , \\ 0 , \qquad \text{otherwise} . \end{cases}$$

LEMMA 2. *Let $\hat{\mu}$ and $\hat{\sigma}$ be the minimum variance unbiased estimates of μ and σ respectively based on the observed values $x_{L(0)}, x_{L(1)}, \cdots$, $x_{L(m-1)}$, then*

$$\hat{\mu} = (m x_{L(0)} - x_{L(m-1)})/(m-1)$$

$$\hat{\sigma} = (x_{L(m-1)} - x_{L(0)})/(m-1)$$

and

$$\mathrm{Var}(\hat{\mu})=m\sigma^2/(m-1)$$

$$\mathrm{Var}(\hat{\sigma})=\sigma^2/(m-1)$$

$$\mathrm{Cov}(\hat{\mu}, \hat{\sigma})=-\sigma^2/(m-1) .$$

PROOF. Let $Y_i=\sigma^{-1}(X_{L(i-1)}-\mu)$, then by Lemma 1, it follows that $\mathrm{E}(Y_i)=i$, $\mathrm{Var}(Y_i)=i$, $i=1, 2, \cdots, m-1$. From (2.4) it follows that for $i>j$

$$i-j=\mathrm{Cov}(Y_i-Y_j)=\mathrm{Var}(Y_i)+\mathrm{Var}(Y_j)-2\,\mathrm{Cov}(Y_iY_j) .$$

Hence

$$\mathrm{Cov}(Y_i-Y_j)=j , \qquad \text{for } i>j .$$

Let $\boldsymbol{Y}$ be the $m\times 1$ vector corresponding to the observed values of $X_{L(i-1)}$, $i=1, \cdots, m$. Then we can write

$$\mathrm{E}(\boldsymbol{Y})=\boldsymbol{\alpha} , \qquad \mathrm{Var}(\boldsymbol{Y})=V , \tag{2.5}$$

where $\boldsymbol{\alpha}'=(1, 2, \cdots, m)$, and $V=(v_{ij})$, $v_{ij}=\min(i, j)$, $i, j=1, 2, \cdots, m$. Now it can easily be shown that

$$\mathrm{E}(\boldsymbol{X})=\mu\mathbf{1}+\sigma\boldsymbol{\alpha} , \qquad \mathrm{Var}(\boldsymbol{X})=\sigma^2 V , \tag{2.6}$$

where $\boldsymbol{X}'=(x_{L(0)}, x_{L(1)}, \cdots, x_{L(m-1)})$, the vector of observed values and $\mathbf{1}$ is a $m\times 1$ vector of units. Let $\Omega=V^{-1}=(v^{ij})$, then

$$v^{ij}=\begin{cases} 2 , & \text{if } i=j=1, 2, \cdots, m-1 , \\ 1 , & \text{if } i=j=m , \\ -1 , & \text{if } |i-j|=1 , \qquad i, j=1, 2, \cdots, m , \\ 0 , & \text{otherwise} . \end{cases} \tag{2.7}$$

The minimum variance linear unbiased estimates of $\hat{\mu}$ and $\hat{\sigma}$ of μ and σ respectively (David [2]) is

$$\hat{\mu}=\boldsymbol{\alpha}'V^{-1}(\mathbf{1}\boldsymbol{\alpha}'-\boldsymbol{\alpha}\mathbf{1}')V^{-1}\boldsymbol{X}/\Delta , \qquad \hat{\sigma}=\mathbf{1}'V^{-1}(\mathbf{1}\boldsymbol{\alpha}'-\boldsymbol{\alpha}\mathbf{1}')V^{-1}\boldsymbol{X}/\Delta ,$$

where

$$\Delta=(\mathbf{1}'V^{-1}\mathbf{1})(\boldsymbol{\alpha}'V^{-1}\boldsymbol{\alpha})-(\mathbf{1}'V^{-1}\boldsymbol{\alpha})^2 ,$$

and

$$\mathrm{Var}(\hat{\mu})=\sigma^2\boldsymbol{\alpha}'V^{-1}\boldsymbol{\alpha}/\Delta ,$$

$$\mathrm{Var}(\hat{\sigma})=\sigma^2\mathbf{1}'V^{-1}\mathbf{1}/\Delta ,$$

$$\text{Cov}(\hat{\mu}, \hat{\sigma}) = -\sigma^2 \mathbf{1}' V^{-1} \boldsymbol{\alpha} / \Delta .$$

It can be shown that

$$\mathbf{1}' V^{-1} = (1, 0, 0, \cdots, 0)$$

$$\boldsymbol{\alpha}' V^{-1} = (0, 0, 0, \cdots, 1) ,$$

$$\boldsymbol{\alpha}' V^{-1} \boldsymbol{\alpha} = m , \qquad \text{and} \qquad \Delta = m-1 .$$

On simplification we get

$$\hat{\mu} = (x_{L(m-1)} - m x_{L(0)})/(m-1) , \qquad \hat{\sigma} = (x_{L(m-1)} - x_{L(0)})/(m-1) ,$$

and

$$\text{Var}(\hat{\mu}) = m\sigma^2/(m-1) ,$$

$$\text{Var}(\hat{\sigma}) = \sigma^2/(m-1) ,$$

$$\text{Cov}(\hat{\mu}, \hat{\sigma}) = -\sigma^2/(m-1) .$$

3. Prediction of $X_{L(s-1)}$

We shall assume $s > m$. Let $\boldsymbol{W}' = (W_1, W_2, \cdots, W_m)$, where $W_i = \text{Cov}(X_{L(i-1)}, X_{L(s-1)})$, $i = 1, 2, \cdots, m$ and $\alpha^* = \sigma^{-1} \text{E}(X_{L(s-1)} - \mu)$. It follows from the results of Goldberger [3] that the best linear unbiased (BLU) predictor of $X_{L(s-1)}$ is $\hat{X}_{L(s-1)}$, where

(3.1) $$\hat{X}_{L(s-1)} = \hat{\text{E}}(X_{L(s-1)}) + \boldsymbol{W}' V^{-1} (\boldsymbol{X} - \hat{\mu} \mathbf{1} - \hat{\sigma} \boldsymbol{\alpha}) ,$$

with

$$\hat{\text{E}}(X_{L(s-1)}) = \hat{\mu} + \hat{\sigma} \alpha^* ,$$

and $\hat{\mu}$ and $\hat{\sigma}$ are as given in Lemma 2. But $\alpha^* = s$, $\boldsymbol{W}' = \boldsymbol{\alpha}'$. It can easily be shown that $\boldsymbol{W}' V^{-1} (\boldsymbol{X} - \hat{\mu} \mathbf{1} - \hat{\sigma} \boldsymbol{\alpha}) = 0$. Hence

(3.2) $$\hat{X}_{L(s-1)} = \hat{\text{E}}(X_{L(s-1)}) = \hat{\mu} + \hat{\sigma} \alpha^* .$$

Substituting the values of $\hat{\mu}$, $\hat{\sigma}$ and α^* in (3.2), we get in terms of observed values

(3.3) $$\hat{X}_{L(s-1)} = ((s-1) x_{L(m-1)} - (s-m) x_{L(0)})/(m-1) .$$

It can be shown that

(3.4) $$\text{E}(\hat{X}_{L(s-1)}) = \text{E}(X_{L(s-1)}) = \mu + s\sigma ,$$

$$\text{Var}(\hat{X}_{L(s-1)}) = \sigma^2 (m + s^2 - 2s)/(m-1) ,$$

and

$$\begin{aligned}\text{MSE}\,(\hat{X}_{L(s-1)})&=\text{E}\,(\hat{X}_{L(s-1)}-X_{L(s-1)})^2\\&=\sigma^2(m+s^2-ms-s)/(m-1)\ .\end{aligned}$$

Let $\tilde{X}_{L(s-1)}$ be the best linear invariant predictor of $X_{L(s-1)}$. From the results of Mann [4] it follows that

$$\tilde{X}_{L(s-1)}=\hat{X}_{L(s-1)}-(C_{12}(1+C_{22})^{-1})\hat{\sigma}\ , \tag{3.5}$$

where

$$C_{12}\sigma^2=\text{Cov}\,(\hat{\sigma},\,(1-\boldsymbol{W}'V^{-1}\mathbf{1})\hat{\mu}+(\alpha^*-\boldsymbol{W}'V^{-1}\boldsymbol{\alpha})\hat{\sigma})$$

$$C_{22}\sigma^2=\text{Var}\,(\hat{\sigma})\ .$$

It can be shown that

$$C_{12}=(s-m)/(m-1)\ , \qquad \text{and} \qquad C_{22}=1/(m-1)\ .$$

On simplification we get

$$\tilde{X}_{L(s-1)}=\hat{X}_{L(s-1)}-\frac{s-m}{m}\hat{\sigma}=\frac{m-s}{m}x_{L(0)}+\frac{s}{m}X_{L(m-1)} \tag{3.6}$$

$$\begin{aligned}&\text{E}\,(\tilde{X}_{L(s-1)})=\text{E}\,(\hat{X}_{L(s-1)})-\frac{s-m}{m}\hat{\sigma}=\mu+\left(s+\frac{m-s}{m}\right)\sigma\\&\text{Var}\,(\tilde{X}_{L(s-1)})=\sigma^2(m^2+ms^2-s^2)\ , \qquad \text{and}\\&\text{MSE}\,(\tilde{X}_{L(s-1)})=\text{E}\,(\tilde{X}_{L(s-1)}-X_{L(s-1)})^2=\text{MSE}\,(\hat{X}_{L(s-1)})-\frac{(s-m)^2}{m(m-1)}\sigma^2\ .\end{aligned} \tag{3.7}$$

It is well known that the best (unrestricted) least squares predictor $\hat{\hat{X}}_{L(s-1)}$ of $X_{L(s-1)}$ is

$$\hat{\hat{X}}_{L(s-1)}=\text{E}\,(X_{L(s-1)}\,|\,X_{L(0)},\,X_{L(1)},\cdots,\,X_{L(m-1)})\ . \tag{3.8}$$

By using Lemma 1, we see that

$$\hat{\hat{X}}_{L(s-1)}=x_{L(m-1)}+(s-m)\sigma\ . \tag{3.9}$$

But $\hat{\hat{X}}_{L(s-1)}$ depends on the unknown parameter σ. If we substitute the minimum variance unbiased estimate $\hat{\sigma}$ for σ in (3.9), then $\hat{\hat{X}}_{L(s-1)}$ becomes equal to $\hat{X}_{L(s-1)}$. Now

$$\begin{aligned}&\text{E}\,(\hat{\hat{X}}_{L(s-1)})=\mu+s\sigma=\text{E}\,(X_{L(s-1)})\ ,\\&\text{Var}\,(\hat{\hat{X}}_{L(s-1)})=m\sigma^2\ , \qquad \text{and}\\&\text{MSE}\,(\hat{\hat{X}}_{L(s-1)})=\text{E}\,(\hat{\hat{X}}_{L(s-1)}-X_{L(s-1)})^2=(s-m)\sigma^2\ .\end{aligned} \tag{3.10}$$

By considering the mean square errors of $\hat{X}_{L(s-1)}$, $\tilde{X}_{L(s-1)}$ and $\hat{\hat{X}}_{L(s-1)}$, we see that MSE $(\hat{\hat{X}}_{L(s-1)}) <$ MSE $(\tilde{X}_{L(s-1)}) <$ MSE $(\hat{X}_{L(s-1)})$.

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