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FUNCTIONAL EQUATION WITH AN ERROR TERM AND THE STABILITY OF SOME CHARACTERIZATIONS OF THE EXPONENTIAL DISTRIBUTION

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1. Introduction and summary

In the previous paper [8], we gave the solution to the functional equation

$$H(x)=\int_0^\infty H(x+y)dG(y)\ , \tag{1}$$

where $G(x)$ is a function of bounded variation with the total variation less than or equal to 1. The solution was applied to some characterization problems of the stable and the exponential distributions. In the present paper we shall consider the extended equation

$$H(x)=\int_0^\infty H(x+y)dG(y)+S(x) \tag{2}$$

where $S(x)$ is an "error term" and is supposed to be small in some sense. In Section 2 we shall derive the boundedness of $H(x)$ assuming some additional conditions on $H(x)$ and $G(x)$. In Section 3 we give a necessary and sufficient condition that a bounded function $H(x)$ satisfies the equation (2). Explicit formulae for $H(x)$ will be given in Section 4. As an application of these results we shall show in Section 5 the stability of some characterizations of the exponential distribution as given by Ferguson [1], Rossberg [7], Ramachandran [6], and others [3], [4], [5], and [8].

2. Boundedness of $H(x)$

Throughout this section we assume that $G(x)$ is a distribution function on the interval $[0, \infty)$ with

$$1<\int_0^\infty e^{\delta x}dG(x)<\infty\ , \qquad \text{for some } \delta>0 \tag{3}$$

and that $H(x)$ is a positive, right continuous function defined for $x \geqq x_0 > -\infty$ and such that for some positive constant λ

$$H(x+y) \leqq e^{\lambda y} H(x) , \qquad x \geqq x_0,\ y \geqq 0 . \tag{4}$$

Under these assumptions it was proved in the previous paper [8] that the following theorem holds.

THEOREM A. *If $H(x)$ satisfies the inequality*

$$H(x) \geqq \int_0^\infty H(x+y)dG(y) , \qquad x \geqq x_0 , \tag{5}$$

then it is bounded.

In this section we shall extend Theorem A and prove

THEOREM 1. *If H and G satisfy the assumptions stated above, then the inequality*

$$H(x) \geqq \int_0^\infty H(x+y)dG(y) - C_1 e^{-\varepsilon x} H(x) , \qquad x \geqq x_0 \tag{6}$$

implies the boundedness of $H(x)$, where C_1 and ε are positive constants.

Theorem 1 is easily obtained from Theorem A if $G(x)$ is concentrated on the interval $[\alpha, \infty)$, where $\alpha > 0$. In fact by letting $x_1 > 0$ sufficiently large we can make

$$A \equiv C_1(1 - \varepsilon^{-\varepsilon\alpha} - C_1 e^{-\varepsilon(x_1+\alpha)})^{-1} > 0 .$$

Put

$$H_0(x) = H(x) + A e^{-\varepsilon x} H(x) .$$

Then for $x \geqq \max\{x_0, x_1\}$

$$\begin{aligned} \int_0^\infty H_0(x+y)dG(y) &= \int_0^\infty H(x+y)dG(y) + Ae^{-\varepsilon x} \int_0^\infty e^{-\varepsilon y} H(x+y)dG(y) \\ &\leqq H(x)\{1 + C_1 e^{-\varepsilon x} + Ae^{-\varepsilon(x+\alpha)}(1 + C_1 e^{-\varepsilon x})\} \\ &\leqq H_0(x) \end{aligned} \tag{7}$$

and the desired result follows from Theorem A. We now proceed to the general case. Let $G^{*n}(x)$ be the n-fold convolution of $G(x)$ and let $R_n(x)$ be the real function defined by

$$H(x) = \int_0^\infty H(x+y)dG^{*n}(y) + R_n(x)H(x) . \tag{8}$$

Then $R_1(x)$ satisfies, by (6)

$$R_1(x) \geqq -C_1 e^{-\varepsilon x} , \qquad x \geqq x^* = \max\{0, x_0\} . \tag{9}$$

LEMMA 1. *For any positive integer m and n, the relation*

$$R_{m+n}(x)H(x)=R_m(x)H(x)+\int_0^\infty R_n(x+y)H(x+y)dG^{*m}(y)\ , \qquad x\geqq x_0 \tag{10}$$

and the inequality

$$R_n(x)\geqq -C_1\sum_{k=0}^{n-1}(1+C_1)^k e^{-\epsilon x}\ , \qquad x\geqq x^* \tag{11}$$

hold. If the condition (9) *is replaced by*

$$|R_1(x)|\leqq C_1 e^{-\epsilon x}\ , \qquad x\geqq x^*\ , \tag{12}$$

then we have

$$|R_n(x)|\leqq C_1\sum_{k=0}^{n-1}(1+C_1)^k e^{-\epsilon x}\ , \qquad x\geqq x^*\ . \tag{13}$$

PROOF. We shall prove Lemma 1 by mathematical induction on m. Substituting $x+y'$ for x in (8) and integrating with respect to $dG(y')$ we obtain

$$\int_0^\infty H(x+y)dG(y)=\int_0^\infty H(x+y)dG^{*n+1}(y) + \int_0^\infty R_n(x+y)H(x+y)dG(y)\ . \tag{14}$$

But by (8) the left-hand side and the first term of the right are equal to $H(x)-R_1(x)H(x)$ and $H(x)-R_{n+1}(x)H(x)$, respectively, so that (14) becomes

$$R_{n+1}(x)H(x)=R_1(x)H(x)+\int_0^\infty R_n(x+y)H(x+y)dG(y)\ . \tag{15}$$

Thus (10) is true for $m=1$ and arbitrary n. Suppose it holds true for some $m\geqq 1$ and arbitrary n, then it also holds for $m+1$ and n. In fact we have from (10)

$$\int_0^\infty R_{m+n}(x+y)H(x+y)dG(y) =\int_0^\infty R_m(x+y)H(x+y)dG(y)+\int_0^\infty R_n(x+y)H(x+y)dG^{*m+1}(y)\ .$$

Using (10) again we obtain the desired result:

$$R_{m+n+1}(x)H(x)=R_{m+1}(x)H(x)+\int_0^\infty R_n(x+y)H(x+y)dG^{*m+1}(y)\ .$$

The inequality (11) is clear if $n=1$. If it is true for some positive n, then from (10) with $m=1$, we have for $x\geqq x^*$

$$R_{n+1}(x)=R_1(x)+H(x)^{-1}\int_0^\infty R_n(x+y)H(x+y)dG(y)$$
$$\geqq -C_1e^{-\varepsilon x}-C_1\sum_{k=0}^{n-1}(1+C_1)^k e^{-\varepsilon x}H(x)^{-1}\int_0^\infty H(x+y)dG(y)$$
$$\geqq -\left\{C_1+C_1\sum_{k=0}^{n-1}(1+C_1)^k(1-R_1(x)\right\}e^{-\varepsilon x}$$
$$\geqq -C_1\sum_{k=0}^{n}(1+C_1)^k e^{-\varepsilon x}\,.$$

The inequality (13) can be obtained similarly. Q.E.D.

Now let $\varepsilon'=\min\{\lambda, \varepsilon\}$ and let ξ be a positive number such that $2\geqq e^{\varepsilon'\xi}>3/2$. Then by the law of large numbers we can find a positive integer p for which $e^{\lambda\xi}G^{*p}(\xi)<1/2$. Then there is a positive number A such that

(16) $$C_2\equiv C_1\sum_{k=0}^{p-1}(1+C_1)^k e^{-\varepsilon' A}<e^{\varepsilon'\xi}-1-e^{\lambda\xi}G^{*p}(\xi)\leqq 1\,.$$

This means that

(17) $$c\equiv e^{(\lambda-\varepsilon')\xi}G^{*p}(\xi)+(1+C_2)e^{-\varepsilon'\xi}<1$$

and that

(18) $$A_1\equiv C_1\sum_{k=0}^{p-1}(1+C_1)^k\leqq e^{\varepsilon' A}\,.$$

LEMMA 2. *The condition* (9) *implies*

(19) $$R_n(x)\geqq -C_0e^{-\varepsilon x}\,,\qquad x\geqq x^{**}\equiv\max\{0, x_0, A\},\ n=1, 2,\cdots,$$

where C_0 is a positive constant which may depend on C_1, λ, ε and $G(x)$, but not on n. Also (12) *implies*

(20) $$|R_n(x)|\leqq C_0e^{-\varepsilon x}\,,\qquad x\geqq x^{**},\ n=1, 2,\cdots.$$

PROOF. We shall first prove by mathematical induction that

(21) $$R_{mp}(x)\geqq -A_m e^{-\varepsilon x}\qquad x\geqq x^{**},\ m=1, 2,\cdots,$$

where

$$A_m=A_1\sum_{k=0}^{m-1}c^k\leqq e^{\varepsilon' A}(1-c)^{-1}\,.$$

For $m=1$, (21) follows from (11) and the definition (18) of A_1. Suppose (21) holds for $m\geqq 1$. Then from (10)

$$R_{(m+1)p}(x)H(x)=R_p(x)H(x)+\int_0^\infty R_{mp}(x+y)H(x+y)dG^{*p}(y)$$

$$\geqq -A_1 e^{-\varepsilon x} H(x) - A_m e^{-\varepsilon x} \int_0^\infty e^{-\varepsilon y} H(x+y) dG^{*p}(y) .$$

But as

$$\int_0^\infty e^{-\varepsilon y} H(x+y) dG^{*p}(y)$$
$$\leqq H(x) \int_0^\xi e^{(\lambda-\varepsilon')y} dG^{*p}(y) + \int_\xi^\infty e^{-\varepsilon' y} H(x+y) dG^{*p}(y)$$
$$\leqq H(x) \{ e^{(\lambda-\varepsilon')\xi} G^{*p}(\xi) + e^{-\varepsilon'\xi}(1-R_p(x)) \} \leqq cH(x) ,$$

we have

$$R_{(m+1)p}(x) \geqq -(A_1 + cA_m) e^{-\varepsilon x} = -A_{m+1} e^{-\varepsilon x} .$$

Thus we have proved (21). Let n be a positive integer which is not a multiple of p. If $n<p$, then (11) and (18) yield

$$R_n(x) \geqq -A_1 e^{-\varepsilon x} , \qquad x \geqq x^{**} . \tag{22}$$

If $n>p$, then there exist positive integers k and l such that

$$n = kp + l , \qquad 1 \leqq l < p .$$

It follows from (10)

$$R_n(x) = R_{kp+l}(x) = R_l(x) + H(x)^{-1} \int_0^\infty R_{kp}(x+y) H(x+y) dG^{*l}(y)$$
$$\geqq -e^{-\varepsilon x}\{A_1 + A_k(1-R_l(x))\} \geqq -\{A_1 + A_k(1+A_1)\} e^{-\varepsilon x} .$$

Thus for all cases (19) holds with

$$C_0 = e^{\varepsilon' A} \left\{ 1 + \frac{1}{1-c}(1+e^{\varepsilon' A}) \right\} .$$

The implication (12)→(20) is obtained similarly. Q.E.D.

PROOF OF THEOREM 1. Let C_3 be any positive number greater than C_0 and let ε'' and η be positive numbers such that

$$C_3 \geqq C_0 + \varepsilon'' C_3 \qquad \text{and} \qquad e^{-\varepsilon'\eta}(1+C_0) < \varepsilon''/2 .$$

We can find a positive integer n such that

$$e^{(\lambda-\varepsilon')\eta} G^{*n}(\eta) < \varepsilon''/2 .$$

For $x \geqq x^{**}$, put

$$H_0(x) = H(x) + C_3 H(x) e^{-\varepsilon x} .$$

Then,

$$\int_0^\infty H_0(x+y)dG^{*n}(y)=\int_0^\infty H(x+y)dG^{*n}(y)+C_3e^{-\epsilon x}\int_0^\infty e^{-\epsilon y}H(x+y)dG^{*n}(y) .$$

But as

$$0\leqq\int_0^\infty H(x+y)dG^{*n}(y)=H(x)\{1-R_n(x)\}\leqq H(x)\{1+C_0e^{-\epsilon x}\}$$

and

$$\begin{aligned}&\int_0^\infty e^{-\epsilon y}H(x+y)dG^{*n}(y)\\ &\qquad\leqq\int_0^\eta e^{-\epsilon' y}H(x+y)dG^{*n}(y)+\int_\eta^\infty e^{-\epsilon' y}H(x+y)dG^{*n}(y)\\ &\qquad\quad\leqq e^{(\lambda-\epsilon')\eta}G^{*n}(\eta)H(x)+e^{-\epsilon'\eta}\{1-R_n(x)\}H(x)\leqq\epsilon''H(x) ,\end{aligned}$$

it follows that

$$\int_0^\infty H_0(x+y)dG^{*n}(y)\leqq H(x)+C_0e^{-\epsilon x}H(x)+\epsilon''C_3e^{-\epsilon x}H(x)\leqq H_0(x) , \tag{23}$$

and the desired result follows from Theorem A. Q.E.D.

3. Solution of the equation with an error term

In this section we assume that $H(x)$ is a real, bounded and right continuous function defined for $x\geqq x_0$. For $k=1$ and 2, let $G_k(x)$ be monotone non-decreasing functions with the set Ω_k of points of increase, i.e., $u\in\Omega_k$ means $G_k(u+d)>G_k(u-d)$ for any $d>0$. We assume that $G(x)\equiv G_1(x)+G_2(x)$ is a distribution function on $[0, \infty)$ not degenerate at $x=0$. Let $S_0(x)$ be a real function such that

$$|S_0(x)|\leqq Ce^{-\epsilon x} , \qquad x\geqq x_0 , \tag{24}$$

where ϵ and C are positive constants. Put

$$c=\int_0^\infty e^{-\epsilon x}dG(x) \qquad (0<c<1) . \tag{25}$$

We shall prove

THEOREM 2. *If $H(x)$ satisfies the functional equation*

$$H(x)=q\int_0^\infty H(x+y)d(G_1(y)-G_2(y))+S_0(x) \tag{26}$$

where $1\geqq q>0$, then for all $x\geqq x_0$ and $u>0$

$$|H(x+u)-H(x)|\leqq C_0e^{-\epsilon x} , \qquad \textit{if } u\in\Omega_1 \tag{27}$$

and

(28) $$|H(x+u)+H(x)|\leqq C_0e^{-\epsilon x}, \qquad \text{if } u\in\Omega_2$$

where

$$C_0=2C/(1-cq) .$$

Let r be an arbitrary positive number and let $f(x)$ be the continuously differentiable density function of a distribution concentrated on the closed interval $[0, r]$. Put

$$\bar{H}(x)\equiv\int_0^r H(x+y)f(y)dy , \qquad x\geqq x_0 .$$

Let $u\in\Omega_1$ and put

$$K(x)\equiv\bar{H}(x+u)-\bar{H}(x)$$

and

$$S(x)\equiv\int_0^r \{S_0(x+u+y)-S_0(x+y)\}f(y)dy .$$

Then there exists a positive constant C_1 which may depend on r and $f(\cdot)$ and such that

(29) $$|\bar{H}(x)-\bar{H}(x')|\leqq C_1|x-x'| , \qquad x, x'\geqq x_0$$

and

(30) $$|K(x)|\leqq C_1 , \quad |H(x)|\leqq C_1 , \qquad x\geqq x_0 .$$

Also we have

(31) $$|S(x)|\leqq 2Ce^{-\epsilon x} , \qquad x\geqq x_0 .$$

Moreover $K(x)$ satisfies the equation

(32) $$K(x)=q\int_0^\infty K(x+y)d(G_1(y)-G_2(y))+S(x) .$$

In particular we have

(33) $$|K(x)|\leqq q\int_0^\infty |K(x+y)|dG(y)+2Ce^{-\epsilon x} .$$

It then follows by mathematical induction that the inequalities

(34) $$|K(x)|\leqq q^n\int_0^\infty |K(x+y)|dG^{*n}(y)+2C(1+cq+\cdots+(cq)^{n-1})e^{-\epsilon x} ,$$

$n=1, 2,\cdots,$ hold.

LEMMA 3.

$$\lim_{n\to\infty} G^{*n}(\sqrt{n})=0 . \tag{35}$$

PROOF. Let a be a positive number such that $G(a)>0$. Let X_1, $X_2,\cdots$ be independent and identically distributed random variables with the common distribution G and let X_n' be defined by

$$X_n'=\begin{cases} X_n & \text{if } X_n\leqq a \\ a & \text{if } X_n>a . \end{cases}$$

Then $X_1', X_2',\cdots$ are also independent and identically distributed and they have a positive mean and a finite variance. As $0\leqq X_n'\leqq X_n$, $n=1, 2,\cdots$ with probability one, we have by the law of large numbers

$$\begin{aligned} 0&\leqq G^{*n}(\sqrt{n})=\Pr\{X_1+X_2+\cdots+X_n\leqq\sqrt{n}\} \\ &\leqq \Pr\{X_1'+X_2'+\cdots+X_n'\leqq\sqrt{n}\}\to 0 . \end{aligned}$$

Q.E.D.

LEMMA 4. *If* $a=\overline{\lim_{x\to\infty}}|K(x)|$ *then*

$$|K(x)|\leqq a+C_0e^{-\epsilon x} , \qquad x\geqq x_0 . \tag{36}$$

PROOF. For any $\varepsilon_1>0$, we can find an $x_1>x_0$ such that $|K(x)|\leqq a+\varepsilon_1$, for all $x>x_1$. Now let $x>x_0$ and take n sufficiently large so that $x+\sqrt{n}\geqq x_1$. Then

$$\begin{aligned} |K(x)|&\leqq q^n\int_0^\infty |K(x+y)|dG^{*n}(y)+C_0e^{-\epsilon x} \\ &\leqq\int_0^{\sqrt{n}} |K(x+y)|dG^{*n}(y)+\int_{\sqrt{n}}^\infty |K(x+y)|dG^{*n}(y)+C_0e^{-\epsilon x} \\ &\leqq C_1G^{*n}(\sqrt{n})+(a+\varepsilon_1)(1-G^{*n}(\sqrt{n}))+C_0e^{-\epsilon x} . \end{aligned}$$

Letting $n\to\infty$

$$|K(x)|\leqq a+\varepsilon_1+C_0e^{-\epsilon x} .$$

As ε_1 is arbitrary Lemma 4 follows. Q.E.D.

LEMMA 5. $a=0$.

PROOF. We assume without loss of generality that

$$a=\overline{\lim}\, K(x)\geqq -\underline{\lim}\, K(x) .$$

If $a>0$, then we could find positive numbers ε_1 and δ, and a positive integer L such that

$$a>3\varepsilon_1>0 , \quad \varepsilon_1>C_1\delta>0 , \quad u>\delta>0 \quad \text{and} \quad L(a-3\varepsilon_1)\geqq 3C_1 .$$

Let A be the closed interval $[u-\delta, u+\delta]$ and $\bar{A}$ its complement. Put

$$\eta=q\int_A dG_1(x) \qquad (>0, \text{ as } u\in\Omega_1 \text{ and } q>0)\,.$$

Then for $x\geqq x_0$,

$$\begin{aligned} K(x)&=q\int_0^\infty K(x+y)d(G_1(y)-G_2(y))+S(x)\\ &\leqq q\int_A K(x+y)dG_1(y)+q\int_{\bar{A}}|K(x+y)|dG_1(y)\\ &\quad+q\int_0^\infty |K(x+y)|dG_2(y)+S(x)\\ &\leqq \eta\sup_{y\in A}K(x+y)+(1-\eta)(a+C_0e^{-\varepsilon x})+2Ce^{-\varepsilon x}\end{aligned}$$

or

(37) $$K(x)\leqq \eta K(x+u_1)+a(1-\eta)+Be^{-\varepsilon x}\,,$$

where $u_1\in A$, and $B=(1-\eta)C_0+2C$. In the same way we obtain

(38) $$K(x+u_1)\leqq\eta K(x+u_1+u_2)+a(1-\eta)+Be^{-\varepsilon(x+u_1)}$$

where $u_2\in A$. Substituting (38) into (37) we get

$$K(x)\leqq\eta^2K(x+u_1+u_2)+a(1-\eta^2)+B(1+\eta)e^{-\varepsilon x}\,.$$

We may repeat this process to obtain

(39) $$K(x)\leqq\eta^kK(x+u_1+\cdots+u_k)+a(1-\eta^k)+B(1+\eta+\cdots+\eta^{k-1})e^{-\varepsilon x}$$

for $x\geqq x_0$ and $k=1,2,\cdots$, where u's lie in the closed interval $A=[u-\delta, u+\delta]$. Now take x^* ($>x_0$) large enough so that

(40) $$Be^{-\varepsilon x^*}\sum_{k=1}^{L}(\eta^{-1}+\eta^{-2}+\cdots+\eta^{-k})\leqq C_1\,.$$

By the definition of a we can find an x_1 ($\geqq x^*$) which satisfies

(41) $$a-\varepsilon_1\eta^L\leqq K(x_1)\,.$$

Inequalities (39) and (41) yield

(42) $$a-\varepsilon_1\eta^{L-k}\leqq K(x_1+u_1+\cdots+u_k)+Be^{-\varepsilon x_1}(\eta^{-1}+\cdots+\eta^{-k})\,.$$

Adding both sides of (42) for $k=1,2,\cdots,L$, we arrive at a contradiction:

$$\begin{aligned} L(a-\varepsilon_1)&\leqq\sum_{k=1}^{L}(a-\varepsilon_1\eta^{L-k})\leqq\sum_{k=1}^{L}K(x_1+u_1+\cdots+u_k)+C_1\\ &=\sum_{k=2}^{L}\{\bar{H}(x_1+u_1+\cdots+u_{k-1}+u)-\bar{H}(x_1+u_1+\cdots+u_k)\}\\ &\quad+\bar{H}(x_1+u_1+\cdots+u_L+u)-\bar{H}(x_1+u_1)+C_1\\ &\leqq(L-1)\delta C_1+3C_1\leqq L(a-2\varepsilon_1)\,.\end{aligned}$$

We conclude that $a=0$. Q.E.D.

PROOF OF THEOREM 2. Lemmas 4 and 5 yield

$$(43)\qquad |K(x)|=\left|\int_0^r \{H(x+u+y)-H(x+y)\}f(y)dy\right|\leqq C_0e^{-\varepsilon x}, \qquad x\geqq x_0 .$$

As r is arbitrary and as $H(x)$ is right continuous (27) follows from (43). In order to derive (28) define the bounded function $K(x)$ by $K(x) \equiv \bar{H}(x+u)+\bar{H}(x)$. We can use the similar argument as above. Q.E.D.

4. Explicit formulae for $H(x)$

Results of the preceding section make it possible to obtain the explicit formulae for $H(x)$.

LEMMA 6. *Let $u\in\Omega_1\cup\Omega_2$, and $u>0$, then there exists a periodic function $\Delta_u(x)$ such that*

$$(44)\qquad \Delta_u(x+u)=\begin{cases}\Delta_u(x) & \text{if } u\in\Omega_1\\ -\Delta_u(x) & \text{if } u\in\Omega_2\end{cases}$$

for all real x and that

$$(45)\qquad |H(x)-\Delta_u(x)|\leqq C_ue^{-\varepsilon x}, \qquad x\geqq x_0 ,$$

where $C_u=C_0(1-e^{-\varepsilon u})^{-1}$.

PROOF. Suppose $u\in\Omega_1$. Then it follows from Theorem 2 that

$$(46)\qquad |H(x+lu)-H(x)|\leqq C_0\sum_{k=0}^{l-1} e^{-\varepsilon ku}e^{-\varepsilon x} \qquad x\geqq x_0,\ l=1, 2,\cdots .$$

In particular $\lim_{l\to\infty} H(x+lu)$ exists for all $x\geqq x_0$ and the inequality (45) is satisfied by the periodic function defined by

$$(47)\qquad \Delta_u(x)=\lim_{l\to\infty} H(x+lu) .$$

When $u\in\Omega_2$ the inequality (46) holds true for $l=2, 4,\cdots$ only, which easily follows from (28) of Theorem 2. The conditions (44) and (45) are satisfied in this case by

$$(48)\qquad \Delta_u(x)=\lim_{l\to\infty} H(x+2lu) . \qquad \text{Q.E.D.}$$

In what follows we distinguish three cases.

Case 1. $\Omega_1\cup\Omega_2$ is not contained in $K(\rho)=\{l\rho\,|\,l=0, 1,\cdots\}$ for any $\rho>0$.

In other cases there exists a unique $\rho>0$ such that $\Omega_1\cup\Omega_2$ is contained in $K(\rho)$ but not in $K(\rho')$ for any $\rho'>\rho$.

Case 2. Either Ω_2 is empty, or Ω_1 is not disjoint from $L(\rho)=\{(2l+1)\rho\,|\, l=0, 1,\cdots\}$, or Ω_2 is not disjoint from $K(2\rho)$.

Case 3. Either Ω_1 is empty, or Ω_1 is contained in $K(2\rho)$ and Ω_2 is in $L(\rho)$.

Now we can state our main theorem.

THEOREM 3. *Suppose the assumptions stated at the beginning of the preceding section are satisfied. If $H(x)$ is a solution of* (26), *then it can be put in the form*

$$H(x)=\Delta(x)+A(x)e^{-\varepsilon x}, \qquad x\geqq x_0, \tag{49}$$

where $A(x)$ is a bounded function and $\Delta(x)$ is a periodic function specified as follows:
Case 1. *$\Delta(x)\equiv\Delta$ is a constant. $\Delta=0$ if $\Omega_2\neq\phi$.*
Case 2. *$\Delta(x)$ is a periodic function with period ρ. $\Delta(x)\equiv 0$ if $\Omega_2\neq\phi$.*
Case 3. *$\Delta(x)$ is a periodic function with period 2ρ and $\Delta(x+\rho)=-\Delta(x)$, for all x.*
For all cases $\Delta(x)\equiv 0$ if $q<1$, and $A(x)$ is bounded by

$$|A(x)|\leqq\frac{C}{1-cq}. \tag{50}$$

PROOF. We can reduce the problem to the case $G_1(0)=G_2(0)=0$. In particular if $G_2(0)>0$ we can rewrite the equation (26) to obtain a similar equation with $G_2(0)=0$ and $q<1$. We may assume therefore $0\notin\Omega_1\cup\Omega_2$ in the cases 2 and 3. Apart from the inequality (50) the assertions for these cases are direct consequences of Lemma 6. Consider the case 1. There exist as least two positive numbers u and v, say, in $\Omega_1\cup\Omega_2$ such that the ratio u/v is an irrational number. Then we have from Theorem 2

$$|H(x)-\Delta_u(x)|\leqq C_u e^{-\varepsilon x} \quad\text{and}\quad |H(x)-\Delta_v(x)|\leqq C_v e^{-\varepsilon x},$$

where Δ's are periodic functions. It follows that

$$\Delta_u(x)=\Delta_v(x)+C(x)e^{-\varepsilon x} \tag{51}$$

where $C(x)$ is a bounded function. Let m be a positive integer and substitute $x+2mu$ in (51). Noting that $\Delta_u(x)$ has period $2u$ we obtain

$$\Delta_u(x)=\Delta_v(x+2mu)+C(x+2mu)e^{-2\varepsilon mu}e^{-\varepsilon x}. \tag{52}$$

It follows that

$$\Delta_u(x)=\lim_{m\to\infty}\Delta_v(x+2mu),$$

which implies that $\Delta_u(x)$ has period $2v$. But as u/v is irrational this is possible only if $\Delta_u(x)$ is a constant. If $w\in\Omega_2$, then we have $\Delta=$constant$=\Delta_w(x)=\Delta_w(x+w)=-\Delta_w(x)=-\Delta$, and we conclude that $\Delta=0$. Suppose $q<1$. We have from (24) and (25)

$$|H(x)|\leqq q\int_0^\infty |H(x+y)|dG(y)+Ce^{-\epsilon x},$$

or more generally

$$|H(x)|\leqq q^n\int_0^\infty |H(x+y)|dG^{*n}(y)+C(1+cq+\cdots+(cq)^{n-1})e^{-\epsilon x},\tag{53}$$

for $n=1, 2, \cdots$. As H is bounded the first term of the right-hand side of (53) goes to 0 as n tends to infinity while the second term is bounded by $(C/(1-cq))e^{-\epsilon x}$. The inequality (50) is easily obtained by substituting the expression (49) in (26). Q.E.D.

COROLLARY. *Let $H(x)$ be a non-negative and right continuous function satisfying the condition* (4) *and let $G(x)$ be a distribution function satisfying* (3). *Let c be given by* (25). *If $H(x)$ is a solution to the functional equation*

$$H(x)=\int_0^\infty H(x+y)dG(y)+R(x)H(x),\qquad x\geqq x_0=0,\tag{54}$$

where $R(x)$ is a real function such that $|R(x)|\leqq R_0e^{-\epsilon x}\leqq((1-c)/4)e^{-\epsilon x}$, then $H(x)$ can be put in the form

$$H(x)=\Delta+A(x)e^{-\epsilon x},\qquad x\geqq 0,\tag{55}$$

where Δ is a non-negative constant and $A(x)$ is bounded by

$$|A(x)|\leqq\frac{2}{1-c}R_0\inf|H(x)|,\qquad x\geqq 0.\tag{56}$$

PROOF. If $H(x_1)=0$ for some $x_1>0$ then $H(x)=0$ for all $x\geqq x_1$ by the condition (4) so that $H(x)$ is bounded. If $H(x)$ is positive then the boundedness follows from Theorem 1. We can then apply Theorem 3 to conclude the $H(x)$ can be put in the form (55). Substituting this expression in (55) we obtain

$$A(x)e^{-\epsilon x}=e^{-\epsilon x}\int_0^\infty A(x+y)e^{-\epsilon y}dG(y)+R(x)(\Delta+A(x)e^{-\epsilon x}),\qquad x\geqq 0.$$

It follows that

$$|A(x)|\leqq c\sup_{x\geqq 0}|A(x)|+R_0\Delta+\frac{1-c}{4}\sup_{x\geqq 0}|A(x)|,\qquad x\geqq 0$$

or

$$\sup_{x\geqq 0}|A(x)|\leqq\frac{4}{3(1-c)}R_0\Delta.\tag{57}$$

On the other hand we have

$$|H(x)|\geqq \Delta-\sup_{x\geqq 0}|A(x)|\geqq\left(1-\frac{4}{3(1-c)}R_0\right)\Delta\geqq\frac{2}{3}\Delta .$$

The inequality (56) follows from this and (57). Q.E.D.

5. Stability of some characterizations of the exponential distribution

Among the continuous distributions on the half interval $(0, \infty)$, the exponential distribution $F(x)=1-e^{-\lambda x}$ $(x\geqq 0)$ has some interesting characteristic properties. Following characterization theorems are known. In Theorems B and C, $X_{k,n}$ denotes the kth smallest observation in a sample $X_1, X_2, \cdots, X_n$ of size n from a distribution F such that $F(0)=0$.

THEOREM B (Ferguson [1]). *Let F be a continuous distribution with a finite mean. If for some k $(1\leqq k<n)$, the conditional expectation of the variable $X_{k+1,n}-X_{k,n}$ given $X_{k,n}=x$ remains constant a.s., then the distribution F is exponential.*

THEOREM C (Rossberg [7]). *If for some k $(1\leqq k<n)$ the variable $X_{k+1,n}-X_{k,n}$ has the same distribution as the smallest of $Y_1, Y_2, \cdots, Y_{n-k}$, a sample of size $n-k$ from F, then F is exponential.*

THEOREM D (Ramachandran [6]-Huang [3]-Shimizu [8]. For related theorems see also [4], [5]). *Let X and Y be independent non-negative random variables such that*

(58) $$\Pr\{Y=0\}<C\equiv\Pr\{X>Y\}<1 ,$$

and that the distribution G_0 of Y is non-lattice. If for all $x\geqq 0$ the relation

(59) $$\Pr\{X>Y+x\,|\,X>Y\}=\Pr\{X>x\}$$

holds then the distribution F of X is exponential.

In this section we are concerned with generalizing these theorems to the cases where the assumptions are not fully satisfied to obtain so-called stability theorems. We measure the distance between two distributions F_1 and F_2 on the half interval $(0, \infty)$ by

$$\Theta(F_1, F_2:\tau)\equiv\sup_{x\geqq 0}|F_1(x)-F_2(x)|e^{\tau x} ,$$

where τ is a positive constant.

In what follows we write $E_\lambda(x)$ to mean the exponential distribution function:

$$E_\lambda(x)=\begin{cases} 0 & \text{if } x<0 \\ 1-e^{-\lambda x} & \text{if } x\geqq 0 \end{cases}$$

and we assume that the real function $R(x)$ is such that

$$|R(x)|\leqq R_0 e^{-\varepsilon x}\,, \qquad \text{for } x\geqq 0\,, \tag{60}$$

where R_0 and ε are positive constants. We shall prove the following stability theorems corresponding to Theorems B, C and D.

THEOREM 4. *Let F be continuous and strictly monotone in the interval $(0, b)$, where*

$$a\equiv \inf\{x\,|\,F(x)>0\}=0 \qquad and \qquad b\equiv \sup\{x\,|\,F(x)<1\}\,.$$

If for some $1\leqq k<n$ there exists a positive constant λ such that

$$\mathrm{E}\,(X_{k+1,n}-X_{k,n}\,|\,X_{k,n}=x)=\lambda^{-1}(1-R(x))\,, \qquad a.s.\,, \tag{61}$$

where $R(x)$ satisfies (60) with

$$R_0\leqq \frac{n-k}{4(1+(n-k)\delta)}\cdot\frac{\varepsilon\delta}{\lambda+\varepsilon}\,, \qquad 0\leqq\delta<1\,, \tag{62}$$

then

$$\Theta(F, E_{\lambda'}:\lambda')\leqq\delta\,, \qquad \lambda'=\lambda/(n-k)\,. \tag{63}$$

THEOREM 5. *Let F be a non-lattice distribution such that $F(0)=0$. Suppose for some $1\leqq k<n$ the relation*

$$\Pr\{X_{k+1,n}-X_{k,n}>x\}=(1-F(x))^{n-k}(1-R(x))\,, \qquad x\geqq 0 \tag{64}$$

holds. Let λ be the unique solution of

$$\binom{n}{k}\int_0^\infty e^{-\lambda(n-k)k}dF^k(x)=1$$

and put

$$c=\binom{n}{k}\int_0^\infty e^{-(\lambda(n-k)+\varepsilon)x}dF^k(x)\,.$$

Then for any $0\leqq\delta<1$,

$$\Theta(F, E_\lambda:\lambda)\leqq\delta \tag{65}$$

provided that

$$|R_0|\leqq\frac{(n-k)(1-c)\delta}{4(1+(n-k)\delta)}\,. \tag{66}$$

THEOREM 6. *Suppose the assumptions of Theorem* D *are satisfied. If there exists a real function* $R(x)$ *satisfying the condition* (60) *such that*

$$\Pr\{X>Y+x\,|\,X>Y\}=\Pr\{X>x\}(1-R(x)), \qquad x\geqq 0, \tag{67}$$

then we have

$$\Theta(F, E_\lambda : \lambda)\leqq\delta, \qquad 0\leqq\delta<1 \tag{68}$$

whenever $R_0\leqq((1-c)/4)\delta$, *where* λ *and* c *are given by*

$$\int_0^\infty e^{-\lambda x}dG_0(x)=C\equiv\Pr\{X>Y\} \qquad \text{and} \qquad c=C^{-1}\int_0^\infty e^{-(\lambda+\varepsilon)x}dG_0(x).$$

PROOF OF THEOREM 4. The continuous version of the conditional expectation $E(X_{k+1,n}\,|\,X_{k,n}=x)$ is given by

$$-\int_x^\infty y d_y\left(\frac{1-F(y)}{1-F(x)}\right)^{n-k}$$

and we have from the condition (61)

$$-\int_x^\infty y d_y\left(\frac{1-F(y)}{1-F(x)}\right)^{n-k}-x=\lambda^{-1}(1-R(x)), \qquad \text{a.s.}$$

In view of the continuity and monotonicity of F we have for $0\leqq x<b$

$$\lambda\int_0^\infty(1-F(x+y))^{n-k}dy=(1-F(x))^{n-k}(1-R(x)). \tag{69}$$

But if $x\geqq b$ then the both sides of (69) are equal to 0 and we conclude that it holds in fact for all $x\geqq 0$. On introducing the non-negative function $H(x)=(1-F(x))^{n-k}e^{\lambda x}$ and the distribution $dG(x)=\lambda e^{-\lambda x}dx$, we obtain from (69) the functional equation (54). The assumptions of the corollary to Theorem 3 are satisfied with $c=\lambda/(\lambda+\varepsilon)$. Thus we can write $F(x)=1-(\Delta+A(x)e^{-\varepsilon x})^{1/(n-k)}e^{-\lambda' x}$, where $\lambda'=\lambda/(n-k)$, Δ is a constant and $A(x)$ satisfies the inequality (56). But as $\inf H(x)\leqq H(0)=1=\Delta+A(0)$, we have

$$F(x)=E_{\lambda'}(x)+B(x)e^{-\lambda' x},$$

where

$$B(x)=1-(\Delta+A(x)e^{-\varepsilon x})^{1/(n-k)}$$

and

$$|B(x)|\leqq|1-(1+A(x)e^{-\varepsilon x}-A(0))^{1/(n-k)}|$$
$$\leqq\frac{2\sup A(x)}{(n-k)(1-2\sup A(x))}\leqq\frac{4R_0}{(n-k)(1-c-4R_0)}\leqq\delta. \qquad \text{Q.E.D.}$$

PROOF OF THEOREM 5. We have

$$\Pr\{X_{k+1,n}-X_{k,n}>x\}=\int_0^\infty \Pr\{X_{k+1,n}>x+y\mid X_{k,n}=y\}d\Pr\{X_{k,n}<y\}$$
$$=\int_0^\infty\left(\frac{1-F(x+y)}{1-F(y)}\right)^{n-k}\binom{n}{k}(1-F(y))^{n-k}dF^k(y)$$
$$=\binom{n}{k}\int_0^\infty(1-F(x+y))^{n-k}dF^k(y)\,.$$

Put $H(x)=(1-F(x))^{n-k}e^{\lambda(n-k)x}$ and let $G(x)$ be the distribution defined by $dG(x)=\binom{n}{k}e^{-\lambda(n-k)x}dF^k(x)$. Then the equation (64) becomes (54) and all the conditions of the corollary to Theorem 3 are satisfied. The rest of the proof is the same as the preceding theorem. Q.E.D.

PROOF OF THEOREM 6. The condition (66) of the theorem is equivalent to

$$\int_0^\infty(1-F(x+y))dG_0(y)=C(1-F(x))(1-R(x))\,,\qquad x\geqq 0\,.$$

Writing $H(x)=(1-F(x))e^{\lambda x}$ and $dG(x)=C^{-1}e^{-\lambda x}dG_0(x)$ this becomes (54). The rest of the proof is the same as the proofs of the preceding two theorems. We omit the detail. Q.E.D.

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