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INVARIANT POLYNOMIALS WITH TWO MATRIX ARGUMENTS EXTENDING THE ZONAL POLYNOMIALS: APPLICATIONS TO MULTIVARIATE DISTRIBUTION THEORY

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1. Introduction

In a recent article [6] the author has defined a class of polynomials $C_\phi^{\kappa,\lambda}(X, Y)$ in the elements of the $m\times m$ symmetric complex matrices X and Y, having the property of invariance under the *simultaneous* transformations

$$(1.1)\qquad X\to H'XH\,,\quad Y\to H'YH\,,\qquad H\in O(m)\,,$$

where $O(m)$ is the group of $m\times m$ orthogonal matrices. These satisfy the basic relationship

$$(1.2)\qquad \int_{O(m)} C_\kappa(AH'XH)C_\lambda(BH'YH)dH=\sum_{\phi\in\kappa.\lambda} C_\phi^{\kappa,\lambda}(A, B)C_\phi^{\kappa,\lambda}(X, Y)/C_\phi(I)\,,$$

where C_κ, C_λ, C_ϕ are zonal polynomials (James [12]), indexed by the ordered partitions κ, λ, ϕ of the nonnegative integers k, l, $f=k+l$ respectively into $\leqq m$ parts. Letting $Gl(m, R)$ denote the group of $m\times m$ real nonsingular matrices, "$\phi\in\kappa.\lambda$" signifies that the irreducible representation of $Gl(m, R)$ indexed by 2ϕ occurs in the decomposition of the Kronecker product $2\kappa\otimes 2\lambda$ of the irreducible representations indexed by 2κ and 2λ. Irreducible representations of $Gl(m, R)$ indexed by ordered partitions of the form 2κ are fundamental to the theory of zonal polynomials. Further properties of the $C_\phi^{\kappa,\lambda}$ are summarized in Section 2.

The present paper applies the polynomials to some problems in multivariate normal distribution theory. The joint distribution of the latent roots of the noncentral Wishart matrix $S\sim W_m(n, \Sigma, \Omega)$ was presented in [6]. In Section 3 expansions are given of multivariate incomplete gamma and beta functions which are relevant to the cumulative distribution functions (c.d.f.'s) of the noncentral Wishart and MANOVA matrices (cf. Constantine [2], p. 1270). A further application is to the noncentral quadratic form (Section 7), since it is shown in Section 4 that certain polynomials in two matrices defined by Hayakawa

[8] and Khatri [13] are expressible in terms of the $C_\phi^{\kappa,\lambda}$. A complete orthogonal set of Laguerre polynomials with two matrix arguments is constructed in Section 5, following the approach of Herz [11] and Constantine [3]. Section 6 presents a number of expansions generally based on the following corollary to (1.2)

$$(1.3)\qquad \int_{O(m)} \operatorname{etr}(AH'XH+BH'YH)dH = \sum_{\kappa,\lambda;\phi}^{\infty} C_\phi^{\kappa,\lambda}(A, B)C_\phi^{\kappa,\lambda}(X, Y)/k!l!C_\phi(I) ,$$

where the summation on the right denotes $\sum_{k,l=0}^{\infty} \sum_{\kappa,\lambda;\phi\in\kappa.\lambda}$. Results for noncentral F in the case of unequal covariance matrices (Pillai [14], Pillai and Sudjana [15]) are obtained in Section 8, and finally the distribution of doubly noncentral multivariate F with equal covariance matrices is derived in Section 9. The latter will be applied in a subsequent paper to consider the effects of moderate nonnormality on the MANOVA tests, following the approach of Davis [5]; this objective constituted the original stimulus for the present investigation.

2. Basic properties of the invariant polynomials $C_\phi^{\kappa,\lambda}(X, Y)$

Proofs of the following results are indicated in [6]. It should be noted that a representation 2ϕ may occur in (1.2) with multiplicity greater than one, so that strictly an additional subscript is required, but we shall omit this for notational convenience, and assume that each ϕ is given the required multiplicity. Furthermore, in such cases the polynomials $C_\phi^{\kappa,\lambda}$ are not uniquely defined, but it is sufficient that they be "orthogonal" in a sense defined in [6] for (1.2) and other basic properties to hold. This non-uniqueness first occurs for polynomials of degree 6, when if $\kappa, \lambda=[2, 1]$, $\phi=[3, 2, 1]$ occurs with multiplicity 3.

$$(2.1)\qquad C_\phi^{\kappa,\lambda}(X, X)=\theta_\phi^{\kappa,\lambda}C_\phi(X) , \qquad \text{where } \theta_\phi^{\kappa,\lambda}=C_\phi^{\kappa,\lambda}(I, I)/C_\phi(I)$$

may be zero.

$$(2.2)\qquad C_\phi^{\kappa,\lambda}(X, I)=\{\theta_\phi^{\kappa,\lambda}C_\phi(I)/C_\kappa(I)\}C_\kappa(X) ,$$

with a corresponding result for $C_\phi^{\kappa,\lambda}(I, Y)$.

$$(2.3)\qquad C_\kappa^{\kappa,0}(X, Y)\overset{\text{def}}{=}C_\kappa(X) , \qquad C_\lambda^{0,\lambda}(X, Y)\overset{\text{def}}{=}C_\lambda(Y) .$$

$$(2.4)\qquad \int_{O(m)} C_\phi^{\kappa,\lambda}(AH'XH, AH'YH)dH=C_\phi^{\kappa,\lambda}(X, Y)C_\phi(A)/C_\phi(I) .$$

If $W\sim W_m(n, \Sigma, 0)$ then

(2.5) $\quad \mathrm{E}_W\, C_\phi^{\kappa,\lambda}(XW, YW)=2^f(n/2)_\phi C_\phi^{\kappa,\lambda}(X\Sigma, Y\Sigma) \qquad (f=k+l)\,,$

(2.6) $\quad \mathrm{E}_W\,\{C_\kappa(XW)C_\lambda(YW)\}=\sum_{\phi\in\kappa.\lambda} 2^f(n/2)_\phi \theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(X\Sigma, Y\Sigma)\,,$

where we note that $(n/2)_\phi$ is constant for equivalent representations 2ϕ.

(2.7) $\quad C_\phi^{\kappa,\lambda}(\alpha X, \beta Y)=\alpha^k\beta^l C_\phi^{\kappa,\lambda}(X, Y)\,, \qquad (\alpha,\ \beta$ complex constants).

The following are consequences of (1.2):

(2.8) $\quad C_\kappa(X)C_\lambda(Y)=\sum_{\phi\in\kappa.\lambda} \theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(X, Y)\,,$

(2.9) $\quad (\mathrm{tr}\, X)^k(\mathrm{tr}\, Y)^l=\sum_{\kappa,\lambda;\phi\in\kappa.\lambda} \theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(X, Y)\,,$

(2.10) $\quad C_\kappa(X)C_\lambda(X)=\sum_{\phi^*\in\kappa.\lambda} g_{\kappa,\lambda}^{\phi^*} C_{\phi^*}(X)\,, \qquad g_{\kappa,\lambda}^{\phi^*}=\sum_{\phi\equiv\phi^*} (\theta_\phi^{\kappa,\lambda})^2\,,$

where $\sum_{\phi^*\in\kappa.\lambda}$ implies that we sum over the inequivalent representations $2\phi^*$ occurring in $2\kappa\otimes 2\lambda$, and $\sum_{\phi\equiv\phi^*}$ denotes summation over the representations equivalent to $2\phi^*$ in $2\kappa\otimes 2\lambda$.

(2.11) $\quad \int_{O(m)} C_\phi^{\kappa,\lambda}(A'H'XHA, B)dH=C_\phi^{\kappa,\lambda}(A'A, B)C_\kappa(X)/C_\kappa(I)\,,$

with a corresponding result for $C_\phi^{\kappa,\lambda}(A, B'H'YHB)$.

Laplace transform:

(2.12) $$\int_{R>0} \mathrm{etr}\,(-RW)|R|^{t-p} C_\phi^{\kappa,\lambda}(ARA', B)dR = \Gamma_m(t, \kappa)|W|^{-t} C_\phi^{\kappa,\lambda}(AW^{-1}A', B)$$

where $p=(m+1)/2$, and $\Gamma_m(t, \kappa)$ is defined in [2]. Similarly for $C_\phi^{\kappa,\lambda}(A, BRB')$. From (2.5), $|R|^{t-p}C_\phi^{\kappa,\lambda}(XR, YR)$ has Laplace transform $\Gamma_m(t, \phi)\cdot |W|^{-t}C_\phi^{\kappa,\lambda}(XW^{-1}, YW^{-1})$. Setting $R=W^{-1/2}HSH'W^{-1/2}$ and integrating over $O(m)$, it follows directly from [3] equation (10) that $|R|^{t-p}C_\phi^{\kappa,\lambda}(XR^{-1}, YR^{-1})$ has Laplace transform $[(-1)^f\Gamma_m(t)/(-t+p)_\phi]|W|^{-t}C_\phi^{\kappa,\lambda}(XW, YW)$.

Binomial expansion:

(2.13) $$C_\phi(X+Y)=\sum_{\kappa,\lambda(\phi\in\kappa.\lambda)} \sum_{\phi'\equiv\phi} \binom{f}{k} \theta_{\phi'}^{\kappa,\lambda} C_{\phi'}^{\kappa,\lambda}(X, Y)\,,$$

where in particular

(2.14) $$C_f(X+Y)=\sum_{k+l=f} \binom{f}{k} C_f^{k,l}(X, Y)\,.$$

Thus $\binom{f}{k}C_f^{k,l}(X, Y)$ is given by the terms of degree k, l in X, Y respectively in the expansion of $C_f(X+Y)$. The polynomials for $\kappa=[1^k]$, $\lambda=[1^l]$, $\phi=[1^f]$ may similarly be obtained from $C_{1^f}(X+Y)$.

If R, S are $r\times r$, $s\times s$ symmetric matrices respectively, $r+s=m$, then it may be shown using (1.3) and comparing coefficients of $(\operatorname{tr} R)^k \cdot(\operatorname{tr} S)^l$ that

$$C_\phi^{\kappa,\lambda}\left(\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}\right)=\theta_\phi^{\kappa,\lambda} z_\phi C_\kappa(R) C_\lambda(S)/z_\kappa z_\lambda \tag{2.15}$$

where $z_\phi=C_\phi(I_m)/2^m(m/2)_\phi$ is the coefficient of $(\operatorname{tr} X)^f$ in $C_\phi(X)$. Hence from (2.13)

$$C_\phi\left(\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}\right)=z_\phi \sum_{\kappa,\lambda(\phi\in\kappa.\lambda)} \binom{f}{k} g_{\kappa,\lambda}^\phi C_\kappa(R) C_\lambda(S)/z_\kappa z_\lambda\,, \tag{2.16}$$

so that Hayakawa's [7] coefficient $a_{\kappa,\lambda}^\phi$ is related to $g_{\kappa,\lambda}^\phi$ by

$$a_{\kappa,\lambda}^\phi=\binom{f}{k} g_{\kappa,\lambda}^\phi z_\phi/z_\kappa z_\lambda\,. \tag{2.17}$$

For a given k, l, the $C_\phi^{\kappa,\lambda}(X, Y)$ are linear combinations of the *distinct* products of traces

$$(\operatorname{tr} X^{a_1} Y^{b_1} X^{c_1}\cdots)^{r_1}(\operatorname{tr} X^{a_2} Y^{b_2} X^{c_2}\cdots)^{r_2}\cdots$$

of total degree k, l in the elements of X, Y respectively. In constructing these, account must be taken of the symmetry of X and Y, and the trace properties $\operatorname{tr} XY=\operatorname{tr} YX$, $\operatorname{tr} Z'=\operatorname{tr} Z$. The number of distinct terms of this type should thus equal the sum of multiplicities of the irreducible representations $2\phi\in 2\kappa\otimes 2\lambda$, for all ordered partitions κ, λ of k, l respectively into $\leq m$ parts. An algorithm for determining these multiplicities is given in Robinson [16] Section 3.3.

The polynomials have been tabulated up to degree $f=k+l=6$, and are available from the author. Polynomials up to $f=5$ are listed in the Appendix. When $\kappa\neq\lambda$ it is convenient for purposes of construction to define $C_\phi^{\kappa,\lambda}(X, Y)=C_\phi^{\lambda,\kappa}(Y, X)$. However, when $\kappa=\lambda$ it may neither be convenient nor possible to insist on symmetry in cases of multiplicity >1; thus we shall not assume symmetry to hold in general. The construction of the polynomials will be discussed in a subsequent paper.

3. Incomplete gamma and beta functions

Incomplete gamma function

Generalizing Constantine [2] equation (60) we have

$$\begin{aligned}\int_0^X \operatorname{etr}(-AS)|S|^{t-p} C_\lambda(BS)dS \\ &= \{\Gamma_m(t)\Gamma_m(p)/\Gamma_m(t+p)\}|X|^t \\ &\quad\cdot\sum_{k=0}^\infty \sum_{\kappa;\phi\in\kappa.\lambda} (t)_\phi \theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(-AX, BX)/k!(t+p)_\phi\,.\end{aligned} \tag{3.1}$$

PROOF. Let $S=X^{1/2}H'THX^{1/2}$, $H\in O(m)$. Expanding the exponential, we average over $O(m)$ and use

$$\int_0^I |T|^{t-p}C_\phi(T)/C_\phi(I)dT=\Gamma_m(t,\phi)\Gamma_m(p)/\Gamma_m(t+p,\phi)\,. \tag{3.2}$$

Equation (3.1) implies an expansion of the c.d.f. of the noncentral Wishart distribution $W_m(n,\Sigma,\Omega)$ (equation (4.9)), and in particular of the largest root.

Incomplete beta function

Equation (61) of [2] may similarly be generalized as follows:

$$\begin{aligned}\int_0^X |R|^{t-p}|I-R|^{u-p}C_\lambda(AR)dR &= \{\Gamma_m(t)\Gamma_m(p)/\Gamma_m(t+p)\}|X|^t \\ &\cdot\sum_{k=0}^{\infty}\sum_{\kappa;\phi\in\kappa.\lambda}(t)_\phi(-u+p)_\kappa\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(X,AX)/k!(t+p)_\phi\,.\end{aligned} \tag{3.3}$$

This yields an expansion for the c.d.f. of the multivariate noncentral beta matrix, and in particular of the largest root.

A further result is

$$\begin{aligned}\int_0^I |R|^{t-p}|I-R|^{u-p}C_\phi^{\kappa,\lambda}(R,I-R)dR \\ =[\Gamma_m(t,\kappa)\Gamma_m(u,\lambda)/\Gamma_m(t+u,\phi)]\theta_\phi^{\kappa,\lambda}C_\phi(I)\,.\end{aligned} \tag{3.4}$$

4. Polynomials of Hayakawa and Khatri

Explicit representations of these polynomials in terms of the $C_\phi^{\kappa,\lambda}$ will be presented in this section. In connection with the multivariate noncentral quadratic form, Hayakawa has defined a polynomial $P_\phi(T,A)$, which may be obtained as the coefficient of $C_\phi(UU')/f!(m/2)_\phi C_\phi(I_n)$ in the generating function ([8] Theorem 7, noting that $(m/2)_\phi C_\phi(I_n)=(n/2)_\phi C_\phi(I_m)$)

$$\begin{aligned}\int_{O(m)}\int_{O(n)} &\operatorname{etr}(-UH_2AH_2'U'+2H_1UH_2A^{1/2}T')dH_1dH_2 \\ &=\int_{O(n)}\operatorname{etr}(-U'UH_2AH_2')\,{}_0F_1(m/2;\,U'UH_2A^{1/2}T'TA^{1/2}H_2')dH_2 \\ &=\sum_{\kappa,\lambda;\phi}^{\infty}\frac{C_\phi(U'U)\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(-A,T'TA)}{k!l!(m/2)_\lambda C_\phi(I_n)}\end{aligned} \tag{4.1}$$

where U, T are $m\times n$, A is $n\times n$ positive definite symmetric. Hence

$$P_\phi(T,A)=(m/2)_\phi\sum_{\kappa,\lambda(\phi\in\kappa.\lambda)}\sum_{\phi'\equiv\phi}\binom{f}{k}\theta_{\phi'}^{\kappa,\lambda}C_{\phi'}^{\kappa,\lambda}(-A,T'TA)/(m/2)_\lambda\,. \tag{4.2}$$

If we set $A=I_n$, then (2.2), (2.10) and Bingham's [1] identity for the generalized binomial coefficient [3]

$$a_{\phi,\lambda}=\binom{f}{l}\sum_{\kappa} g^{\phi}_{\kappa,\lambda} \tag{4.3}$$

implies that

$$P_\phi(T, I_n)=(-1)^f L_\phi^{n/2-p}(TT') \tag{4.4}$$

([7] equation (35)), where L_ϕ^t is Constantine's [3] generalized Laguerre polynomial. Similarly, using (2.11), we obtain

$$\begin{aligned}\int_{O(n)} P(TH, A)dH&=\int_{O(n)} P(T, HAH')dH \\ &=(-1)^f \frac{C_\phi(A)}{C_\phi(I_n)} L_\phi^{n/2-p}(TT')\end{aligned} \tag{4.5}$$

([7] equation (38)). Hayakawa [9] has tabulated the P_ϕ up to $f=4$; further tabulation could be based on (4.2).

Also in connection with the noncentral quadratic form, Khatri [13] has defined a two-matrix generalization $L_\phi^t(X, A)$ of $L_\phi^t(X)$ (X, A are $m\times m$ symmetric), such that $|X|^t L_\phi^t(X, A)$ has Laplace transform with respect to X

$$\begin{aligned}&\Gamma_m(t+p, \phi)|Z|^{-t-p}C_\phi((I-Z^{-1})A) \\ &\quad=\Gamma_m(t+p, \phi)|Z|^{-t-p}\sum_{\kappa,\lambda(\phi\in\kappa.\lambda)}\sum_{\phi'\equiv\phi}\binom{f}{k}\theta^{\kappa,\lambda}_{\phi'}C^{\kappa,\lambda}_{\phi'}(A, -Z^{-1}A)\end{aligned} \tag{4.6}$$

by (2.13), which may be inverted by (2.12) to yield

$$L_\phi^t(X, A)=(t+p)_\phi \sum_{\kappa,\lambda(\phi\in\kappa.\lambda)}\sum_{\phi'\equiv\phi}\binom{f}{k}\theta^{\kappa,\lambda}_{\phi'}C^{\kappa,\lambda}_{\phi'}(A, -XA)/(t+p)_\lambda\ . \tag{4.7}$$

From (4.2), expressions for L_ϕ^t may be obtained by replacing $m/2$ by $t+p$ in $(-1)^f P_\phi$. The three-matrix polynomials defined by Crowther [4] and Khatri [13] are not obtainable using the methods of this paper.

Khatri's expansion of the Wishart density $W_m(n, \Sigma, \Omega)$ ([13] equation (4.10)) may be obtained directly using (2.8) and (4.7),

$$\begin{aligned}&W_m(n, \Sigma, 0)\,\mathrm{etr}\,(-\Omega)\,{}_0F_1(n/2; (\Sigma^{-1/2}/2)S\Sigma^{-1/2}\Omega) \\ &\quad=W_m(n, \Sigma, 0)\sum_{\kappa,\lambda;\phi}^{\infty}\theta^{\kappa,\lambda}_{\phi}C^{\kappa,\lambda}_{\phi}(-\Omega, \Sigma^{-1/2}S\Sigma^{-1/2}\Omega/2)/k!l!(n/2)_\lambda \\ &\quad=W_m(n, \Sigma, 0)\sum_{f=0}^{\infty}\sum_{\phi}L_\phi^{n/2-p}(\Sigma^{-1/2}S\Sigma^{-1/2}/2, -\Omega)/f!(n/2)_\phi\ .\end{aligned} \tag{4.8}$$

The c.d.f. of this distribution may be obtained from (3.1) in the form

$$(4.9)\quad P\{S<X\}=\{\Gamma_m(p)/2^{mn/2}\Gamma_m(n/2+p)|\Sigma|^{n/2}\}\operatorname{etr}(-\Omega)|X|^{n/2} \cdot \sum_{f=0}^{\infty}\sum_{\phi}(-1)^f L_\phi^{n/2-p}(\Omega, \Sigma^{-1/2}X\Sigma^{-1/2}/2)/f!(n/2+p)_\phi \, .$$

Taking $X=\lambda_1 I$ we obtain the c.d.f. of the largest root λ_1.

5. A complete system of generalized Laguerre polynomials with two matrix arguments

From (4.7) or Khatri's original definition, the $L_\phi^t(X, A)$ are to be regarded as a multivariate generalization of $a^f L_f^t(x)$, rather than of $L_k^t(x)L_l^u(y)$ as would be required for functions invariant under (1.1). In the present section we present such a generalization along the lines of Herz [11] and Constantine [3]. Define

$$(5.1)\quad L_{\kappa,\lambda;\phi}^{t,u}(X, Y)=\operatorname{etr}(X+Y)\int_{R>0}\int_{S>0}\operatorname{etr}(-R-S)|R|^t|S|^u C_\phi^{\kappa,\lambda}(R, S) \cdot A_t(RX)A_u(SY)dRdS\,, \qquad (t, u>-1)$$

where A_t is Herz's Bessel function (see above references), noting that the $L_{\kappa,\lambda;\phi}^{t,u}$ will depend on the particular basis $\{C_\phi^{\kappa,\lambda}\}$, in the spirit of Herz's definition.

Laplace transform. Noting that ([11] equation (15))

$$(5.2)\quad \int_{R>0}\operatorname{etr}(-RW)|R|^t A_t(RM)dR=|W|^{-t-p}\operatorname{etr}(-MW^{-1})\,,$$

we obtain using (2.12)

$$(5.3)\quad \int_{R>0}\int_{S>0}\operatorname{etr}(-RW-SZ)|R|^t|S|^u L_{\kappa,\lambda;\phi}^{t,u}(R, S)dRdS = \Gamma_m(t+p, \kappa)\Gamma_m(u+p, \lambda)|W|^{-t-p}|Z|^{-u-p}C_\phi^{\kappa,\lambda}(I-W^{-1}, I-Z^{-1})\,.$$

This yields an explicit representation of the polynomials, analogous to [3] equation (20). Let

$$(5.4)\quad C_\phi^{\kappa,\lambda}(I+X, I+Y)/C_\phi(I)=\sum_{r=0}^{k}\sum_{s=0}^{l}\sum_{\rho,\sigma;\tau} a_{\rho,\sigma;\tau}^{\kappa,\lambda;\phi}C_\tau^{\rho,\sigma}(X, Y)/C_\tau(I)$$

where ρ, σ, τ are partitions of r, s, $r+s$ respectively. Again using (2.12),

$$(5.5)\quad L_{\kappa,\lambda;\phi}^{t,u}(X, Y)=(t+p)_\kappa(u+p)_\lambda C_\phi(I)\sum_{r=0}^{k}\sum_{s=0}^{l}\sum_{\rho,\sigma;\tau}\frac{(-1)^{r+s}a_{\rho,\sigma;\tau}^{\kappa,\lambda;\phi}}{(t+p)_\rho(u+p)_\sigma} \cdot \frac{C_\tau^{\rho,\sigma}(X, Y)}{C_\tau(I)}\,.$$

Generating function. $L^{t,u}_{\kappa,\lambda;\phi}(X, Y)$ is the coefficient of $C^{\kappa,\lambda}_{\phi}(W, Z)/k!l!C_{\phi}(I)$ in the expansion of

$$(5.6)\quad |I-W|^{-t-p}|I-Z|^{-u-p} \cdot \int_{O(m)} \mathrm{etr}\{-XH'W(I-W)^{-1}H-YH'Z(I-Z)^{-1}H\}dH .$$

A relation with Khatri's polynomial proved by taking Laplace transforms is

$$(5.7)\quad \int_{O(m)} L^{t}_{\kappa}(HAH', X)L^{u}_{\lambda}(HBH', Y)dH = \sum_{\phi\in\kappa.\lambda} C^{\kappa,\lambda}_{\phi}(X, Y)L^{t,u}_{\kappa,\lambda;\phi}(A, B)/C_{\phi}(I) .$$

Orthogonality. Multiplying (5.6) by $\mathrm{etr}(-X-Y)|X|^{t}|Y|^{u}C^{\rho,\sigma}_{\tau}(X, Y)$ and integrating over $X>0$, $Y>0$, we find that

$$(5.8)\quad \int_{X>0}\int_{Y>0} \mathrm{etr}(-X-Y)|X|^{t}|Y|^{u}C^{\rho,\sigma}_{\tau}(X, Y)L^{t,u}_{\kappa,\lambda;\phi}(X, Y)dXdY$$

is the coefficient of $C^{\kappa,\lambda}_{\phi}(W, Z)/k!l!C_{\phi}(I)$ in the expansion of

$$(5.9)\quad \Gamma_{m}(t+p, \rho)\Gamma_{m}(u+p, \sigma)C^{\rho,\sigma}_{\tau}(I-W, I-Z) .$$

A similar argument to [3] Theorem 2 shows that (5.8) is zero for $k\geqq r$, $l\geqq s$ unless $(\kappa, \lambda; \phi)=(\rho, \sigma; \tau)$ and hence that $L^{t,u}_{\kappa,\lambda;\phi}$ is orthogonal to all $L^{t,u}_{\rho,\sigma;\tau}$ with respect to $\mathrm{etr}(-X-Y)|X|^{t}|Y|^{u}$ for $k\geqq r$, $l\geqq s$ and $(\rho, \sigma; \tau)\neq(\kappa, \lambda; \phi)$, with

$$(5.10)\quad \int_{X>0}\int_{Y>0} \mathrm{etr}(-X-Y)|X|^{t}|Y|^{u}\{L^{t,u}_{\kappa,\lambda;\phi}(X, Y)\}^{2}dXdY = k!l!\Gamma_{m}(t+p, \kappa)\Gamma_{m}(u+p, \lambda)C_{\phi}(I) .$$

Completeness. On the basis of (5.10), we could follow Herz [11] Sections 3 and 4 in considering the Hilbert space $\mathcal{L}^{2}_{t,u}$ of functions $f(X, Y)$ defined for $X>0$, $Y>0$, such that

$$(5.11)\quad \|f\|^{2}_{t,u}=\int_{R>0}\int_{S>0} |f(R, S)|^{2}|R|^{t}|S|^{u}dRdS<\infty ,$$

and show that the "(t, u)-Hankel transform" of f, $g=\mathcal{U}_{t,u}f$ say, where

$$(5.12)\quad g(X, Y)\sim\int_{R>0}\int_{S>0} A_{t}(RX)A_{u}(SY)|R|^{t}|S|^{u}f(R, S)dRdS ,$$

has the properties of a Watson transform; i.e. $\mathcal{U}_{t,u}$ is unitary, self-adjoint and self-inversive on $\mathcal{L}^{2}_{t,u}$. In (5.12), $\sim$ signifies equality whenever the integral is absolutely convergent, otherwise a limit in the above norm.

The Laplace transforms F, G of f, g respectively are related by

(5.13) $$G(W, Z)=|W|^{-t-p}|Z|^{-u-p}F(W^{-1}, Z^{-1}) .$$

Hence if we set

(5.14) $$l_{\kappa,\lambda;\phi}^{t,u}(X, Y)=\mathrm{etr}\,(-X-Y)L_{\kappa,\lambda;\phi}^{t,u}(2X, 2Y)$$

it follows from (5.1) that $l_{\kappa,\lambda;\phi}^{t,u} \in \mathcal{L}_{t,u}^2$, while from (5.3) and (5.13) $l_{\kappa,\lambda;\phi}^{t,u}$ and $(-1)^f \mathcal{U}_{t,u} l_{\kappa,\lambda;\phi}^{t,u}$ each has double Laplace transform

(5.15) $$\Gamma_m(t+p, \kappa)\Gamma_m(u+p, \lambda)|I+W|^{-t-p}|I+Z|^{-u-p} \cdot C_\phi^{\kappa,\lambda}\left(\frac{W-I}{W+I}, \frac{Z-I}{Z+I}\right) .$$

Thus the $l_{\kappa,\lambda;\phi}^{t,u}$ are orthogonal eigenfunctions of $\mathcal{U}_{t,u}$ with eigenvalues $(-1)^f$. We may now show that they are complete in the closed subspace $\tilde{\mathcal{L}}_{t,u}^2$ consisting of the functions in $\mathcal{L}_{t,u}^2$ invariant under (1.1) ([11] p. 501). Suppose that $f \in \tilde{\mathcal{L}}_{t,u}^2$ and that

(5.16) $$\int_{R>0}\int_{S>0} l_{\kappa,\lambda;\phi}^{t,u}(R, S)f(R, S)|R|^t|S|^u dRdS=0 \qquad \text{for all } \kappa, \lambda; \phi .$$

Then since each $C_\phi^{\kappa,\lambda}(R, S)$ can be expressed as a linear combination of $L_{\kappa,\lambda;\phi}^{t,u}$ and lower degree Laguerre polynomials,

(5.17) $$\omega_{\kappa,\lambda;\phi}=\int_{R>0} \mathrm{etr}\,(-R-S)C_\phi^{\kappa,\lambda}(R, S)f(R, S)|R|^t|S|^u dRdS=0 \qquad \text{for all } \kappa, \lambda; \phi .$$

Hence, since f satisfies (1.1) its double Laplace transform can be written

(5.18) $$\begin{aligned} F(W, Z)&=\int_{R>0}\int_{S>0} \mathrm{etr}\,(-R-S)f(R, S)|R|^t|S|^u \\ &\quad\cdot \int_{O(m)} \mathrm{etr}\,\{(I-W)H'RH+(I-Z)H'SH\}dH \\ &=\sum_{\kappa,\lambda;\phi}^{\infty} \omega_{\kappa,\lambda;\phi}C_\phi^{\kappa,\lambda}(I-W, I-Z)/k!l!C_\phi(I)=0 \end{aligned}$$

for $0 \leq \mathrm{Re}\, W$, $\mathrm{Re}\, Z<I$. But F is complex analytic for $\mathrm{Re}\, W>0$, $\mathrm{Re}\, Z>0$; hence $F=0$, and f is a null function.

6. Some useful expansions

(a) Multiplying both sides of (1.3) by $\mathrm{etr}\,(-X)$,

(6.1) $$\mathrm{etr}\,(-X) \sum_{\kappa,\lambda;\phi}^{\infty} \frac{C_\phi^{\kappa,\lambda}(A, B)C_\phi^{\kappa,\lambda}(X, Y)}{k!l!C_\phi(I)} = \sum_{\kappa,\lambda;\phi}^{\infty} \frac{C_\phi^{\kappa,\lambda}(A-I, B)C_\phi^{\kappa,\lambda}(X, Y)}{k!l!C_\phi(I)} .$$

A number of expansions may be derived from (6.1) by taking and inverting Laplace transforms.

Substitute $X\to Z^{1/2}XZ^{1/2}$, $Y\to Z^{1/2}YZ^{1/2}$, multiply by $\operatorname{etr}(-Z)|Z|^{a-p}$ and integrate over $Z>0$; then

$$(6.2)\qquad |I+X|^{-a}\sum_{\kappa,\lambda;\phi}^{\infty}(a)_\phi C_\phi^{\kappa,\lambda}(A,B)C_\phi^{\kappa,\lambda}(X(I+X)^{-1},Y(I+X)^{-1})/k!l!C_\phi(I)$$
$$=\sum_{\kappa,\lambda;\phi}^{\infty}(a)_\phi C_\phi^{\kappa,\lambda}(A-I,B)C_\phi^{\kappa,\lambda}(X,Y)/k!l!C_\phi(I)\,.$$

On the other hand, replacing B by B^{-1}, multiplying by $|B|^{-u}$ and inverting the Laplace transform yields

$$(6.3)\qquad \operatorname{etr}(-X)\sum_{\kappa,\lambda;\phi}^{\infty}C_\phi^{\kappa,\lambda}(A,B)C_\phi^{\kappa,\lambda}(X,Y)/k!l!(u)_\lambda C_\phi(I)$$
$$=\sum_{\kappa,\lambda;\phi}^{\infty}C_\phi^{\kappa,\lambda}(A-I,B)C_\phi^{\kappa,\lambda}(X,Y)/k!l!(u)_\lambda C_\phi(I)\,.$$

To obtain a result used later, set $Y=X$, $B\to -AB$ in the R.H.S., multiply by $\operatorname{etr}\{-B(I+Z)\}|B|^{u-p}/\Gamma_m(u)$ and integrate over $B>0$; using the binomial expansion (2.13), the transform is

$$(6.4)\qquad |I+Z|^{-u}{}_0F_0^{(m)}(I-AZ(I+Z)^{-1},-X)\,.$$

This may be expanded in terms of Constantine's Laguerre polynomials; applying the binomial expansion again, and inverting the Laplace transform, we obtain

$$(6.5)\qquad \operatorname{etr}(-B)\sum_{\kappa,\lambda;\phi}^{\infty}C_\phi(X)\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(A-I,-AB)/k!l!(u)_\lambda C_\phi(I)$$
$$=|A|^{-u}\sum_{\kappa,\lambda;\phi}^{\infty}L_\phi^{u-p}(-X)\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(I-A^{-1},-A^{-1}B)/k!l!(u)_\lambda C_\phi(I)\,.$$

(b) We now show that $C_\phi^{\kappa,\lambda}(I+A,B)$ has an expansion of the form

$$(6.6)\qquad C_\phi^{\kappa,\lambda}(I+A,B)/C_\phi(I)=\sum_{r=0}^{k}\sum_{\phi;\rho\in\rho.\lambda}b_{\rho,\lambda;\tau}^{\kappa,\lambda;\phi}C_\tau^{\rho,\lambda}(A,B)/C_\tau(I)\,.$$

PROOF. There certainly exists an expansion

$$(6.7)\qquad C_\phi^{\kappa,\lambda}(I+A,B)/C_\phi(I)=\sum_{r=0}^{k}\sum_{s=0}^{l}\sum_{\rho,\sigma;\tau}b_{\rho,\sigma;\tau}^{\kappa,\lambda;\phi}(A,B)/C_\tau(I)\,,$$

so that from (2.11)

$$\int_{O(m)}C_\phi^{\kappa,\lambda}(I+A,B'H'XHB)dH/C_\phi(I)$$
$$=\sum_{r=0}^{k}\sum_{s=0}^{l}\sum_{\rho,\sigma;\tau}b_{\rho,\sigma;\tau}^{\kappa,\lambda;\phi}C_\tau^{\rho,\sigma}(A,B'B)C_\sigma(X)/C_\sigma(I)C_\tau(I)\,.$$

But by (2.11) the L.H.S. is also

$$(6.8)\qquad C_\phi^{\kappa,\lambda}(I+A,\,B'B)C_\lambda(X)/C_\lambda(I)C_\phi(I)$$
$$=\{C_\lambda(X)/C_\lambda(I)\}\sum_{r=0}^{k}\sum_{s=0}^{l}\sum_{\rho,\sigma;\tau} b_{\rho,\sigma;\tau}^{\kappa,\lambda;\phi}C_\tau^{\rho,\sigma}(A,\,B'B)/C_\tau(I)\ ,$$

whence if $b_{\rho,\sigma;\tau}^{\kappa,\lambda;\phi}\neq 0$ we must have $\sigma=\lambda$. Q.E.D.

Thus, we may define another generalized Laguerre polynomial

$$(6.9)\qquad L_{\kappa,\lambda;\phi}^{t}(X,\,Y)=\operatorname{etr}(X)\int_{R>0}\operatorname{etr}(-R)|R|^{t}C_\phi^{\kappa,\lambda}(R,\,Y)A_t(RX)dR$$

with Laplace transform

$$(6.10)\qquad \int_{X>0}\operatorname{etr}(-XW)|X|^{t}L_{\kappa,\lambda;\phi}^{t}(X,\,Y)dX$$
$$=\Gamma_m(t+p,\,\kappa)|W|^{-t-p}C_\phi^{\kappa,\lambda}(I-W^{-1},\,Y)$$

so that

$$(6.11)\quad L_{\kappa,\lambda;\phi}^{t}(X,\,Y)=(t+p)_\kappa C_\phi(I)\sum_{r=0}^{k}\sum_{\rho;\tau\in\rho.\lambda}(-1)^r b_{\rho,\lambda;\tau}^{\kappa,\lambda;\phi}C_\tau^{\rho,\lambda}(X,\,Y)/(t+p)_\rho C_\tau(I)\ .$$

Replace A, B in (6.2) by A^{-1}, B^{-1} and multiply by $|A|^{-t}|B|^{-u}$; inverting the Laplace transform

$$(6.12)\quad |I+X|^{-a}\sum_{\kappa,\lambda;\phi}^{\infty}(a)_\phi C_\phi^{\kappa,\lambda}(A,\,B)C_\phi^{\kappa,\lambda}(X(I+X)^{-1},\,Y(I+X)^{-1})/k!l!(t)_\kappa(u)_\lambda C_\phi(I)$$
$$=\sum_{\kappa,\lambda;\phi}^{\infty}(a)_\phi C_\phi^{\kappa,\lambda}(-X,\,Y)L_{\kappa,\lambda;\phi}^{t-p}(A,\,B)/k!l!(t)_\kappa(u)_\lambda C_\phi(I)\ .$$

(c) Multiplying (1.3) by $\operatorname{etr}(-X-Y)$,

$$(6.13)\qquad \operatorname{etr}(-X-Y)\sum_{\kappa,\lambda;\phi}^{\infty}C_\phi^{\kappa,\lambda}(A,\,B)C_\phi^{\kappa,\lambda}(X,\,Y)/k!l!C_\phi(I)$$
$$=\sum_{\kappa,\lambda;\phi}^{\infty}C_\phi^{\kappa,\lambda}(A-I,\,B-I)C_\phi^{\kappa,\lambda}(X,\,Y)/k!l!C_\phi(I)\ .$$

Replace A, B by A^{-1}, B^{-1}, multiply by $|A|^{-t}|B|^{-u}$ and invert the Laplace transform:

$$(6.14)\quad \operatorname{etr}(-X-Y)\int_{O(m)}{}_0F_1(t;\,AHXH')\,{}_0F_1(u;\,BHYH')dH$$
$$=\sum_{\kappa,\lambda;\phi}^{\infty}(-1)^f C_\phi^{\kappa,\lambda}(X,\,Y)L_{\kappa,\lambda;\phi}^{t-p,u-p}(A,\,B)/k!l!(t)_\kappa(u)_\lambda C_\phi(I)\ .$$

(d) Expanding each side of $|I-(X+Y)|^{-a}=|I+Z|^{-a}|I-(X+Z+Y)\cdot(I+Z)^{-1}|^{-a}$ and using (2.13),

$$(6.15)\qquad \sum_{\kappa,\lambda;\phi}^{\infty}(a)_{\phi}\theta_{\phi}^{\kappa,\lambda}C_{\phi}^{\kappa,\lambda}(X, Y)/k!l! \\ =|I+Z|^{-a}\sum_{\kappa,\lambda;\phi}^{\infty}(a)_{\phi}\theta_{\phi}^{\kappa,\lambda}C_{\phi}^{\kappa,\lambda}\left(\frac{X+Z}{I+Z}, \frac{Y}{I+Z}\right)\Big/k!l!\ .$$

Replace Y by Y^{-1}, multiply by $|Y|^{-u}$ and invert the Laplace transform:

$$(6.16)\qquad \sum_{\kappa,\lambda;\phi}^{\infty}(a)_{\phi}\theta_{\phi}^{\kappa,\lambda}C_{\phi}^{\kappa,\lambda}(X, Y)/k!l!(u)_{\lambda} \\ =|I+Z|^{-a}\sum_{\kappa,\lambda;\phi}^{\infty}(a)_{\phi}\theta_{\phi}^{\kappa,\lambda}C_{\phi}^{\kappa,\lambda}\left(\frac{X+Z}{I+Z}, \frac{Y}{I+Z}\right)\Big/k!l!(u)_{\lambda}\ .$$

Taking $X=-Z$,

$$(6.17)\qquad |I+Z|^{-a}{}_1F_1(a; u; Y(I+Z)^{-1}) \\ =\sum_{\kappa,\lambda;\phi}^{\infty}(a)_{\phi}\theta_{\phi}^{\kappa,\lambda}C_{\phi}^{\kappa,\lambda}(-Z, Y)/k!l!(u)_{\lambda} \\ =\sum_{f=0}^{\infty}\sum_{\phi}(-1)^{f}(a)_{\phi}L_{\phi}^{u-p}(YZ^{-1}, Z)/f!(u)_{\phi}\ .$$

In particular, if $a=u$,

$$(6.18)\qquad |I+Z|^{-a}\operatorname{etr}[YZ(I+Z)^{-1}]=\sum_{f=0}^{\infty}\sum_{\phi}(-1)^{f}L_{\phi}^{a-p}(Y, Z)/f!\ .$$

Taking $Z=-A^{1/2}H'XHA^{1/2}$ and integrating over $O(m)$ yields Khatri's generating function ([13] equation (2.26)). Equation (6.18) also provides an expansion of the noncentral Wishart distribution ([2] equation (34)).

Setting $Y=A^{1/2}H'XHA^{1/2}$ in (6.18) and integrating over $O(m)$, we obtain

$$(6.19)\qquad |I-Z|^{-a}C_{\lambda}(A(I-Z)^{-1})=(a)_{\lambda}^{-1}\sum_{k=0}^{\infty}\sum_{\kappa,\phi\in\kappa.\lambda}(a)_{\phi}\theta_{\phi}^{\kappa,\lambda}C_{\phi}^{\kappa,\lambda}(Z, A)/k!\ .$$

7. Noncentral quadratic form

Suppose that X is $N_{m,n}(M, I_n\otimes\Sigma)$ and A is $n\times n$ positive definite symmetric. The quadratic form $\Sigma^{-1/2}XAX'\Sigma^{-1/2}=YY'$, where $Y=\Sigma^{-1/2}\cdot XA^{1/2}$, has the same latent roots $\lambda_1\geqq\cdots\geqq\lambda_m\geqq 0$ as $\tilde{Y}\tilde{Y}'$, where $\tilde{Y}=HYK$ ($H\in O(m)$, $K\in O(n)$). Integrating the density of $\tilde{Y}$ over $O(m)$, $O(n)$ we obtain

$$(7.1)\qquad f(\tilde{Y})=(2\pi)^{-mn/2}|A|^{-m/2}\operatorname{etr}(-\Omega)\sum_{\kappa,\lambda;\phi}^{\infty}C_{\phi}(\tilde{Y}\tilde{Y}'/2)\theta_{\phi}^{\kappa,\lambda}C_{\phi}^{\kappa,\lambda} \\ \cdot(-A^{-1}, M'\Sigma^{-1}MA^{-1}/2)/k!l!(m/2)_{\lambda}C_{\phi}(I_n) \\ =(2\pi)^{-mn/2}|A|^{-m/2}\operatorname{etr}(-\Omega)\sum_{f=0}^{\infty}\sum_{\phi}C_{\phi}(\tilde{Y}\tilde{Y}'/2)$$

$$\cdot P_\phi\left(\frac{1}{\sqrt{2}}\Sigma^{-1/2}M,\ A^{-1}\right)\Big/ f!(n/2)_\phi C_\phi(I_m)$$

by (4.2) (Hayakawa [8], Theorem 8). The density of $\Lambda=\operatorname{diag}(\lambda_i)$ follows directly by the usual method. From (6.3), the joint density of the roots may be written in the following form for real $q>0$,

$$(7.2)\qquad f(\Lambda)=[\pi^{m^2/2}/\Gamma_m(m/2)]|q\Lambda|^{-m/2}\operatorname{etr}(-\Omega)W_m(n,\ q^{-1}I,\ 0)\prod_{i<j}(\lambda_i-\lambda_j)$$
$$\cdot\sum_{\kappa,\lambda;\phi}^{\infty}\theta_\phi^{\kappa,\lambda}C_\phi(q\Lambda/2)C_\phi^{\kappa,\lambda}(I-q^{-1}\Lambda^{-1},\ q^{-1}M'\Sigma^{-1}M\Lambda^{-1}/2)$$
$$/k!l!(m/2)_\lambda C_\phi(I_n)\ ,$$

(Hayakawa [9], Theorem 2). Setting

$$(7.3)\qquad m\to n\ ,\qquad A\to q^{-1}\Lambda^{-1}\ ,\qquad B=M'\Sigma^{-1}M/2\ ,$$
$$X=\begin{pmatrix}-q\Lambda/2 & 0\\ 0 & 0\end{pmatrix},\qquad u=m/2$$

in (6.5), and noting rank $(\Lambda)=m$, we obtain Khatri's [13] equation (5.2) in the form

$$(7.4)\qquad [\pi^{m^2/2}/\Gamma_m(m/2)]W_m(n,\ q^{-1}I,\ 0)\prod_{i<j}(\lambda_i-\lambda_j)$$
$$\cdot\sum_{\kappa,\lambda,\phi}^{\infty}L_\phi^{n/2-p}(q\Lambda/2)\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(I-q\Lambda,\ -q\Lambda M'\Sigma^{-1}M/2)$$
$$/k!l!(m/2)_\lambda C_\phi(I_n)\ ,$$

expressible also in terms of the P_ϕ. It may be noted that this expression facilitates the derivation of Hayakawa's [10] asymptotic expansion of the distribution of $\sqrt{n/2m}\log|q\Lambda/n|$, since the characteristic function may be obtained from

$$(7.5)\qquad \mathrm{E}_\Lambda\{|q\Lambda/n|^\theta\}=[\Gamma_n(n/2+\theta)/(n/2)^{m\theta}\Gamma_n(n/2)]$$
$$\cdot\sum_{\kappa,\lambda;\phi}^{\infty}e_\phi(m/2)_\phi\theta_\phi^{\kappa,\lambda}C_\phi^{\kappa,\lambda}(I-B,\ -B\tilde{\Omega})/k!l!(m/2)_\lambda$$

where

$$(7.6)\qquad B=q\Lambda\ ,\qquad \tilde{\Omega}=M'\Sigma^{-1}M/2\ ,$$
$$e_\phi=\mathrm{E}_W\{L_\phi^{n/2-p}(W/2)/(n/2)_\phi C_\phi(I_m)\}\ ,\qquad W\sim W_m(n+2\theta,\ I,\ 0)\ ,$$
$$=\sum_{r=0}^{f}\sum_{\rho}(-1)^r a_{\phi,\rho}(n/2+\theta)_\rho/(n/2)_\rho\ .$$

With $\theta=it\sqrt{n/2m}$, we have $e_\phi=O(n^{-f/2})$, at least for small f.

8. Noncentral F with unequal covariance matrices

In the present section we consider the joint distribution of the latent roots $\lambda_1 \geqq \cdots \geqq \lambda_m \geqq 0$ of $F = S_2^{-1/2} S_1 S_2^{-1/2}$, where $S_1 \sim W_m(n_1, \Sigma_1, \Omega)$ and $S_2 \sim W_m(n_1, \Sigma_2, 0)$, $\Sigma_1 \neq \Sigma_2$. This distribution has been investigated by Pillai [14], and Pillai and Sudjana [15], under certain "randomness" assumptions on the parameter matrices which facilitate the application of zonal polynomials, and also provide scope for an exact study of the robustness of some standard test criteria against nonnormality and inequality of covariance matrices.

Pillai [14] showed that the roots of F have the same distribution as those of $\tilde{F}$ with density function

$$(8.1) \quad f(\tilde{F}) = C_1(m; n_1, n_2) \operatorname{etr}(-\Omega) |\Psi|^{-n_1/2} |\tilde{F}|^{n_1/2-p} |I + \Psi^{-1}\tilde{F}|^{-(n_1+n_2)/2} \cdot {}_1F_1((n_1+n_2)/2, n_1/2; \Phi\tilde{F}(I+\Psi^{-1}\tilde{F})^{-1})$$

where

$$(8.2) \quad \Psi = \Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2}, \qquad \Phi = \Psi^{-1/2} \Omega \Psi^{-1/2},$$

$$C_1(m; n_1, n_2) = \Gamma_m((n_1+n_2)/2) / \Gamma_m(n_1/2) \Gamma_m(n_2/2).$$

From (6.17), (8.1) may be expressed in terms of Khatri's Laguerre polynomials of two matrix arguments,

$$(8.3) \quad f(\tilde{F}) = C_1(m; n_1, n_2) \operatorname{etr}(-\Omega) |\Psi|^{-n_1/2} |\tilde{F}|^{n_1/2-p} \cdot \sum_{\kappa, \lambda; \phi}^{\infty} ((n_1+n_2)/2)_\phi \theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(-\Psi^{-1}\tilde{F}, \Phi\tilde{F}) / k! l! (n_1/2)_\lambda$$

$$(8.4) \quad = C_1(m; n_1, n_2) \operatorname{etr}(-\Omega) |\Psi|^{-n_1/2} |\tilde{F}|^{n_1/2-p} \cdot \sum_{f=0}^{\infty} \sum_{\phi} (-1)^f ((n_1+n_2)/2)_\phi L_\phi^{n_1/2-p}(\Omega, \Psi^{-1/2}\tilde{F}\Psi^{-1/2}) / f! (n_1/2)_\phi .$$

Applying (6.16), with $Z = q\tilde{F}$ for suitable real q, we obtain

$$(8.5) \quad f(\tilde{F}) = C_1(m; n_1, n_2) \operatorname{etr}(-\Omega) |\Psi|^{-n_1/2} |\tilde{F}|^{n_1/2-p} |I + q\tilde{F}|^{-(n_1+n_2)/2} \cdot \sum_{\kappa, \lambda; \phi}^{\infty} ((n_1+n_2)/2)_\phi \theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}((I - q^{-1}\Psi^{-1})\tilde{B}, q^{-1}\Phi\tilde{B}) / k! l! (n_1/2)_\lambda$$

where $\tilde{B} = q\tilde{F}(I + q\tilde{F})^{-1}$. Expansions for the distribution of $\Lambda = \operatorname{diag}(\lambda_i)$ may be obtained from (8.3), (8.4) and (8.5).

It may be shown by taking a Laplace transform with respect to B, or from (6.3) with $X = Y$ using Bingham's identity (4.3), that

$$(8.6) \quad \sum_{\kappa, \lambda (\phi \in \kappa \cdot \lambda)} \binom{f}{k} \theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(I - A, B) / (t)_\lambda$$

$$=C_\phi(I)\sum_{\rho,\sigma;\nu} a_{\phi,\nu}\binom{n}{r}\theta_\nu^{\rho,\sigma}C_\nu^{\rho,\sigma}(-A, B)/(t)_\sigma C_\nu(I) .$$

Hence from (8.5)

$$(8.7) \quad f(\Lambda)=C_2(m; n_1, n_2)\,\mathrm{etr}\,(-\Omega)|\Psi|^{-n_1/2}|\Lambda|^{n_1/2-p}|I+q\Lambda|^{-(n_1+n_2)/2} \cdot \prod_{i<j}(\lambda_i-\lambda_j)\sum_{f=0}^{\infty}\sum_\phi \{((n_1+n_2)/2)_\phi C_\phi(q\Lambda/(I+q\Lambda))/f!\} \cdot \sum_{n=0}^{f}\sum_\nu a_{\phi,\nu}(-1)^n L_\nu^{n_1/2-p}(\Omega, q^{-1}\Psi^{-1})/(n_1/2)_\nu C_\nu(I) ,$$

where $C_2(m; n_1, n_2)=\pi^{m^2/2}C_1(m; n_1, n_2)/\Gamma_m(m/2)$. Taking Ω and Ψ "random" in Pillai's sense, we obtain his [14] equations (3.2) and (3.7) from (8.3) and (8.7), respectively, using (2.11).

Hotelling trace $T=\mathrm{tr}\,F$. For $S>0$ let $s=\mathrm{tr}\,S$, $S=sS_1$; then $dS=s^{mp-1}dS_1$, where $S_1=(s_{ij}^{(1)})$, $dS_1=\prod ds_{ij}^{(1)}$ $(i\leq j;\ (i, j)\neq(m, m))$, and

$$(8.8) \quad \int_{S_1} L_\phi^t(X, A^{1/2}SA^{1/2})|S_1|^{b-p}dS_1=s^f\Gamma_m(b, \phi)L_\phi^t(X, A)/\Gamma(bm+f) .$$

This may be obtained using an unpublished result of A. T. James

$$(8.9) \quad \int_{S_1} C_\phi(AS_1)|S_1|^{b-p}dS_1=\Gamma_m(b, \phi)C_\phi(A)/\Gamma(bm+f) .$$

Hence from (8.4),

$$(8.10) \quad f(T)=C_3(m; n_1, n_2)\,\mathrm{etr}\,(-\Omega)|\Psi|^{-n_1/2}T^{mn_1/2-1} \cdot \sum_{f=0}^{\infty}[(-T)^f/f!(mn_1/2)_f]\sum_\phi((n_1+n_2)/2)_\phi L_\phi^{n_1/2-p}(\Omega, \Psi^{-1}) ,$$

where $C_3(m; n_1, n_2)=\Gamma_m((n_1+n_2)/2)/\Gamma(mn_1/2)\Gamma_m(n_2/2)$; this extends Constantine [3] equation (1) to the case of unequal covariance matrices.

Using (3.3), the c.d.f. of the largest root f_1 of F may be obtained from (3.1) in the form

$$(8.11) \quad \mathrm{P}\{f_1<x\}=C_4(m; n_1, n_2)\,\mathrm{etr}\,(-\Omega)|\Psi|^{-n_1/2}(q^{-1}\beta)^{mn_1/2} \cdot \sum_{f=0}^{\infty}\sum_\phi[(n_1/2)_\phi C_\phi(\beta I)/f!(n_1/2+p)_\phi] \cdot \sum_{\kappa,\lambda(\phi\in\kappa.\lambda)}\binom{f}{k}g_{\kappa,\lambda}^\phi(-n_2/2+p)_\kappa((n_1+n_2)/2)_\lambda A_\lambda$$

where $\beta=qx/(1+qx)$, $q>0$ real, $C_4(m; n_1, n_2)=\Gamma_m((n_1+n_2)/2)\Gamma_m(p)/\Gamma_m(n_2/2)\cdot\Gamma_m(n_1/2+p)$, and

$$(8.12) \quad A_\lambda=\sum_{\rho,\sigma(\lambda\in\rho.\sigma)}\binom{l}{r}\theta_\lambda^{\rho,\sigma}C_\lambda^{\rho,\sigma}(I-q^{-1}\Psi^{-1}, q^{-1}\Phi)/(n_1/2)_\sigma .$$

9. Doubly noncentral F

Finally, we consider the latent roots of multivariate F when $S_1 \sim W_m(n_1, I, \Omega_1)$, $S_2 \sim W_m(n_2, I, \Omega_2)$.

(a) Case $n_1 \geqq m$. F has the same roots as $\tilde{F} = \tilde{S}_2^{-1/2}\tilde{S}_1\tilde{S}_2^{-1/2}$, where $\tilde{S}_1 = H'S_1H$, $\tilde{S}_2 = H'S_2H$; averaging over $O(m)$,

$$(9.1) \quad f(\tilde{S}_1, \tilde{S}_2) = \mathcal{K}(m; n_1, n_2) \operatorname{etr}(-\Omega_1 - \Omega_2) \cdot \operatorname{etr}[-(\tilde{S}_1 + \tilde{S}_2)/2] |\tilde{S}_1|^{n_1/2-p} |\tilde{S}_2|^{n_2/2-p} \cdot \int_{O(m)} {}_0F_1(n_1/2; \Omega_1 H'\tilde{S}_1 H/2) {}_0F_1(n_2/2; \Omega_2 H'\tilde{S}_2 H/2) dH$$

$$(9.2) \quad = \mathcal{K}(m; n_1, n_2) \operatorname{etr}(-\Omega_1 - \Omega_2) \cdot \operatorname{etr}[-(\tilde{S}_1 + \tilde{S}_2)/2] |\tilde{S}_1|^{n_1/2-p} |\tilde{S}_2|^{n_2/2-p} \cdot \sum_{\kappa,\lambda;\phi}^{\infty} C_\phi^{\kappa,\lambda}(\Omega_1, \Omega_2) C_\phi^{\kappa,\lambda}(\tilde{S}_1/2, \tilde{S}_2/2) / k! l! (n_1/2)_\kappa (n_2/2)_\lambda C_\phi(I)$$

$$(9.3) \quad = \mathcal{K}(m; n_1, n_2) \operatorname{etr}[-(\tilde{S}_1 + \tilde{S}_2)/2] |\tilde{S}_1|^{n_1/2-p} |\tilde{S}_2|^{n_2/2-p} \cdot \sum_{\kappa,\lambda;\phi}^{\infty} (-1)^f C_\phi^{\kappa,\lambda}(\Omega_1, \Omega_2) L_{\kappa,\lambda;\phi}^{n_1/2-p, n_2/2-p}(S_1/2, S_2/2) / k! l! (n_1/2)_\kappa (n_2/2)_\lambda C_\phi(I)$$

by (6.14), where $\mathcal{K}(m; n_1, n_2) = [2^{m(n_1+n_2)/2} \Gamma_m(n_1/2) \Gamma_m(n_2/2)]^{-1}$. Hence we obtain

$$(9.4) \quad f(\tilde{F}) = C_1(m; n_1, n_2) \operatorname{etr}(-\Omega_1 - \Omega_2) |\tilde{F}|^{n_1/2-p} |I + \tilde{F}|^{-(n_1+n_2)/2} \cdot \sum_{\kappa,\lambda;\phi}^{\infty} ((n_1+n_2)/2)_\phi C_\phi^{\kappa,\lambda}(\Omega_1, \Omega_2) C_\phi^{\kappa,\lambda}(\tilde{F}(I+\tilde{F})^{-1}, (I+\tilde{F})^{-1}) / k! l! (n_1/2)_\kappa (n_2/2)_\lambda C_\phi(I)$$

$$(9.5) \quad = C_1(m; n_1, n_2) |\tilde{F}|^{n_1/2-p} |I + \tilde{F}|^{-(n_1+n_2)/2} \cdot \sum_{\kappa,\lambda;\phi}^{\infty} (-1)^f C_\phi^{\kappa,\lambda}(\Omega_1, \Omega_2) \operatorname{E}_S \{L_{\kappa,\lambda;\phi}^{n_1/2-p, n_2/2-p}(\tilde{F}S/2, S/2)\} / k! l! (n_1/2)_\kappa (n_2/2)_\lambda C_\phi(I)$$

where $S \sim W_m(n_1+n_2, (I+\tilde{F})^{-1}, 0)$. From (5.5),

$$(9.6) \quad \operatorname{E}_S \{L_{\kappa,\lambda;\phi}^{n_1/2-p, n_2/2-p}(\tilde{F}S/2, S/2)\} = (n_1/2)_\kappa (n_2/2)_\lambda C_\phi(I) \sum_{\rho,\sigma;\tau} (-1)^{r+s} a_{\rho,\sigma;\tau}^{\kappa,\lambda;\phi} ((n_1+n_2)/2)_\tau \cdot C_\tau^{\rho,\sigma}(\tilde{F}(I+\tilde{F})^{-1}, (I+\tilde{F})^{-1}) / (n_1/2)_\rho (n_2/2)_\sigma C_\tau(I) .$$

Expressions for the joint distributions of the roots of F follow from (9.4) and (9.5). From (9.4) and (6.12)

$$(9.7) \quad f(\tilde{F}) = C_1(m; n_1, n_2) \operatorname{etr}(-\Omega_1 - \Omega_2) |\tilde{F}|^{n_1/2-p}$$

$$\cdot \sum_{\kappa,\lambda;\phi}^{\infty} ((n_1+n_2)/2)_\phi C_\phi^{\kappa,\lambda}(-\tilde{F}, I) L_{\kappa,\lambda;\phi}^{n_1/2-p}(\Omega_1, \Omega_2)$$
$$/k!l!(n_1/2)_\kappa(n_2/2)_\lambda C_\phi(I) .$$

Hence using (8.9) we obtain the distribution of Hotelling's $T=\operatorname{tr} F$ in the doubly noncentral case,

$$f(T)=C_3(m; n_1, n_2) \operatorname{etr}(-\Omega_1-\Omega_2) T^{mn_1/2-1} \sum_{k=0}^{\infty} \{(-T)^k/k!(mn_1/2)_k\}$$
$$\cdot \sum_{l=0}^{\infty} \sum_{\kappa,\lambda;\phi} ((n_1+n_2)/2)_\phi \theta_\phi^{\kappa,\lambda} L_{\kappa,\lambda;\phi}^{n_1/2-p}(\Omega_1, \Omega_2)/l!(n_2/2)_\lambda . \tag{9.8}$$

which also generalizes [3] equation (1).

(b) Case $n_1 \leqq m$. Consider $F=X_1'S_2^{-1}X_1$ where $X_1 \sim N_{m,n_1}(M_1, I_{mn_1})$, $M_1M_1'/2=\Omega_1$ and $S_2 \sim W_m(n_2, I, \Omega_2)$. The roots of F are invariant under $X_1 \to HX_1K$, $S_2 \to HS_2H'$ ($H \in O(m)$, $K \in O(n_1)$), so averaging over H, K

$$f(X_1, S_2)=C_5(m; n_1, n_2) \operatorname{etr}(-\Omega_1-\Omega_2)$$
$$\cdot \operatorname{etr}(-X_1X_1'/2-S_2/2)|S_2|^{n_2/2-p}$$
$$\cdot \int_{O(m)} {}_0F_1(n_1/2; \Omega_1HX_1X_1'H'/2) {}_0F_1(n_2/2; \Omega_2HS_2H'/2)dH , \tag{9.9}$$

where $C_5(m; n_1, n_2)=[2^{m(n_1+n_2)/2}\pi^{mn_1/2}\Gamma_m(n_2/2)]^{-1}$. Let $Y=S_2^{-1/2}X_1$, so that $F=Y'Y$ and

$$f(Y, S_2)=C_5(m; n_1, n_2) \operatorname{etr}(-\Omega_1-\Omega_2) \operatorname{etr}\{-S_2(I+YY')/2\}$$
$$\cdot |S_2|^{(n_1+n_2)/2-p} \int_{O(m)} {}_0F_1(n_1/2; \Omega_1HS_2^{1/2}YY'S_2^{1/2}H'/2)$$
$$\cdot {}_0F_1(n_2/2; \Omega_2HS_2H'/2)dH . \tag{9.10}$$

Now

$$YY'=J_YF^*J_Y' , \tag{9.11}$$

where $J_Y=[YF^{-1/2}|J_2] \in O(m)$, $F^*=\begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$, and F^* is $m \times m$. Making the transformation

$$S_2^*=J_Y'S_2J_Y , \qquad H^*=HJ_Y \tag{9.12}$$

and integrating over $S_2^*>0$, we finally obtain

$$f(F)=[\Gamma_m((n_1+n_2)/2)/\Gamma_{n_1}(m/2)\Gamma_m(n_2/2)] \operatorname{etr}(-\Omega_1-\Omega_2)$$
$$\cdot |F|^{(m-n_1-1)/2}|I+F^*|^{-(n_1+n_2)/2}$$
$$\cdot \mathrm{E}_{S_2^*} \int_{O(m)} {}_0F_1(n_1/2; \Omega_1HS_2^{*1/2}F^*S_2^{*1/2}H'/2)$$
$$\cdot {}_0F_1(n_2/2; \Omega_2HS_2^*H'/2)dH \tag{9.13}$$

where $S_2^* \sim W_m(n_1+n_2, (I+F^*)^{-1}, 0)$. Forms corresponding to (9.4), (9.5) and (9.7) may now be obtained.

Appendix

Orthonormal polynomials $z_\phi^{-1/2} C_\phi^{\kappa,\lambda}(X, Y)$, where $z_\phi=\chi_{[2\phi]}(1)/1\cdot 3\cdots(2f-1)$. C=multiplier. (X)=tr X.

$f\ k\ l$			
2 1 1	κ, λ	1, 1	
	ϕ	2	1^2
	C^2	1/3	2/3
	(XY)	2	-1
	$(X)(Y)$	1	1

$f\ k\ l$					
3 2 1	κ, λ	2, 1		1^2, 1	
	ϕ	3	21	21	1^3
	C^2	1/15	4/15	1/3	1/3
	(X^2Y)	8	-2	-2	2
	$(X^2)(Y)$	2	2	-1	-1
	$(XY)(X)$	4	-1	2	-2
	$(X)^2(Y)$	1	1	1	1

$f\ k\ l$								
4 3 1	κ, λ	3, 1		21, 1			1^3, 1	
	ϕ	4	31	31	2^2	21^2	21^2	1^4
	C^2	1/105	2/35	2/15	2/15	1/3	1/5	2/15
	(X^3Y)	48	-8	-8	-2	4	4	-6
	$(X^3)(Y)$	8	8	-2	-2	-2	2	2
	$(X^2Y)(X)$	24	-4	6	-6	0	-4	6
	$(X^2)(XY)$	12	-2	-2	7	-2	-2	3
	$(X^2)(X)(Y)$	6	6	1	1	1	-3	-3
	$(XY)(X)^2$	6	-1	4	1	-2	2	-3
	$(X)^3(Y)$	1	1	1	1	1	1	1

$f\ k\ l$									
4 2 2	κ, λ	2, 2			$2, 1^2$		$1^2, 1^2$		
	ϕ	4	31	2^2	31	21^2	2^2	21^2	1^4
	C^2	1/105	8/189	8/135	2/27	4/27	2/27	32/135	2/15
	(X^2Y^2)	32	4	−8	−8	4	4	1	−4
	$(XYXY)$	16	−12	6	0	0	−6	3	−2
	$(X^2Y)(Y)$	16	2	−4	8	−4	−4	−1	4
	$(XY^2)(X)$	16	2	−4	−4	2	−4	−1	4
	$(X^2)(Y^2)$	4	4	4	−2	−2	1	1	1
	$(XY)^2$	8	−6	3	0	0	6	−3	2
	$(X^2)(Y)^2$	2	2	2	2	2	−1	−1	−1
	$(XY)(X)(Y)$	8	1	−2	4	−2	4	1	−4
	$(Y^2)(X)^2$	2	2	2	−1	−1	−1	−1	−1
	$(X)^2(Y)^2$	1	1	1	1	1	1	1	1

$f\ k\ l$													
5 4 1	κ, λ	4, 1		31, 1			$2^2, 1$		$21^2, 1$			1^4	
	ϕ	5	41	41	32	31^2	32	2^21	31^2	2^21	21^3	21^3	1^5
	C^2	1/945	8/945	1/35	16/315	1/9	2/45	4/45	8/63	8/45	8/35	4/45	2/45
	(X^4Y)	384	−48	−48	−8	16	−8	4	16	4	−12	−12	24
	$(X^4)(Y)$	48	48	−8	−8	−8	−2	−2	4	4	4	−6	−6
	$(X^3Y)(X)$	192	−24	32	−18	0	−24	12	−12	6	2	12	−24
	$(X^3)(XY)$	64	−8	−8	22	−8	−8	4	4	−8	4	4	−8
	$(X^2Y)(X^2)$	96	−12	−12	−2	4	28	−14	−8	−2	6	6	−12
	$(X^3)(X)(Y)$	32	32	4	4	4	−8	−8	−2	−2	−2	8	8
	$(X^2Y)(X)^2$	48	−6	22	−8	−2	4	−2	4	−8	4	−6	12
	$(X^2)^2(Y)$	12	12	−2	−2	−2	7	7	−2	−2	−2	3	3
	$(X^2)(XY)(X)$	48	−6	8	13	−8	4	−2	−8	7	−1	−6	12
	$(X^2)(X)^2(Y)$	12	12	5	5	5	2	2	−1	−1	−1	−6	−6
	$(XY)(X)^3$	8	−1	6	1	−2	4	−2	4	1	−3	2	−4
	$(X)^4(Y)$	1	1	1	1	1	1	1	1	1	1	1	1

5 3 2	κ, λ	3, 2			21, 2				1^3, 2		3, 1^2		21, 1^2				1^3, 1^2		
	ϕ	5	41	32	41	32	31^2	2^21	31^2	21^3	41	31^2	32	31^2	2^21	21^3	2^21	21^3	1^5
	C^2	1/945	4/675	8/525	1/50	8/225	1/18	4/45	1/21	4/63	1/90	1/30	2/45	32/315	1/9	1/7	1/15	1/9	2/45
	(X^3Y^2)	192	48	−32	−32	−12	0	12	16	−12	−48	16	12	6	0	−8	−8	0	12
	(X^2YXY)	192	−96	24	−16	4	16	−8	0	0	0	0	−20	10	4	−4	12	−12	12
	$(X^3Y)(Y)$	96	24	−16	−16	−6	0	6	8	−6	48	−16	−12	−6	0	8	8	0	−12
	$(X^2Y^2)(X)$	96	24	−16	24	−16	8	−4	−16	12	−24	8	−4	−7	8	0	8	0	−12
	$(XYXY)(X)$	48	−24	6	16	−4	−16	8	0	0	0	0	−10	5	2	−2	−6	6	−6
	$(X^3)(Y^2)$	16	16	16	−4	−4	−4	−4	4	4	−8	−8	2	2	2	2	−2	−2	−2
	$(X^2Y)(XY)$	96	−48	12	−8	2	8	−4	0	0	0	0	20	−10	−4	4	−12	12	−12
	$(XY^2)(X^2)$	48	12	−8	−8	22	−8	−2	−8	6	−12	4	−2	4	−8	4	4	0	−6
	$(X^3)(Y)^2$	8	8	8	−2	−2	−2	−2	2	2	8	8	−2	−2	−2	−2	2	2	2
	$(X^2Y)(X)(Y)$	48	12	−8	12	−8	4	−2	−8	6	24	−8	4	7	−8	0	−8	0	12
	$(XY^2)(X)^2$	24	6	−4	16	6	0	−6	8	−6	−6	2	−6	−3	0	4	−4	0	6
	$(X^2)(XY)(Y)$	24	6	−4	−4	11	−4	−1	−4	3	12	−4	2	−4	8	−4	−4	0	6
	$(XY)^2(X)$	24	−12	3	8	−2	−8	4	0	0	0	0	10	−5	−2	2	6	−6	6
	$(X^2)(Y^2)(X)$	12	12	12	2	2	2	2	−6	−6	−6	−6	−1	−1	−1	−1	3	3	3
	$(X^2)(X)(Y)^2$	6	6	6	1	1	1	1	−3	−3	6	6	1	1	1	1	−3	−3	−3
	$(XY)(X)^2(Y)$	12	3	−2	8	3	0	−3	4	−3	6	−2	6	3	0	−4	4	0	−6
	$(Y^2)(X)^3$	2	2	2	2	2	2	2	2	2	−1	−1	−1	−1	−1	−1	−1	−1	−1
	$(X)^3(Y)^2$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

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