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THE DISTRIBUTION OF THE CHARACTERISTIC ROOTS OF $S_1S_2^{-1}$ UNDER VIOLATIONS IN THE COMPLEX CASE AND POWER COMPARISONS OF FOUR TESTS

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Summary

The joint density function of the latent roots of $S_1S_2^{-1}$ under violations is obtained where S_1 has a complex non-central Wishart distribution $W_c(p, n_1, \Sigma_1, \Omega)$ and S_2, an independent complex central Wishart, $W_c(p, n_2, \Sigma_2, 0)$. The density and moments of Hotelling's trace are also derived under violations. Further, the non-null distributions of the following four criteria in the two-roots case are studied for tests of three hypotheses: Hotelling's trace, Pillai's trace, Wilks' criterion and Roy's largest root. In addition, tabulations of powers are carried out and power comparisons for tests of each of three hypotheses based on the four criteria are made in the complex case extending such work of Pillai and Jayachandran in the classical Gaussian case. The findings in the complex Gaussian are generally similar to those in the classical.

1. Introduction

Consider the test of the following three hypotheses: 1) equality of covariance matrices in two p-variate complex normal populations, 2) equality of p-dimensional mean vectors in l p-variate complex normal populations having a common covariance matrix, and 3) independence between a p-set and a q-set of variates in a $(p+q)$-variate complex normal population. In order to study the robustness of tests of 1) and 3) when the assumption of normality is violated and 2) when that of a common covariance matrix is disturbed, the density of the characteristic roots of $S_1S_2^{-1}$ is studied, where S_1 has a complex non-central

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Wishart distribution and S_2 has an independently distributed complex central Wishart distribution. Further, the density function and moments of Hotelling's trace are also obtained under violations. In the real case, the density of the characteristic roots of $S_1S_2^{-1}$ under violations has been studied by Pillai [19].

Pillai and Sudjana [20] have carried out numerical study of the robustness of tests of 1) and 2) in the real two-roots case based on the four test criteria: i) Hotelling's trace, ii) Pillai's trace, iii) Wilks' criterion and iv) Roy's largest root. Pillai and Hsu [22], have made a similar study for 3). Earlier, Pillai and Jayachandran [16], [17] have made power comparisons of the tests of the three hypotheses based on the above four criteria in the two-roots case. In this paper a similar power comparison study is attempted in the complex case. For this numerical study, the non-null distributions of the four criteria in the two-roots complex case are obtained for each of the three hypotheses. The tabulations of the powers are made for various values of the parameters and selected degrees of freedom and are available in [21]. Finally, some findings from the tabulations have been discussed.

2. Preliminaries

In order to study the distribution problem of the characteristic roots of $S_1S_2^{-1}$ in the sequel, we introduce in this section a few notations and lemmas. Let A, B etc. be $p\times p$ Hermitian matrices. We call A positive definite or $A>0$ if $lA\bar{l}'>0$ for any $1\times p$ complex matrix $l\neq 0$, and $\tilde{C}_\kappa(A)$, [7], the complex zonal polynomial of A corresponding to partition $\kappa=(k_1,\cdots,k_p)$ of k where $k_1\geqq\cdots\geqq k_p\geqq 0$ and $\sum_{i=1}^{p} k_i=k$. Further we denote

$$(2.1)\qquad [a]_\kappa=\prod_{i=1}^{p}(a-i+1)_{k_i}\,,\qquad \text{where } (a)_k=a(a+1)\cdots(a+k-1)\,,$$

$$(2.2)\qquad \tilde{\Gamma}_p(a)=\pi^{p(p-1)/2}\prod_{i=1}^{p}\Gamma(a-i+1)$$

$$(2.3)\qquad \tilde{\Gamma}_p(a,\kappa)=\pi^{p(p-1)/2}\prod_{i=1}^{p}\Gamma(a+k_i-i+1)\,,$$

and

$$(2.4)\qquad \tilde{\Gamma}_p(a,-\kappa)=\pi^{p(p-1)/2}\prod_{i=1}^{p}\Gamma(a-p-k_i+i)\,.$$

We define complex hypergeometric functions of matrix argument as

$$(2.5)\qquad {}_n\tilde{F}_q(a_1,\cdots,a_n;b_1,\cdots,b_q;A)=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{[a_1]_\kappa\cdots[a_n]_\kappa}{[b_1]_\kappa\cdots[b_q]_\kappa}\frac{\tilde{C}_\kappa(A)}{k!}\,,$$

$$(2.6)\quad {}_n\tilde{F}_q(a_1,\cdots,a_n;b_1,\cdots,b_q;\boldsymbol{A},\boldsymbol{B})=\sum_{k=1}^{\infty}\sum_{\kappa}\frac{[a_1]_\kappa\cdots[a_n]_\kappa}{[b_1]_\kappa\cdots[b_q]_\kappa}\frac{\tilde{C}_\kappa(\boldsymbol{A})\tilde{C}_\kappa(\boldsymbol{B})}{k!\tilde{C}_\kappa(\boldsymbol{I})},$$

where $\boldsymbol{I}$ denotes the $p\times p$ identity matrix.

For special cases of (2.5) by (89), (90) of [7] we have

$$(2.7)\quad {}_0\tilde{F}_0(\boldsymbol{A})=\exp(\operatorname{tr}\boldsymbol{A})\quad\text{and}$$

$$(2.8)\quad {}_1\tilde{F}_0(a;\boldsymbol{A})=|\boldsymbol{I}-\boldsymbol{A}|^{-a}.$$

The splitting formula: (See (92) of [7]) is given by

$$(2.9)\quad \int_{U(p)}\tilde{C}_\kappa(\boldsymbol{AUB\bar{U}'})d\boldsymbol{U}=\frac{\tilde{C}_\kappa(\boldsymbol{A})\tilde{C}_\kappa(\boldsymbol{B})}{\tilde{C}_\kappa(\boldsymbol{I})},$$

where $U(p)$ is the unitary group of $p\times p$ Hermitian matrices.

LEMMA 2.1. *If $f(A)dA$ is the probability density of a Hermitian matrix variate $\boldsymbol{A}$ $(p\times p)$, then the distribution of the diagonal matrix $\boldsymbol{W}$ of the latent roots of $\boldsymbol{A}$, $\boldsymbol{A}=\boldsymbol{UW\bar{U}'}$ is*

$$(2.10)\quad \int_{U(p)}f(\boldsymbol{UA\bar{U}'})(d\boldsymbol{U})\frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)}\prod_{i>j}^{p}(\omega_i-\omega_j)^2d\omega_1\cdots d\omega_p$$

(see (93) of [7]).

LEMMA 2.2. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two $p\times p$ Hermitian matrices, then*

$$(2.11)\quad \int_{\bar{A}'=A>0}(\exp(-\operatorname{tr}\boldsymbol{A}))|A|^{a-p}\tilde{C}_\kappa(\boldsymbol{AB})d\boldsymbol{A}=\tilde{\Gamma}_p(a)[a]_\kappa\tilde{C}_\kappa(B)$$

(see (86) of [7]).

LEMMA 2.3. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $p\times p$ Hermitian matrices, then*

$$(2.12)\quad \int_{\bar{A}'=A>0}(\exp(-\operatorname{tr}\boldsymbol{A}))|\boldsymbol{A}|^{a-p}\tilde{C}_\kappa(\boldsymbol{BA}^{-1})d\boldsymbol{A}=\tilde{\Gamma}_p(a,-\kappa)\tilde{C}_\kappa(\boldsymbol{B})$$

(see (54) of [9]).

LEMMA 2.4. *Let $\boldsymbol{Z}$: $p\times p$ be a Hermitian matrix with characteristic roots $z_1\geqq z_2\geqq\cdots\geqq z_p$ such that the absolute values of z_i $(i=1,\cdots,p)$ are less than or equal to 1. Then*

$$(2.13)\quad \sum_{k=0}^{\infty}\sum_{\kappa}\tilde{L}_\kappa^r(\boldsymbol{S})\tilde{C}_\kappa(\boldsymbol{Z})/[k!\tilde{C}_\kappa(\boldsymbol{I})]$$
$$=|\boldsymbol{I}-\boldsymbol{Z}|^{-r-p}\int_{U(p)}\exp(-\operatorname{tr}\boldsymbol{SUZ}(\boldsymbol{I}-\boldsymbol{Z})^{-1}\boldsymbol{\bar{U}'})d\boldsymbol{U},$$

where $\boldsymbol{S}$ is an arbitrary $p\times p$ Hermitian matrix and

$$(2.14)\quad \tilde{L}_\kappa^r(\boldsymbol{S})=\exp(\operatorname{tr}\boldsymbol{S})\int_{\boldsymbol{R}>0}[\tilde{\Gamma}_p(r+p)]^{-1}{}_0\tilde{F}_1(r+p;-\boldsymbol{RS})$$

$$\cdot (\exp(-\operatorname{tr} \boldsymbol{R}))|\boldsymbol{R}|^r \tilde{C}_\kappa(\boldsymbol{R}) d\boldsymbol{R} \qquad (see\ [11]) .$$

LEMMA 2.5. *For any positive definite $p\times p$ Hermitian matrix $\boldsymbol{S}$, we have*

$$(2.15)\quad |\tilde{L}_\kappa^{n-p}(\boldsymbol{S})| \leqq [n]_\kappa \tilde{C}_\kappa(\boldsymbol{I}) \exp(\operatorname{tr} \boldsymbol{S}) , \quad \textit{where } n \geqq p .$$

PROOF. First consider (2.15) when $n=p$. By (91) of James [7], $|{}_0\tilde{F}_1(p; -\boldsymbol{RS})| \leqq 1$, then from (2.11) and (2.14) we get

$$(2.16)\quad \tilde{L}_\kappa^0(\boldsymbol{S}) \leqq [p]_\kappa \tilde{C}_\kappa(\boldsymbol{I}) \exp(\operatorname{tr} \boldsymbol{S}) .$$

Next, we want to show that

$$(2.17)\quad \tilde{L}_\kappa^\beta(\boldsymbol{S})/[\tilde{C}_\kappa(\boldsymbol{I})k!] = \sum_{t+i=k} \sum_\tau \frac{[\beta-r]_\tau}{t!} \sum_\nu \frac{\tilde{L}_\nu^r(\boldsymbol{S})\tilde{g}_{\nu,\tau}^\kappa}{i!\tilde{C}_\nu(\boldsymbol{I})} ,$$

where ν is a partition of i, τ is a partition of t and $\tilde{g}_{\nu,\tau}^\kappa$ is defined by

$$\tilde{C}_\tau(\boldsymbol{S})\tilde{C}_\nu(\boldsymbol{S}) = \sum_\kappa \tilde{g}_{\tau,\nu}^\kappa \tilde{C}_\kappa(\boldsymbol{S}) .$$

To show (2.17), multiply $|\boldsymbol{I}-\boldsymbol{Z}|^{-(\beta-r)}$ on both sides of (2.13). The right hand side becomes

$$(2.18)\quad \sum_{k=0}^\infty \sum_\kappa \tilde{L}_\kappa^\beta(\boldsymbol{S})\tilde{C}_\kappa(\boldsymbol{Z})/[k!C_\kappa(\boldsymbol{I})] .$$

The left hand side is $|\boldsymbol{I}-\boldsymbol{Z}|^{-(\beta-r)} \sum_{k=0}^\infty \sum_\kappa \tilde{L}_\kappa^r(\boldsymbol{S})\tilde{C}_\kappa(\boldsymbol{Z})/[k!\tilde{C}_\kappa(\boldsymbol{I})]$. Now, expanding $|\boldsymbol{I}-\boldsymbol{Z}|^{-(\beta-r)} = \sum_{t=0}^\infty \sum_\tau ([\beta-r]_\tau \tilde{C}_\tau(\boldsymbol{Z})/t!)$, then the left hand side becomes

$$(2.19)\quad \sum_{k=0}^\infty \sum_\kappa \frac{\tilde{L}_\kappa^r(\boldsymbol{S})}{k!\tilde{C}_\kappa(\boldsymbol{I})} \sum_{t=0}^\infty \sum_\tau \frac{[\beta-r]_\tau}{t!} \sum_\nu \tilde{g}_{\kappa,\tau}^\nu \tilde{C}_\nu(\boldsymbol{Z}) .$$

Comparing the corresponding coefficients of $\tilde{C}_\kappa(\boldsymbol{Z})$ in (2.18) and (2.19), we get (2.17). Putting $r=0$ in (2.17) and using (2.16), we get

$$(2.20)\quad \frac{\tilde{L}_\kappa^\beta(\boldsymbol{S})}{k!\tilde{C}_\kappa(\boldsymbol{I})} \leqq \sum_{t+i=k} \sum_\tau \frac{[\beta]_\tau}{t!} \frac{\tilde{g}_{\nu,\tau}^\kappa [p]_\nu \exp(\operatorname{tr} \boldsymbol{S})}{i!} = K \exp(\operatorname{tr} \boldsymbol{S}) ,$$

where K can be easily seen to be the coefficient of $\tilde{C}_\kappa(\boldsymbol{Z})$ in the expansion of $|\boldsymbol{I}-\boldsymbol{Z}|^{-\beta}|\boldsymbol{I}-\boldsymbol{Z}|^{-p}$, i.e., $|\boldsymbol{I}-\boldsymbol{Z}|^{-\beta-p}$. Hence $K=[\beta+p]_\kappa/k!$, putting $\beta=n-p$, we get (2.15).

LEMMA 2.6. *Let $\boldsymbol{A}$ be a $p\times p$ Hermitian matrix and $\boldsymbol{Z}=(z_{kj}+iz'_{kj})$ be a complex $p\times p$ matrix with z_{kj} and z'_{kj} real, and let them be non-singular. Define $\boldsymbol{V}$ and $\boldsymbol{W}$ as follows:*

$$v_{kj}=\frac{1}{2}(z_{kj}+z_{jk})\,,\quad v'_{kj}=\frac{1}{2}(z'_{kj}-z'_{jk})\quad\text{and}\quad \boldsymbol{V}=(v_{kj}+iv'_{kj})$$

$$w_{kj}=\frac{1}{2}(z'_{kj}+z'_{jk})\,,\quad w'_{kj}=\frac{1}{2}(z_{jk}-z_{kj})\quad\text{and}\quad \boldsymbol{W}=(w_{kj}+iw'_{kj})\,.$$

Then $\boldsymbol{Z}=\boldsymbol{V}+i\boldsymbol{W}$ where $\boldsymbol{V}$ and $\boldsymbol{W}$ are Hermitian. Now, if $f(\boldsymbol{A})$ is an analytic function such that

$$(2.21)\quad \int_{\boldsymbol{A}>0}(\exp(-\operatorname{tr}\boldsymbol{AZ}))f(\boldsymbol{A})d\boldsymbol{A}=g(\boldsymbol{Z})$$

for all $\boldsymbol{V}>0$ and $g(\boldsymbol{Z})$ satisfies the following conditions:

(i) $\int_{\boldsymbol{V}=\boldsymbol{X}_0>0}|g(\boldsymbol{Z})|d\boldsymbol{Z}<\infty$ *for all $\boldsymbol{X}_0>0$, and*

(ii) $\lim_{\boldsymbol{X}_0\to\infty}\int_{\boldsymbol{V}=\boldsymbol{X}_0}|g(\boldsymbol{Z})|d\boldsymbol{Z}=0,$

then we have the Cauchy inversion formula:

$$(2.22)\quad f(\boldsymbol{A})=\frac{(2)^{p(p-1)}}{(2\pi i)^{p^2}}\int_{\boldsymbol{V}=\boldsymbol{X}_0>0}(\exp(\operatorname{tr}\boldsymbol{AZ}))g(\boldsymbol{Z})d\boldsymbol{Z}\,.$$

PROOF. This follows from the inverse Laplace transform theorem for several variables. We only pay attention to $\operatorname{tr}\boldsymbol{AZ}=\operatorname{tr}\boldsymbol{A}(\boldsymbol{V}+i\boldsymbol{W})=\operatorname{tr}\boldsymbol{AV}+i\operatorname{tr}\boldsymbol{AW}$. Let $\boldsymbol{A}=(a_{kj}+ib_{kj})$, where a_{kj}, b_{kj} are real, then by the Hermitian property

$$\begin{aligned}(2.23)\quad \operatorname{tr}\boldsymbol{AZ}&=\sum_{k,j}(a_{kj}+ib_{kj})(v_{kj}+iv'_{kj})+i\sum_{k,j}(a_{kj}+ib_{kj})(w_{kj}+iw'_{kj})\\&=[\sum_{k}a_{kk}v_{kk}+2\sum_{k>j}a_{kj}v_{kj}+2\sum_{k>j}b_{kj}v'_{kj}]\\&\quad+i[\sum_{k}a_{kk}w_{kk}+2\sum_{k>j}a_{kj}w_{kj}+2\sum_{k>j}b_{kj}w'_{kj}]\,.\end{aligned}$$

So, in (2.21), since $\boldsymbol{A}>0$, $\boldsymbol{V}>0$, we have the real part of $\operatorname{tr}\boldsymbol{AZ}$ positive, which satisfies the condition for Laplace transform. Furthermore, there are p^2 variables in (2.21), namely a_{kj} for all $k\geqq j$ and b_{kj} for all $k>j$. From (2.23) we can easily see that we have to transform from W_{kj}, W'_{kj} to $2W_{kj}$, $2W'_{kj}$, clearly, the Jacobian is $2^{p(p-1)}$. So, the constant for inversion formula must be $2^{p(p-1)}/(2\pi i)^{p^2}$. The lemma follows from the inverse Laplace transform theorem (see Herz [6]).

Now let $K=2^{p(p-1)}/(2\pi i)^{p^2}$. Define

$$(2.24)\quad \tilde{L}_{\nu}^{n_1-p}(\boldsymbol{\Omega})=K\cdot\tilde{\Gamma}_p(n_1,\nu)\int_{\boldsymbol{V}=\boldsymbol{X}_0>0}(\exp(\operatorname{tr}\boldsymbol{Z}))|\boldsymbol{Z}|^{-n_1}\tilde{C}_{\nu}(\boldsymbol{I}-\boldsymbol{Z}^{-1}\boldsymbol{\Omega})d\boldsymbol{Z}\,.$$

This definition will be equivalent to (2.14).

Now, from Herz [6], $g(\boldsymbol{Z})=|\boldsymbol{Z}|^{-n}$ with $n\geqq 1$ satisfies (i) and (ii) of the above lemma and hence we have the following equation:

$$(2.25)\quad {}_0\tilde{F}_1(n_1;\, \boldsymbol{\Omega}^{1/2}\boldsymbol{A}_1\boldsymbol{\Omega}^{1/2}) = \tilde{\Gamma}_p(n_1)\cdot K \int_{V=X_0>0} (\exp(\operatorname{tr}\boldsymbol{Z}))|\boldsymbol{Z}|^{-n_1} \cdot (\exp(\operatorname{tr}\boldsymbol{Z}^{-1}))\boldsymbol{\Omega}^{1/2}\boldsymbol{A}_1\boldsymbol{\Omega}^{1/2}d\boldsymbol{Z}\,.$$

LEMMA 2.7. *Let H be a $p\times p$ positive definite Hermitian matrix, then the Jacobian of the transformation $\boldsymbol{H}\to\boldsymbol{H}^{-1}$ is $|\boldsymbol{H}|^{2p}$.*

PROOF. The lemma follows directly from the relation $d\boldsymbol{H}^{-1} = -\boldsymbol{H}^{-1}(d\boldsymbol{H})\boldsymbol{H}^{-1}$.

3. The distribution of the latent roots of $S_1S_2^{-1}$

Let S_1 $(p\times p)$ have a complex non-central Wishart distribution with n_1 d.f. and non-centrality $\boldsymbol{\Omega}$ and covariance matrix $\boldsymbol{\Sigma}_1$, denoted by $W_c(p, n_1, \boldsymbol{\Sigma}_1, \boldsymbol{\Omega})$ and S_2, an independent complex central Wishart distribution with n_2 d.f. and covariance matrix $\boldsymbol{\Sigma}_2$, $W_c(p, n_2, \boldsymbol{\Sigma}_2, 0)$. The densities of S_1 and S_2 are respectively given by, [7],

$$(3.1)\quad [\tilde{\Gamma}_p(n_1)|\boldsymbol{\Sigma}_1|^{n_1}]^{-1}(\exp(-\operatorname{tr}\boldsymbol{\Omega}))_0\tilde{F}_1(n_1,\, \boldsymbol{\Sigma}_1^{-1/2}\boldsymbol{\Omega}\boldsymbol{\Sigma}_1^{-1/2}\boldsymbol{S}_1) \cdot (\exp(\operatorname{tr}\boldsymbol{\Sigma}_1^{-1}\boldsymbol{S}_1))|\boldsymbol{S}_1|^{n_1-p}\,,$$

and

$$(3.2)\quad [\tilde{\Gamma}_p(n_2)|\boldsymbol{\Sigma}_2|^{n_2}]^{-1}(\exp(-\operatorname{tr}\boldsymbol{\Sigma}_2^{-1}\boldsymbol{S}_2))|\boldsymbol{S}_2|^{n_2-p}\,,$$

where all the matrices are Hermitian.

Let $\boldsymbol{R} = \operatorname{diag}(r_1, \cdots, r_p)$ where $0 < r_1 \leqq \cdots \leqq r_p < \infty$ are the latent roots of $R = \boldsymbol{S}_2^{-1/2}\boldsymbol{S}_1\boldsymbol{S}_2^{-1/2}$. (The same notation is used to denote a matrix $\boldsymbol{A}$ both before and after diagonalization as is a practice with many authors.) Now we prove the following theorem:

THEOREM 3.1. *Under the assumption that $\boldsymbol{\Lambda} = \boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1}$ is "random" ("Random" as defined by Pillai* [19] *denotes partial random i.e. diagonalization of the matrix by a unitary matrix $\boldsymbol{U}$ and integration of $\boldsymbol{U}$, in other words, putting a Haar prior on $\boldsymbol{U}$ leaving the latent roots non-random. See Appendix C.) the joint density of $r_1, r_2, \cdots, r_p$ is given by*

$$(3.3)\quad D(p, n_1, n_2)(\exp(-\operatorname{tr}\boldsymbol{\Omega}))|\boldsymbol{\Lambda}|^{-n_1}|\boldsymbol{R}|^{n_1-p}|\boldsymbol{I}+\lambda\boldsymbol{R}|^{-(n_1+n_2)} \cdot \prod_{i>j}^{p}(r_i-r_j)^2 \sum_{k=0}^{\infty}\sum_{\kappa} \frac{\tilde{C}_\kappa(\lambda\boldsymbol{R}(\boldsymbol{I}+\lambda\boldsymbol{R})^{-1})[n_1+n_2]_\kappa}{k!} \cdot \sum_{n=0}^{k}\sum_{\nu} \frac{(-\lambda^{-n})\tilde{a}_{\kappa,\nu}\tilde{C}_\nu(\boldsymbol{\Lambda}^{-1})\tilde{L}_\nu^{n_1-p}(\boldsymbol{\Omega})}{\tilde{C}_\nu(\boldsymbol{I})\tilde{C}_\nu(\boldsymbol{I})[n_1]_\nu}\,,$$

where $D(p, n_1, n_2) = \pi^{p(p-1)}\tilde{\Gamma}_p(n_1+n_2)/\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)\tilde{\Gamma}_p(p)$ where λ is a positive number, and $\boldsymbol{R} = \operatorname{diag}(r_1, \cdots, r_p)$ and $\tilde{a}_{\kappa,\nu}$ is defined by

$$(3.4)\qquad \tilde{C}_\kappa(I+A)/\tilde{C}_\kappa(I)=\sum_{n=0}^{k}\sum_{\nu}\tilde{a}_{\kappa,\nu}\tilde{C}_\nu(A)/\tilde{C}_\nu(I)\ ,$$

where ν *is a partition of* n. $\tilde{L}_\nu^{n_1-p}(\Omega)$ *is defined in Lemma* 2.6.

PROOF. By (3.1) and (3.2), the joint density of S_1 and S_2 is given by

$$[\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)|\Sigma_1|^{n_1}|\Sigma_2|^{n_2}]^{-1}\exp(-\operatorname{tr}\Omega)(\exp(-\operatorname{tr}\Sigma_1^{-1}S_1))|S_1|^{n_1-p}$$
$$\cdot {}_0\tilde{F}_1(n_1, \Sigma_1^{-1/2}\Omega\Sigma_1^{-1/2}S_1)(\exp(-\operatorname{tr}\Sigma_2^{-1}S_2))|S_2|^{n_2-p}\ .$$

Now make the transformation $A_1=\Sigma_1^{-1/2}S_1\Sigma_1^{-1/2}$ and $A_2=\Sigma_1^{-1/2}S_2\Sigma_1^{-1/2}$. The Jacobian is given by $|\Sigma_1|^{2p}$ and the joint density of A_1, A_2 is obtained as

$$(3.5)\qquad [\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)]^{-1}\exp(-\operatorname{tr}\Omega)\exp(-\operatorname{tr}A_1)\exp(-\operatorname{tr}\Lambda A_2)$$
$$\cdot|A_1|^{n_1-p}|A_2|^{n_2-p}|\Lambda|^{n_2}{}_0\tilde{F}_1(n_1; \Omega A_1)\ .$$

Now substitute (2.25) into (3.5) and transform $B_1=\Lambda^{1/2}A\Lambda^{1/2}$, $B_2=\Lambda^{1/2}A_2\cdot\Lambda^{1/2}$ and further transform $B_1=B_1$, $B_2=B_1^{1/2}R_1B_1^{1/2}$, then integrating B_1 over $B_1>0$ we get

$$(3.6)\qquad [K/\tilde{\Gamma}_p(n_2)](\exp(-\operatorname{tr}\Omega))|\Lambda|^{-n_1}|R_1|^{n_2-p}\int_{V=X_0>0}(\exp(\operatorname{tr}Z))|Z|^{-n_1}$$
$$\cdot\ \tilde{\Gamma}_p(n_1+n_2)|R_1+\Lambda^{-1/2}(I-\Omega^{1/2}Z^{-1}\Omega^{1/2})\Lambda^{-1/2}|^{-(n_1+n_2)}dZ\ .$$

Transform R_1 back to $W=R_1^{-1}$ (using Lemma 2.7). Let $W=UR\bar{U}'$ where U is unitary and R is diagonal, then integrate U over unitary group. We get:

$$(3.7)\qquad \frac{K\pi^{p(p-1)}\tilde{\Gamma}_p(n_1+n_2)}{\tilde{\Gamma}_p(n_2)\tilde{\Gamma}_p(p)}(\exp(-\operatorname{tr}\Omega))|\Lambda|^{-n_1}|R|^{n_1-p}\prod_{i>j}^{p}(r_i-r_j)^2$$
$$\cdot\int_{V=X_0>0}(\exp(\operatorname{tr}Z))|Z|^{-n_1}$$
$$\cdot\int_{U(p)}|I+UR\bar{U}'\Lambda^{-1/2}(I-\Omega^{1/2}Z^{-1}\Omega^{1/2})\Lambda^{-1/2}|^{-(n_1+n_2)}dUdZ\ .$$

Now, in view of the identities

$$(3.8)\qquad \begin{aligned}&|I+UR\bar{U}'B|=|I+\lambda R|\times|I-(I-\lambda^{-1}B)U(\lambda R)(I+\lambda R)^{-1}\bar{U}'|\ ,\\ &\int_{U(p)}|I-UV_1\bar{U}'V_2|^{-a}dU={}_1\tilde{F}_0(a; V_1, V_2)\ .\end{aligned}$$

Substituting (3.8) into (3.7) and expanding ${}_1\tilde{F}_0$, we get

$$(3.9)\qquad \frac{K\pi^{p(p-1)}\tilde{\Gamma}_p(n_1+n_2)}{\tilde{\Gamma}_p(n_2)\tilde{\Gamma}_p(p)}(\exp(-\operatorname{tr}\Omega))|\Lambda|^{-n_1}|R|^{n_1-p}|I+\lambda R|^{-(n_1+n_2)}$$
$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{C}_\kappa(\lambda R(I+\lambda R)^{-1})[n_1+n_2]_\kappa}{k!}\prod_{i>j}^{p}(r_i-r_j)^2$$

$$\cdot \sum_{n=0}^{k} \sum_{\nu} (-\lambda)^{-n} \tilde{a}_{\kappa,\nu} \int_{V=X_0>0} (\exp(\operatorname{tr} \boldsymbol{Z})) |\boldsymbol{Z}|^{-n_1}$$

$$\cdot \frac{\tilde{C}_\nu(\boldsymbol{\Lambda}^{-1}(\boldsymbol{I}-\boldsymbol{\Omega}^{1/2}\boldsymbol{Z}^{-1}\boldsymbol{\Omega}^{1/2}))}{\tilde{C}_\nu(\boldsymbol{I})} d\boldsymbol{Z} .$$

Now transform $\boldsymbol{\Lambda} \to \boldsymbol{U}\boldsymbol{\Lambda}\bar{\boldsymbol{U}}'$ and integrating $\boldsymbol{U}$ over $U(p)$ we get

$$\int_{U(p)} \tilde{C}_\nu(\boldsymbol{U}\boldsymbol{\Lambda}^{-1}\bar{\boldsymbol{U}}'(\boldsymbol{I}-\boldsymbol{\Omega}^{1/2}\boldsymbol{Z}^{-1}\boldsymbol{\Omega}^{1/2}))d\boldsymbol{U} = \frac{C_\nu(\boldsymbol{\Lambda}^{-1})C_\nu(\boldsymbol{I}-\boldsymbol{\Omega}^{1/2}\boldsymbol{Z}^{-1}\boldsymbol{\Omega}^{1/2})}{\tilde{C}_\nu(\boldsymbol{I})} . \tag{3.10}$$

Further, substitute (3.10) in (3.9) and integrate with respect to $\boldsymbol{Z}$ using (2.24). We get Theorem 3.1.

In order to discuss the convergence of series (3.3) let us note that series (3.3) (excluding the factors outside the summation) is dominated by the series

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{[n_1+n_2]_\kappa \tilde{C}_\kappa(\lambda \boldsymbol{R}(\boldsymbol{I}+\lambda \boldsymbol{R})^{-1})}{k!} \sum_{n=0}^{k} \sum_{\nu} \frac{\tilde{a}_{\kappa,\nu}\tilde{C}_\nu(-\lambda^{-1}\boldsymbol{\Lambda}^{-1}) \exp(\operatorname{tr} \boldsymbol{\Omega})}{\tilde{C}_\nu(\boldsymbol{I})}$$

in view of Lemma 2.5. Further, by (3.4)

$$\sum_{n=0}^{k} \sum_{\nu} \tilde{a}_{\kappa,\nu}\tilde{C}_\nu(-\lambda^{-1}\boldsymbol{\Lambda}^{-1})/\tilde{C}_\nu(\boldsymbol{I}) = \tilde{C}_\kappa(\boldsymbol{I}-\lambda^{-1}\boldsymbol{\Lambda}^{-1})/\tilde{C}_\kappa(\boldsymbol{I}) .$$

Hence (3.3) is dominated termwise by

$$D(p, n_1, n_2)|\boldsymbol{\Lambda}|^{-n_1}|\boldsymbol{R}|^{n_1-p}|\boldsymbol{I}+\lambda \boldsymbol{R}|^{-(n_1+n_2)} \prod_{i>j} (r_i - r_j)^2$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[n_1+n_2]_\kappa}{k!\tilde{C}_\kappa(\boldsymbol{I})} \tilde{C}_\kappa(\boldsymbol{I}-\lambda^{-1}\boldsymbol{\Lambda}^{-1})\tilde{C}_\kappa(\lambda \boldsymbol{R}(\boldsymbol{I}+\lambda \boldsymbol{R})^{-1}) ,$$

which is independent of $\boldsymbol{\Omega}$ and $\tilde{a}_{\kappa,\nu}$ coefficients and is in fact the joint density function of the characteristic roots of $\boldsymbol{S}_1\boldsymbol{S}_2^{-1}$ in the complex Gaussian case given by Khatri [10]. The choice of λ can be made to improve the convergence of the series in question. The distributional form above is also useful for testing the hypothesis $\lambda\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$.

Special cases of Theorem 3.1

(a) For $\boldsymbol{\Omega} = \boldsymbol{0}$ from (2.24)

$$\frac{\tilde{L}_\nu^{n_1-p}(\boldsymbol{\Omega})}{\tilde{C}_\nu(\boldsymbol{I})} = \tilde{\Gamma}_p(n_1, \nu)K \int_{V=X_0>0} (\exp(\operatorname{tr} \boldsymbol{Z}))|\boldsymbol{Z}|^{-n_1} \frac{\tilde{C}_\nu(\boldsymbol{I}-\boldsymbol{Z}^{-1}\boldsymbol{\Omega})}{\tilde{C}_\nu(\boldsymbol{I})} d\boldsymbol{Z} \tag{3.11}$$

$$= \tilde{\Gamma}_p(n_1, \nu)K \sum_{d=0}^{n} \sum_{\delta} \frac{\tilde{a}_{\nu,\delta}(-1)^i}{\tilde{C}_\nu(\boldsymbol{I})} \int_{V=X_0>0} (\exp(\operatorname{tr} \boldsymbol{Z}))|\boldsymbol{Z}|^{-n_1}$$

$$\cdot \tilde{C}_\delta(\boldsymbol{Z}^{-1}\boldsymbol{\Omega})d\boldsymbol{Z}$$

$$= \tilde{\Gamma}_p(n_1)[n_1]_\nu \sum_{d=0}^{n} \sum_{\delta} \frac{(-1)^d \tilde{a}_{\nu,\delta}}{\tilde{C}_\delta(\boldsymbol{I})} \frac{1}{\tilde{\Gamma}_p(n_1)} \tilde{C}_\delta(\boldsymbol{\Omega}) ,$$

where ν is a partition of n and δ a partition of d. Hence, we have

$$\frac{\tilde{L}_\nu^{n_1-p}(0)}{\tilde{C}_\nu(\boldsymbol{I})[n_1]_\nu}=1 .$$

Now putting $\lambda=1$ and $\boldsymbol{\Omega}=\boldsymbol{0}$ in (3.3) we get

$$(3.12)\quad D(p, n_1, n_2)|\boldsymbol{A}|^{-n_1}|\boldsymbol{R}|^{n_1-p}|\boldsymbol{I}+\boldsymbol{R}|^{-(n_1+n_2)}\prod_{i>j}^{p}(r_i-r_j)^2 \cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{C}_\kappa(\boldsymbol{R}(\boldsymbol{I}+\boldsymbol{R})^{-1})[n_1+n_2]_\kappa}{k!}\sum_{n=0}^{k}\sum_{\nu}\frac{(-1)^n\tilde{a}_{\kappa,\nu}\tilde{C}_\nu(\boldsymbol{A}^{-1})}{\tilde{C}_\nu(\boldsymbol{I})}$$
$$=D(p, n_1, n_2)|\boldsymbol{A}|^{-n_1}|\boldsymbol{R}|^{n_1-p}|\boldsymbol{I}+\boldsymbol{R}|^{-(n_1+n_2)}\prod_{i>j}^{p}(r_i-r_j)^2 \cdot {}_1\tilde{F}_0(n_1+n_2, \boldsymbol{I}-\boldsymbol{A}^{-1}, \boldsymbol{R}(\boldsymbol{I}+\boldsymbol{R})^{-1}) .$$

This is exactly the same as Khatri's result [10].
(b) For $\boldsymbol{A}=\boldsymbol{I}$ and $\lambda=1$ in (3.3), and $\nu=\delta=\kappa$, we get the result of James [7], Khatri [10]

$$(3.13)\quad D(p, n_1, n_2)(\exp(-\operatorname{tr}\boldsymbol{\Omega}))\,{}_1\tilde{F}_1(n_1+n_2; n_1; \boldsymbol{\Omega}, \boldsymbol{R}(\boldsymbol{I}+\boldsymbol{R})^{-1}) \cdot|\boldsymbol{R}|^{n_1-p}|\boldsymbol{I}+\boldsymbol{R}|^{-(n_1+n_2)}\prod_{i>j}(r_i-r_j)^2 .$$

$\boldsymbol{\Omega}$ completely random

Let us consider now $\boldsymbol{\Omega}$ as a random matrix $\boldsymbol{\Sigma}_1^{-1/2}\boldsymbol{MYY'M'\Sigma}_1^{-1/2}$, where $\boldsymbol{YY'}$ has a complex central Wishart distribution $W_c(q, n_3, \boldsymbol{\Sigma}_3, 0)$, i.e.

$$(3.14)\quad [\tilde{\Gamma}_q(n_3)|\boldsymbol{\Sigma}_3|^{n_3}]^{-1}(\exp(-\operatorname{tr}\boldsymbol{\Sigma}_3^{-1}\boldsymbol{YY'}))|\boldsymbol{YY'}|^{n_3-q} .$$

Substituting (3.11) in (3.3) we get

$$(3.15)\quad D(p, n_1, n_2)(\exp(-\operatorname{tr}\boldsymbol{\Omega}))|\boldsymbol{A}|^{-n_1}|\boldsymbol{R}|^{n_1-p}|\boldsymbol{I}+\lambda\boldsymbol{R}|^{-(n_1+n_2)}\prod_{i>j}^{p}(r_i-r_j)^2 \cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{C}_\kappa(\lambda\boldsymbol{R}(\boldsymbol{I}+\lambda\boldsymbol{R}))[n_1+n_2]_\kappa}{k!}\sum_{n=0}^{k}\sum_{\nu}\frac{(-\lambda^{-1})^n\tilde{a}_{\kappa,\nu}\tilde{C}_\nu(\boldsymbol{A}^{-1})}{\tilde{C}_\nu(\boldsymbol{I})} \cdot\sum_{d=0}^{n}\sum_{\delta}\frac{(-1)^d\tilde{a}_{\nu,\delta}\tilde{C}_\delta(\boldsymbol{\Omega})}{\tilde{C}_\delta(\boldsymbol{I})} .$$

We let $\boldsymbol{\Omega}=\boldsymbol{\Sigma}_1^{-1/2}\boldsymbol{MYY'M'\Sigma}_1^{-1/2}$. Multiplying (3.14) by (3.15) and using (2.11) to integrate out $\boldsymbol{YY'}$ we get the joint density of $r_1,\cdots, r_p$ as

$$(3.16)\quad D(p, n_1, n_2)|\boldsymbol{A}|^{-n_1}|\boldsymbol{R}|^{n_1-p}|\boldsymbol{I}+\lambda\boldsymbol{R}|^{-(n_1+n_2)}|\boldsymbol{I}+\boldsymbol{\Omega}_1|^{-n_3}\prod_{i>j}^{p}(r_i-r_j)^2 \cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{C}_\kappa(\lambda\boldsymbol{R}(\boldsymbol{I}+\lambda\boldsymbol{R}))[n_1+n_2]_\kappa}{k!}\sum_{n=0}^{k}\sum_{\nu}\frac{(-\lambda^{-1})^n\tilde{a}_{\kappa,\nu}\tilde{C}_\nu(\boldsymbol{A}^{-1})}{\tilde{C}_\nu(\boldsymbol{I})}$$

$$\cdot \sum_{d=0}^{n} \sum_{\delta} \frac{(-1)^d \tilde{a}_{\nu,\delta}[n_3]_\delta \tilde{C}_\delta((\boldsymbol{I}+\boldsymbol{\Omega}_1)^{-1}\boldsymbol{\Omega}_1)}{[n_1]_\delta \tilde{C}_\delta(\boldsymbol{I})} .$$

If we let $\boldsymbol{Q}^2=(\boldsymbol{I}+\boldsymbol{\Omega}_1)^{-1}\boldsymbol{\Omega}_1$, $\lambda=1$, $n_1+n_2=n_3$, $\nu=\delta=k$, $\boldsymbol{\Lambda}=\boldsymbol{I}$ and $\boldsymbol{R}'=(\boldsymbol{I}+\boldsymbol{R})^{-1}\boldsymbol{R}=\mathrm{diag}(r_1',\cdots,r_p')$ where $r_i'=r_i/(1+r_i)$, by (3.16) we get

$$(3.17)\quad D(p, n_1, n_2)|\boldsymbol{R}'|^{n_1-p}|\boldsymbol{I}-\boldsymbol{R}'|^{n_3-n_1-p}|\boldsymbol{I}-\boldsymbol{Q}^2|^{n_3} \cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{([n_3]_\kappa)^2 \tilde{C}_\kappa(\boldsymbol{R}')\tilde{C}_\kappa(\boldsymbol{Q}^2)}{[n_1]_\kappa k! \tilde{C}_\kappa(\boldsymbol{I})} \prod_{i>j}(r_i'-r_j')^2 .$$

This is the same as the joint density of the canonical correlations in James [7] and Khatri [10].

4. Density function and moments of $U^{(p)}$

Density function of $U^{(p)}$

Let $U^{(p)}=U=\lambda \operatorname{tr} \boldsymbol{S}_1\boldsymbol{S}_2^{-1}$ where $\lambda>0$ and $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ are as defined in the previous section, then we have the following theorem:

THEOREM 4.1. *Under the assumption that* $\boldsymbol{\Omega}$ *is "random", the density function of* $U^{(p)}$ *is given by*

$$(4.1)\quad f(U)=[\tilde{\Gamma}_p(n_1+n_2)/\tilde{\Gamma}_p(n_2)]|\lambda\boldsymbol{\Lambda}|^{-n_1}(\exp(-\operatorname{tr}\boldsymbol{\Omega}))U^{n_1p-1} \cdot \sum_{k=0}^{\infty}\sum_{\kappa} \frac{[n_1+n_2]_\kappa(-U)^k \tilde{C}_\kappa(\lambda^{-1}\boldsymbol{\Lambda}^{-1})\tilde{L}_\kappa^{n_1-p}(\boldsymbol{\Omega})}{k!\tilde{C}_\kappa(\boldsymbol{I})\Gamma(n_1p+k)} ,$$

where $\boldsymbol{\Lambda}=\mathrm{diag}(\lambda_1,\cdots,\lambda_p)$, $\lambda_1,\cdots,\lambda_p$ *being the latent roots of* $\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1}$, $0<\lambda_1\leqq\cdots\leqq\lambda_p<\infty$ *and* $U\leqq\lambda\lambda_1$.

PROOF. We start from (3.5). The Laplace transform of $U=\lambda \operatorname{tr} \boldsymbol{S}_1\boldsymbol{S}_2^{-1}=\lambda\operatorname{tr}\boldsymbol{A}_1\boldsymbol{A}_2^{-1}$ is given by

$$\mathrm{E}(e^{-tU})=[\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)]^{-1}(\exp(-\operatorname{tr}\boldsymbol{\Omega}))|\boldsymbol{\Lambda}|^{n_2} \cdot \int_{\boldsymbol{A}_2>0}(\exp(-\operatorname{tr}\boldsymbol{\Lambda}\boldsymbol{A}_2))|\boldsymbol{A}_2|^{n_2-p} \cdot \left(\int_{\boldsymbol{A}_1>0}(\exp(-\operatorname{tr}\boldsymbol{A}_1))|\boldsymbol{A}_1|^{n_1-p}(\exp(-t\lambda\operatorname{tr}\boldsymbol{A}_1\boldsymbol{A}_2^{-1})) \cdot {}_0\tilde{F}_1(n_1;\boldsymbol{\Omega}\boldsymbol{A}_1)d\boldsymbol{A}_1\right)d\boldsymbol{A}_2 .$$

Now using (2.11) to integrate out $\boldsymbol{A}_1$, we get

$$(4.2)\quad \mathrm{E}(e^{-tU})=g(t)=[\tilde{\Gamma}_p(n_2)]^{-1}(\exp(-\operatorname{tr}\boldsymbol{\Omega}))|\boldsymbol{\Lambda}|^{n_2} \cdot \int_{\boldsymbol{A}_2>0}(\exp(-\operatorname{tr}\boldsymbol{\Lambda}\boldsymbol{A}_2))|\boldsymbol{A}_2|^{n_2-p}|\boldsymbol{I}+t\lambda\boldsymbol{A}_2^{-1}|^{-n_1} \cdot (\exp(\operatorname{tr}(\boldsymbol{I}+t\lambda\boldsymbol{A}_2^{-1})\boldsymbol{\Omega}))d\boldsymbol{A}_2$$

$$=[\tilde{\Gamma}_p(n_2)]^{-1}(\exp(-\mathrm{tr}\,\boldsymbol{\Omega}))|\boldsymbol{\Lambda}|^{n_2}$$
$$\cdot\int_{\boldsymbol{A}_2>0}(\exp(-\mathrm{tr}\,\boldsymbol{\Lambda}\boldsymbol{A}_2))|\boldsymbol{A}_2|^{n_2-p}|t\lambda\boldsymbol{A}_2^{-1}|^{-n_1}|\boldsymbol{I}+(t\lambda)^{-1}\boldsymbol{A}_2|^{-n_1}$$
$$\cdot(\exp(\mathrm{tr}\,\boldsymbol{\Omega}(t\lambda)^{-1}\boldsymbol{A}_2(\boldsymbol{I}+(t\lambda)^{-1}\boldsymbol{A}_2)^{-1}))d\boldsymbol{A}_2\ .$$

Further let $\boldsymbol{\Omega}\rightarrow\boldsymbol{U}\boldsymbol{\Omega}\bar{\boldsymbol{U}}'$ and using (2.13) to integrate $\boldsymbol{U}$ over $U(p)$, (4.2) becomes

$$\text{(4.3)}\quad g(t)=[\tilde{\Gamma}_p(n_2)]^{-1}(\exp(-\mathrm{tr}\,\boldsymbol{\Omega}))|\boldsymbol{\Lambda}|^{n_2}\int_{\boldsymbol{A}_2>0}(\exp(-\mathrm{tr}\,\boldsymbol{\Lambda}\boldsymbol{A}_2))|\boldsymbol{A}_2|^{n_2-p}$$
$$\cdot\ |(t\lambda)^{-1}\boldsymbol{A}_2|^{n_1}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{L}_\kappa^{n_1-p}(\boldsymbol{\Omega})\tilde{C}_\kappa(-(t\lambda)^{-1}\boldsymbol{A}_2)}{k!\tilde{C}_\kappa(\boldsymbol{I})}d\boldsymbol{A}_2\ .$$

Now the density function of $U^{(p)}$, $f(U)$, is given by the inverse Laplace transform of $g(t)$, i.e.

$$\text{(4.4)}\quad f(U)=\frac{1}{2\pi i}\int_{R(t)>0}e^{tU}g(t)dt\ .$$

Noting that

$$\text{(4.5)}\quad (2\pi i)U^{n_1p+k-1}/\Gamma(n_1p+k)=\int_{R(t)>0}e^{tU}t^{-n_1p-k}dt\ ,$$

substitute (4.3) in (4.4), then using (4.5) we get

$$\text{(4.6)}\quad f(U)=[\tilde{\Gamma}_p(n_2)]^{-1}(\exp(-\mathrm{tr}\,\boldsymbol{\Omega}))|\boldsymbol{\Lambda}|^{n_2}\int_{\boldsymbol{A}_2>0}(\exp(-\mathrm{tr}\,\boldsymbol{\Lambda}\boldsymbol{A}_2))|\boldsymbol{A}_2|^{n_2-p}$$
$$\cdot\ U^{n_1p-1}|\lambda^{-1}\boldsymbol{A}_2|^{n_1}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{L}_\kappa^{n_1-p}(\boldsymbol{\Omega})\tilde{C}_\kappa(-\lambda^{-1}U\boldsymbol{A}_2)}{k!\tilde{C}_\kappa(\boldsymbol{I})\Gamma(n_1p+k)}d\boldsymbol{A}_2\ .$$

Using (2.15), the integral in (4.6) is bounded by

$$\tilde{\Gamma}_p(n_1+n_2)U^{n_1p-1}\lambda^{-n_1}(\exp(\mathrm{tr}\,\boldsymbol{\Omega}))\,{}_1F_0(n_1+n_2;\,-U(\lambda\boldsymbol{\Lambda})^{-1})\ ,$$

which is convergent when all the absolute values of the latent roots of $-\boldsymbol{U}(\lambda\boldsymbol{\Lambda})^{-1}$ are less than 1, this is true by our assumption $U\leqq\lambda\lambda_1$. Now, using (2.11) to integrate out $\boldsymbol{A}_2$ in (4.6), we get (4.1).

THEOREM 4.2. *Under the assumption $\boldsymbol{\Lambda}$ is "random", if $n_2\geqq p+k$ then the kth moment of $U^{(p)}$ is given by*

$$\text{(4.7)}\quad \mathrm{E}(U^k)=[\tilde{\Gamma}_p(n_2)]^{-1}\lambda^k\sum_{\kappa}\tilde{\Gamma}_p(n_2,\,-\kappa)\tilde{L}_\kappa^{n_1-p}(-\boldsymbol{\Omega})\frac{\tilde{C}_\kappa(\boldsymbol{\Lambda})}{\tilde{C}_\kappa(\boldsymbol{I})}\ .$$

PROOF. Since $U^k=(\lambda\,\mathrm{tr}\,\boldsymbol{S}_1\boldsymbol{S}_2^{-1})^k=\lambda^k\sum_{\kappa}\tilde{C}_\kappa(\boldsymbol{A}_1\boldsymbol{A}_2^{-1})$, then by (3.7) we have

$$\mathrm{E}(U^k)=[\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)]^{-1}(\exp(-\mathrm{tr}\,\boldsymbol{\Omega}))|\boldsymbol{\Lambda}|^{n_2}$$

$$\cdot \int_{A_1>0} (\exp(-\operatorname{tr} A_1))|A_1|^{n_1-p} {}_0\tilde{F}_1(n_1; \Omega A_1)$$
$$\cdot \left(\int_{A_2>0} (\exp(-\operatorname{tr} \Lambda A_2))|A_2|^{n_2-p}\lambda^k \sum_\kappa \tilde{C}_\kappa(A_1A_2^{-1})dA_2\right)dA_1 .$$

Using (2.12) to integrate out A_2, we get

$$\text{(4.8)} \quad \mathrm{E}(U^k) = [\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)]^{-1}(\exp(-\operatorname{tr} \Omega))\lambda^k$$
$$\cdot \sum_\kappa \int_{A_1>0} (\exp(-\operatorname{tr} A_1))|A_1|^{n_1-p} {}_0\tilde{F}_1(n_1; \Omega A_1)$$
$$\cdot \tilde{\Gamma}_p(n_2, -\kappa)\tilde{C}_\kappa(A_1\Lambda)dA_1 .$$

Now, making Λ "random" in (4.8), i.e. transforming $\Lambda \to U\Lambda\bar{U}'$ and integrating U over $U(p)$, by (2.9), we get (4.7).

Special cases for $\mathrm{E}(U^k)$

(i) If we let $\Lambda = I$ and $\lambda = 1$ in (4.7), we get

$$\text{(4.9)} \quad \mathrm{E}(U^k) = [\tilde{\Gamma}_p(n_2)]^{-1} \sum_\kappa \tilde{\Gamma}_p(n_2, -\kappa)\tilde{L}_\kappa^{n_1-p}(-\Omega) .$$

(4.9) can be shown to be (46) of Khatri [11] when his result is corrected for a factor $(-1)^k$.

(ii) If we let $\Omega = 0$ and $\lambda = 1$ in (4.7), we get

$$\text{(4.10)} \quad \mathrm{E}(U^k) = [\tilde{\Gamma}_p(n_2)]^{-1} \sum_\kappa \tilde{\Gamma}_p(n_2, -\kappa)[n_1]_\kappa \tilde{C}_\kappa(\Lambda) .$$

(4.10) can be easily shown to be (48) of Khatri [11].

5. Numerical results

In order to make power comparisons of the tests of each of the three hypotheses (see Introduction) based on the four criteria, we tabulated the powers by using the exact distributions of all four criteria in the two-roots case (which can be derived easily from (3.12), (3.13) and (3.16) by using the relation $\tilde{C}_k\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \sum_{r+2s=k} \tilde{b}_k(r, s)(a+b)^r(ab)^s$ where $\tilde{b}_k(r, s)$ are given in Appendix B). We discuss below the c.d.f.'s of the four criteria in the three cases as used for computation. In computing from these c.d.f.'s, zonal polynomials of degree 0 to 6 were used. Before computing the tail probability for specific values of the parameters, the total probability in that case over the whole range of the respective statistics for all terms included in the formula was calculated and the number of decimals included in the tables was determined depending on the number of places of accuracy obtained in the total probability, at least as many decimal places as in the table. Moreover, the total probability was computed by cumulating successively the probability contribution for each term from $k=0$ to 6 and noting the successive reduction in the contribution for each term. The c.d.f.'s used for computation are as follows:

1) For hypothesis 1), we have

$$\Pr\{U^{(2)}\leqq u\}=D(\lambda_1\lambda_2)^{-n_1}\sum_{k=0}^{\infty}\sum_{\kappa}\mathcal{V}_{k,\kappa}\sum_{r+2s=k}\tilde{b}_\kappa(r, s)\tilde{H}_{rs}(u) ,$$

$$\Pr\{V^{(2)}\leqq v\}=D(\lambda_1\lambda_2)^{-n_1}\sum_{k=0}^{\infty}\sum_{\kappa}\mathcal{V}_{k,\kappa}\sum_{r+2s=k}\tilde{b}_\kappa(r, s)\tilde{K}_{rs}(v) ,$$

$$\Pr\{W^{(2)}\leqq w\}=D(\lambda_1\lambda_2)^{-n_1}\sum_{k=0}^{\infty}\sum_{\kappa}\mathcal{V}_{k,\kappa}\sum_{r+2s=k}\tilde{b}_\kappa(r, s)\tilde{G}_{rs}(w) ,$$

and

$$\Pr\{L_{(2)}\leqq l\}=D(\lambda_1\lambda_2)^{-n_1}\sum_{k=0}^{\infty}\sum_{\kappa}\mathcal{V}_{k,\kappa}\sum_{r+2s=k}\tilde{b}_\kappa(r, s)\tilde{P}_{rs}(l) .$$

$\mathcal{V}_{k,\kappa}$ and $\tilde{b}_\kappa(r, s)$ are given in Appendix A.

2) For hypothesis 2), we have

$$\Pr\{U^{(2)}\leqq u\}=De^{-(\omega_1+\omega_2)}\sum_{k=0}^{\infty}\sum_{\kappa}\mathcal{M}_{k,\kappa}\sum_{r+2s=k}\tilde{b}_\kappa(r, s)\tilde{H}_{rs}(u) ,$$

$$\Pr\{V^{(2)}\leqq v\}=De^{-(\omega_1+\omega_2)}\sum_{k=0}^{\infty}\sum_{\kappa}\mathcal{M}_{k,\kappa}\sum_{r+2s=k}\tilde{b}_\kappa(r, s)\tilde{K}_{rs}(v) ,$$

$$\Pr\{W^{(2)}\leqq w\}=De^{-(\omega_1+\omega_2)}\sum_{k=0}^{\infty}\sum_{\kappa}\mathcal{M}_{k,\kappa}\sum_{r+2s=k}\tilde{b}_\kappa(r, s)\tilde{G}_{rs}(w) ,$$

and

$$\Pr\{L_{(2)}\leqq l\}=De^{-(\omega_1+\omega_2)}\sum_{k=0}^{\infty}\sum_{\kappa}\mathcal{M}_{k,\kappa}\sum_{r+2s=k}\tilde{b}_\kappa(r, s)\tilde{P}_{rs}(l) .$$

$\mathcal{M}_{k,\kappa}$ are given in Appendix A.

3) For hypothesis 3), we have

$$\Pr\{U^{(2)}\leqq u\}=D[(1-\rho_1^2)(1-\rho_2^2)]^n\sum_{k=0}^{\infty}\sum_{\kappa}\mathcal{F}_{k,\kappa}\sum_{r+2s=k}\tilde{b}_\kappa(r, s)\tilde{H}_{rs}(u) ,$$

$$\Pr\{V^{(2)}\leqq v\}=D[(1-\rho_1^2)(1-\rho_2^2)]^n\sum_{k=0}^{\infty}\sum_{\kappa}\mathcal{F}_{k,\kappa}\sum_{r+2s=k}\tilde{b}_\kappa(r, s)\tilde{K}_{rs}(v) ,$$

$$\Pr\{W^{(2)}\leqq w\}=D[(1-\rho_1^2)(1-\rho_2^2)]^n\sum_{k=0}^{\infty}\sum_{\kappa}\mathcal{F}_{k,\kappa}\sum_{r+2s=k}\tilde{b}_\kappa(r, s)\tilde{G}_{rs}(w) ,$$

and

$$\Pr\{L_{(2)}\leqq l\}=D[(1-\rho_1^2)(1-\rho_2^2)]^n\sum_{k=0}^{\infty}\sum_{\kappa}\mathcal{F}_{k,\kappa}\sum_{r+2s=k}\tilde{b}_\kappa(r, s)\tilde{P}_{rs}(l) .$$

$\mathcal{F}_{k,\epsilon}$ are given in Appendix A. $\tilde{H}_{rs}(u)$, $\tilde{K}_{rs}(v)$, $\tilde{G}_{rs}(w)$ and $\tilde{P}_{rs}(l)$ are available in [22].

Powers are tabulated for values of $\alpha=.05$, $m=0, 1, 2, 5$, $n=5, 15, 30, 40$, where $m=n_1-2$, $n=n_2-2$ and for various values of the parameters. For hypothesis 1) the tabulations are presented for $f_i=\lambda_i-1$, $i=1, 2$. All these tabulations and a table of upper/lower 5% points used for computing the powers are available in [22]. The following findings seem to emerge from the tabulations:

1. There is general agreement in the behaviour of the powers of each criterion in regard to the tests of the three hypotheses.
2. Relative performances of the criteria for each of the three hypotheses are also in general about the same in the three cases.
3. For small deviations from hypothesis, in general, Power$(V^{(2)})\geqq$ Power$(W^{(2)})\geqq$Power$(U^{(2)})\geqq$Power$(L_{(2)})$.
4. For constant sum of the roots, Power$(V^{(2)})$ increases generally as the two roots tend to be equal, Power$(W^{(2)})$ increases in most cases but those of $U^{(2)}$ and $L_{(2)}$ decrease.
5. For large deviations from the hypothesis, when the values of the roots are far apart, Power$(U^{(2)})\geqq$Power$(W^{(2)})\geqq$Power$(V^{(2)})$. But Power$(V^{(2)})\geqq$Power$(W^{(2)})\geqq$Power$(U^{(2)})\geqq$Power$(L_{(2)})$ when the values of the roots are close.
6. Power$(L_{(2)})$ stays below those of the other three criteria except for large deviations in which case Power$(L_{(2)})$ seems to exceed those of others when there is only one non-zero deviation.
7. The findings are in general agreement with those discussed by Pillai and Jayachandran in the real case [16], [17].
8. The powers for tests of hypotheses 1) and 2) are larger in the complex case than in the real for the same m, n and parameter value; on the other hand, for test of hypothesis 3) it is just the opposite.

It should be pointed out that, in the real case, for test 2), the admissibility of $U^{(p)}$ and $L_{(p)}$ has been established by Ghosh [5] for large values of the parameters in the alternative hypotheses i.e. against unrestricted alternatives and Schwartz [24] that of $V^{(p)}$ in the same sense. Kiefer and Schwartz [12] have shown that $V^{(p)}$ test is admissible Bayes, fully invariant, similar and unbiased. They have also shown that $W^{(p)}$ is admissible Bayes, under a restriction, although admissibility could be established without this restriction. Further, sufficient conditions on the procedure for the power function to be a monotonically increasing function of each of the parameters, for 1) are obtained by Anderson and Das Gupta, [2]; for 2), by Das Gupta, Anderson and Mudholkar, [3]; and for 3), by Anderson and Das Gupta, [1]. Furthermore, for 2) and 3) Mudholkar [14] has shown that the power functions of the members of a class of invariant tests based on statistics, which are symmetric gauge functions of increasing convex functions of the maximal invariants, are monotone increasing functions of relevant noncentrality parameters.

Tests for 2) and 3) based on $W^{(p)}$ and for 1) to 3) based on $U^{(p)}$ and $L_{(p)}$ have been shown by the above authors to have monotonicity property of power with respect to each population root. The monotonicity of the power function of $L_{(p)}$ has been demonstrated earlier by Roy and Mikhail [13], [23]. More recently Perlman [15] has shown that the power functions of the tests of 2) and 3) based on $V^{(p)}$ are monotonically increasing in each noncentrality parameter provided that the cut off point is not too large. Eaton and Perlman [4] have shown the Schur-convexity of the power functions of the test for 2) based on $U^{(p)}$ and $L_{(p)}$. He also proves the finding of Pillai and Jayachandran from numerical studies that the power functions of $L_{(p)}$ and $U^{(p)}$ increase as the rank of the centrality matrix decreases from p to 1.

Pillai and Li [18] have extended to the complex case the results on the monotonicity of power proved in the real case by the authors Anderson and Das Gupta [2]; Das Gupta, Anderson and Mudholkar [3], Anderson and Das Gupta [1], and Mudholkar [14]. However these results relate only to the monotonicity of power and do not serve to compare the powers of the tests of any of the three hypotheses based on the four criteria. Hence the relevance of this numerical study.

In regard to robustness of these criteria against violations of assumptions in tests of 1) to 3) in the complex case it may be conjectured that the findings in the real case may possibly apply in the complex case as well. This conjecture is brought forth here in view of the finding 7 above that the results of this paper on power comparisons are in general agreement with those discussed by Pillai and Jayachandran in the real case [16], [17]. In the event the conjecture is true the $V^{(p)}$-test may have some advantage over others [20], [22], in regard to the violation of the assumption of equality of covariance matrices in test of hypothesis 2) and that of normality in the test of hypothesis 3). However a separate study is needed in this direction for the verification of the conjecture.

Appendix A

$\mathcal{V}_{k,\kappa}$, $\mathcal{M}_{k,\kappa}$ and $\mathcal{F}_{k,\kappa}$ coefficients

$$E_{0,(0)}=1$$

$$E_{1,(1)}=\frac{1}{2}a_1d_{(1)}$$

$$E_{2,(2)}=(a_1^2-a_2)d_{(2)}/6$$

$$E_{2,(1,1)}=\frac{1}{2}a_2d_{(1,1)}$$

$$E_{3,(3)}=(a_1^3-2a_1a_2)d_{(3)}/24$$

$$E_{3,(2,1)}=a_1a_2d_{(2,1)}/12$$

$$E_{4,(4)}=(a_1^4-3a_1^2a_2+a_2^2)d_{(4)}/120$$

$$E_{4,(3,1)}=(a_1^2a_2-a_2^2)d_{(3,1)}/72$$

$$E_{4,(2,2)}=a_2^2d_{(2,2)}/24$$

$$E_{5,(5)}=(a_1^5-4a_1^3a_2+3a_1a_2^2)d_{(5)}/720$$

$$E_{5,(4,1)}=(a_1^3a_2-2a_1a_2^2)d_{(4,1)}/480$$

$$E_{5,(3,2)}=a_1a_2^2d_{(3,2)}/240$$

$$E_{6,(6)}=(a_1^6-5a_1^4a_2+6a_1^2a_2^2-a_2^3)d_{(6)}/5040$$

$$E_{6,(5,1)}=(a_1^4a_2-3a_1^2a_2^2+a_2^3)d_{(5,1)}/3600$$

$$E_{6,(4,2)}=(a_1^2a_2^2-a_2^3)d_{(4,2)}/2160$$

$$E_{6,(3,3)}=a_2^3d_{(3,3)}/720$$

where (i) $E_{k,\kappa}=\mathcal{V}_{k,\kappa}$ if we let $a_1=2-(1/\lambda_1+1/\lambda_2)$, $a_2=(1-1/\lambda_1)(1-1/\lambda_2)$ and $d_\kappa=[n_1+n_2]_\kappa$
(ii) $E_{k,\kappa}=\mathcal{M}_{k,\kappa}$ if we let $a_1=\omega_1+\omega_2$, $a_2=\omega_1\omega_2$ and $d_\kappa=[n_1+n_2]_\kappa/[n_1]_\kappa$
(iii) $E_{k,\kappa}=\mathcal{F}_{k,\kappa}$ if we let $a_1=\rho_1+\rho_2$, $a_2=\rho_1\rho_2$ and $d_\kappa=([n]_\kappa)^2/[n_1]_\kappa$.

Appendix B

The constants $\tilde{b}_\kappa(r, s)$ up to $k=6$ for the c.d.f. of the criteria in the two-roots case

κ	r \ s	0	1	2	κ	r \ s	0	1	2	3
(1)	1	1			(5)	5	1			
(2)	0		−1			3		−4		
	2	1				1			3	
(1^2)	0		1		(4,1)	3		4		
(3)	3	1				1			−8	
	1		−2		(3,2)	1			5	
(2,1)	1		2		(6)	6	1			
(4)	4	1				4		−5		
	2		−3			2			6	
	0			1		0				−1
(3,1)	2		3		(5,1)	1		5		
	0			−3		2			−15	
(2^2)	0			2		0				5
					(4,2)	2			9	
						0				−9
					(3^2)	0				5

Appendix C

For test concerning the noncentrality matrix $\boldsymbol{\Omega}$, $\boldsymbol{\Lambda}$ is assumed to be $\boldsymbol{I}$. However if we wish to see whether tests concerning $\boldsymbol{\Omega}$ are robust against the violation of the assumption of equality of covariance matrices we need to investigate the powers of the tests where $\boldsymbol{\Lambda}\neq\boldsymbol{I}$.

Here we discuss the consequence of the "partial randomness". In order to compute the power for studying the robustness we need to specify latent roots of both $\boldsymbol{\Sigma}_2^{-1/2}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1/2}$ and $\boldsymbol{\Sigma}_1^{-1/2}\boldsymbol{\mu}\boldsymbol{\mu}'\boldsymbol{\Sigma}_1^{-1/2}$ i.e. $\lambda_1,\cdots,\lambda_p$, and $w_1,\cdots,w_p$, respectively. Assume no "partial randomness" i.e. consider the usual classical invariant tests. Write $\boldsymbol{\Sigma}_2^{-1/2}\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1/2}=\sum_{i=1}^{p}\lambda_i\boldsymbol{\beta}_i\boldsymbol{\beta}_i'$, where λ_i is the ith characteristic root and $\boldsymbol{\beta}_i$, the corresponding characteristic vector and similarly $\boldsymbol{\Sigma}_1^{-1/2}\boldsymbol{\mu}\boldsymbol{\mu}'\boldsymbol{\Sigma}_1^{-1/2}=\sum_{i=1}^{p}w_i\boldsymbol{\Gamma}_i\boldsymbol{\Gamma}_i'$. Let us consider $\boldsymbol{\Gamma}_i$ and $\boldsymbol{\beta}_i$ known $(i=1,\cdots,p)$. Then

$$\boldsymbol{\mu}\boldsymbol{\mu}'=\left(\boldsymbol{\Sigma}_2^{1/2}\left(\sum_{i=1}^{p}\lambda_i\boldsymbol{\beta}_i\boldsymbol{\beta}_i'\right)\boldsymbol{\Sigma}_2^{1/2}\right)^{1/2}\left(\sum_{i=1}^{p}w_i\boldsymbol{\Gamma}_i\boldsymbol{\Gamma}_i'\right)\left(\boldsymbol{\Sigma}_2^{1/2}\left(\sum_{i=1}^{p}\lambda_i\boldsymbol{\beta}_i\boldsymbol{\beta}_i'\right)\boldsymbol{\Sigma}_2^{1/2}\right)^{1/2}$$

which expresses $\boldsymbol{\mu}\boldsymbol{\mu}'$ as a function of $\boldsymbol{\Sigma}_2$. If "partial random", then

$\mu\mu'$ is a function of Σ_2 and β_i's. This shows that the classical case is contained in the "partial random" case.

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