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SOME NEW PROPERTIES OF THE BECHHOFER-KIEFER-SOBEL STOPPING RULE

Y. L. TONG

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1. Introduction

For $k\geq 2$ consider k Koopman-Darmois populations with pdf's

$$f(x_i, \theta_i)=\exp\{T(x_i)Q(\theta_i)+R(x_i)+S(\theta_i)\}, \tag{1.1}$$

where T, Q, R, S are real-valued functions and $\theta_i \in \Omega$ (the parameter space) for all i. Let

$$\boldsymbol{\theta}=(\theta_1, \theta_2, \cdots, \theta_k) \tag{1.2}$$

be the vector of parameters and let

$$\theta_{[1]}\leq\theta_{[2]}\leq\cdots\leq\theta_{[k]} \tag{1.3}$$

denote the ordered parameters. For selecting the population associated with the largest parameter $\theta_{[k]}$ Bechhofer-Kiefer-Sobel proposed a "Basic Ranking Procedure" in their book ([3], p. 114). The procedure, to be identified as the BKS Procedure in this paper, represents a multihypothesis extension of Wald's Sequential Probability Ratio Test [13]. It calls for sampling one vector at a time, and the stopping rule depends on a statistic $Q^*(\xi_m)$ which in turn depends on the sequential probability ratio. Let us denote

$$\boldsymbol{X}_j=(X_{1j}, X_{2j}, \cdots, X_{kj}) \tag{1.4}$$

and consider a sequence of independent random variables $\{\boldsymbol{X}_j\}_{j=1}^{\infty}$ with a common pdf $f(\boldsymbol{x}, \boldsymbol{\theta})=\prod_{i=1}^{k} f(x_i, \theta_i)$. Let $\delta^*>0$ and $P^* \in \left(\frac{1}{k}, 1\right)$ be predetermined real numbers. For every $m\geq 1$ define

$$Y_{im}=\sum_{j=1}^{m} T(X_{ij}), \qquad i=1, \cdots, k. \tag{1.5}$$

Let

$$Y_{[1]m}\leq Y_{[2]m}\leq\cdots\leq Y_{[k]m} \tag{1.6}$$

be the ordered Y's; and define

$$Q^*(\xi_m)=\exp(\delta^* Y_{[k]m})\Big/\sum_{i=1}^{k}\exp(\delta^* Y_{[i]m})\,. \tag{1.7}$$

Then the stopping variable N under the BKS Procedure is

$$N=\text{the smallest } n \text{ such that } Q^*(\xi_n)\geq P^*. \tag{1.8}$$

A more general theorem in their book ([3], p. 129) implies that the probability of a correct selection is at least P^* whenever $Q(\theta_{[k]})-Q(\theta_{[k-1]})\geq\delta^*$ holds ($Q(\theta)$ is assumed to be strictly increasing in θ).

The behavior of $\mathrm{E}\,N$ (the average sample number) has been studied rather extensively. In particular, a general lower bound on $\mathrm{E}\,N$ for multihypothesis testing problems ([3], p. 33 or [9]) can be applied; separate results on the behavior of $\mathrm{E}\,N$ as $\delta^*\to 0$ and as $P^*\to 1$ were given ([3], Sections 6.3 and 6.4). On the other hand, the problem of obtaining upper bounds on $\mathrm{E}\,N$ was stated as an open problem ([3], p. 336), and the distribution and the limiting behavior of the stopping variable N itself have not yet been investigated.

In this paper we obtain some new properties of both N and $\mathrm{E}\,N$. We first obtain bounds on the stopping variable N. The behavior of N can then be obtained from the behavior of its bounds. In particular, it is shown that, when $\theta_{[k]}>\theta_{[k-1]}$ holds, then the probability $\mathrm{P}_{\boldsymbol{\theta}}[N>n]$ converges to zero exponentially. Because of this fact we can obtain upper bounds on $\mathrm{E}_{\boldsymbol{\theta}}\,N$ (which depends on $\boldsymbol{\theta}$, δ^* and P^*). A limiting behavior of N as $\delta^*\to 0$ or as $P^*\to 1$ is also given. In particular, it's shown that under the slippage configuration

$$\lim_{\delta^*\to 0} N\Big/\left\{\frac{1}{\lambda_*}C(\delta^*, P^*)\right\}=\lim_{P^*\to 1} N\Big/\left\{\frac{1}{\lambda_*}C(\delta^*, P^*)\right\}=1 \quad \text{a.s.},$$

$$\lim_{\delta^*\to 0} \mathrm{E}_{\boldsymbol{\theta}}\, N\Big/\left\{\frac{1}{\lambda_*}C(\delta^*, P^*)\right\}=\lim_{P^*\to 1} \mathrm{E}_{\boldsymbol{\theta}}\, N\Big/\left\{\frac{1}{\lambda_*}C(\delta^*, P^*)\right\}=1$$

holds when suitable conditions are satisfied, where

$$\lambda_*=\mathrm{E}_{\theta_{[k]}}T(X)-\mathrm{E}_{\theta_{[k-1]}}T(X)\,, \qquad C(\delta^*, P^*)=\frac{1}{\delta^*}\ln\left[\frac{(k-1)P^*}{1-P^*}\right].$$

2. Some general inequalities and bounds on N

Because of the identity

$$1/Q^*(\xi_m)=1+\sum_{i=1}^{k-1}\exp\{-\delta^* U_{i,m}\}\,, \tag{2.1}$$

where $U_{i,m}=Y_{[k]m}-Y_{[i]m}$ $(i=1,\cdots,k-1)$, the stopping variable N defined in (1.8) can be rewritten as

$$N = \text{the smallest } n \text{ such that } Q_n^{**} \leq (1-P^*)/P^* \tag{2.2}$$

for

$$Q_m^{**} = \sum_{i=1}^{k-1} \exp\{-\delta^* U_{i,m}\}\,. \tag{2.3}$$

Denoting $\bar{Y}_m = \frac{1}{k}\sum_{i=1}^{k} Y_{im}$, clearly we have

$$\bar{U}_m = \frac{1}{k-1}\sum_{i=1}^{k-1} U_{i,m} = \frac{k}{k-1}(Y_{[k]m} - \bar{Y}_m)\,.$$

Now for

$$C(\delta^*, P^*) = \frac{1}{\delta^*}\ln\left[\frac{(k-1)P^*}{1-P^*}\right] \tag{2.4}$$

consider two auxiliary stopping variables N_1 and N_2 given by

$$N_1 = \text{the smallest } n \text{ such that } \bar{U}_n \geq C(\delta^*, P^*)\,, \tag{2.5}$$

$$N_2 = \text{the smallest } n \text{ such that } U_{k-1,n} \geq C(\delta^*, P^*)\,. \tag{2.6}$$

Those two stopping variables are obtained from (2.2) when the $U_{i,m}$'s are replaced by $\bar{U}_m$ and $U_{k-1,m}$, respectively. In practice we may not actually use them. Our main reason for introducing them is that they provide bounds and help establish the desired results.

THEOREM 2.1. *The inequality*

$$N_1 \leq N \leq N_2 \tag{2.7}$$

holds a.s. for every parameter vector $\boldsymbol{\theta}$.

PROOF. It suffices to show that

$$(k-1)\exp(-\delta^* \bar{U}_m) \leq Q_m^{**} \leq (k-1)\exp(-\delta^* U_{k-1,m}) \tag{2.8}$$

holds a.s. The inequality on the r.h.s. follows immediately from $U_{1,m} \geq \cdots \geq U_{k-1,m} \geq 0$ a.s. To show the other inequality we simply use the fact that given $(k-1)$ nonnegative real numbers their geometric mean is bounded above by their arithmetic mean.

Because of this theorem bounds on N can then be established from bounds on N_1 and N_2.

3. Upper bounds on $\mathrm{P}[N>n]$ and on $\mathrm{E}N$

We first impose a condition on the density function $f(x,\theta)$ given in (1.1).

Condition A. (a) $Q(\theta)$ is strictly increasing in θ; (b) $\mathrm{E}_\theta T(X)=\mu(\theta)$ (say) exists for all $\theta\in\Omega$ and $\mu(\theta)$ is strictly increasing in θ.

We immediately see that the family of densities $\{f(x,\theta):\theta\in\Omega\}$ has the monotone likelihood ratio property in $T(x)$ under (a). It follows that the corresponding family of distributions of $T(x)$ is a stochastically increasing family and $\mu(\theta)$ is nondecreasing in θ (see [8], Chapter 3). Hence if $\theta_1\neq\theta_2$ implies $\mu(\theta_1)\neq\mu(\theta_2)$, then (b) in Condition A follows immediately from (a). Denote

$$\lambda_i=\mu(\theta_{[k]})-\mu(\theta_{[i]})\qquad\text{for } i=1,\cdots,k-1 \tag{3.1}$$

and

$$\bar{\lambda}=\frac{1}{k-1}\sum_{i=1}^{k-1}\lambda_i\,. \tag{3.2}$$

Clearly $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{k-1}>0$ holds whenever $\theta_{[k]}>\theta_{[k-1]}$ holds. For notational convenience λ_{k-1} will be denoted by λ_*.

For $i=1,\cdots,k$ let us define $Y_{(i)m}$ the corresponding Y statistic given in (1.5) from the population with parameter $\theta_{[i]}$. It is obvious that for every $n=1,2,\cdots$

$$[N_2>n]=\bigcap_{m=1}^{n}\bigcup_{i=1}^{k-1}[U_{i,m}<C]\subset\bigcup_{i=1}^{k-1}[U_{i,n}<C]\,,$$

where $C=C(\delta^*,P^*)$ was defined in (2.4). Therefore for every fixed $\boldsymbol{\theta}$

$$\begin{aligned}\mathrm{P}_{\boldsymbol{\theta}}[N_2>n]&\leq 1-\mathrm{P}_{\boldsymbol{\theta}}[U_{k-1,n}>C]\\&\leq 1-\mathrm{P}_{\boldsymbol{\theta}}\left[\bigcap_{i=1}^{k-1}\left\{Z_{i,n}>\frac{C}{n}\right\}\right],\end{aligned} \tag{3.3}$$

where $Z_{i,n}=\frac{1}{n}(Y_{(k)n}-Y_{(i)n})$ $(i=1,\cdots,k-1)$. Now for every n the random variables $Z_{1,n},\cdots,Z_{k-1,n}$ are conditionally independent (for given value of $Y_{(k)n}$). Following from a similar argument used in [10] or Theorem 5.2.4 in [12] we have

$$\mathrm{P}_{\boldsymbol{\theta}}[N_2>n]\leq 1-\prod_{i=1}^{k-1}\mathrm{P}_{(\theta_{[i]},\theta_{[k]})}\left[Z_{i,n}>\frac{C}{n}\right]. \tag{3.4}$$

Note that for every $i=1,\cdots,k-1$, $\mathrm{E}\,Z_{i,n}=\lambda_i$, $Z_{i,n}\to\lambda_i$ a.s. as $n\to\infty$ and the distribution of $Z_{i,n}$ depends on $\boldsymbol{\theta}$ only through $(\theta_{[i]},\theta_{[k]})$.

We now give a result concerning large deviations for a stochastically increasing family of distribution functions. Let $\{F(v, \theta): \theta \in \Omega\}$ be a family of distributions with moment generating functions (m.g.f.) $\{\phi_\theta(t): \theta \in \Omega\}$, where

$$\phi_\theta(t) = \int \exp(tv) dF(v, \theta) . \tag{3.5}$$

Let $\{V_{1j}\}$, $\{V_{2j}\}$ be two sequences of independent and identically-distributed random variables with distributions $F(v, \theta')$ and $F(v, \theta'')$, respectively. For arbitrary but fixed $\varepsilon > 0$ define

$$\psi(t) = e^{-\varepsilon t} \phi_{\theta''}(t) \phi_{\theta'}(-t) , \tag{3.6}$$

which is the m.g.f. of $(V_{21} - V_{11} - \varepsilon)$.

THEOREM 3.1. *Assume that* $\phi_\theta(t)$ *(hence* $\psi(t)$*) exists for* $t \in (a, b)$ *for some* $a < 0$, $b > 0$ *(possibly* $\pm\infty$*) for all* θ, *and define*

$$\rho = \rho(\varepsilon, \theta', \theta'') = \inf_{t<0} \psi(t) . \tag{3.7}$$

If $\mathrm{E}_{(\theta', \theta'')}(V_{21} - V_{11}) > \varepsilon$, *then* (a) $\rho \in (0, 1)$ *and*

$$p_n = \mathrm{P}_{(\theta', \theta'')} \left[\frac{1}{n} \left(\sum_{j=1}^{n} V_{2j} - \sum_{j=1}^{n} V_{1j} \right) \leq \varepsilon \right] \leq \rho^n \tag{3.8}$$

holds for $n = 1, 2, \cdots$. (b) *If in addition the family of distribution functions is stochastically increasing, i.e., if for* $\theta' < \theta''$, $F(v, \theta') \geq F(v, \theta'')$ *holds for all* v, *then* ρ *is monotonically decreasing in* θ'' *and monotonically increasing in* θ' *for all* $\theta' < \theta''$. (c) *For fixed* θ' *and* θ'', ρ *is monotonically increasing in* ε.

PROOF. The proof of (a) follows immediately from Chernoff's Theorem (see [4] or [1]). To prove (b) we rewrite $\psi(t)$ in the form of

$$\psi(t) = e^{-\varepsilon t} (\mathrm{E}_{\theta''} \exp(tV_{21})) (\mathrm{E}_{\theta'} \exp(-tV_{11})) .$$

Clearly $\exp(tV_{21})$ (or $\exp(-tV_{11})$) is monotonically decreasing (or increasing) in V_{21} (or V_{11}) for $t < 0$. Therefore, applying a result in Lehmann ([8], p. 112), $\psi(t)$ is monotonically decreasing in θ'' and monotonically increasing in θ'. The proof now follows from the definition of ρ. The proof of (c) is similar.

Theorem 3.1 asserts that p_n converges to zero exponentially in n whenever the difference of the means is greater than ε. Also, if the two probability distributions move away from each other (i.e., the distance between θ' and θ'' becomes larger), then the upper bound on p_n will approach to zero at a faster rate (with a smaller ρ).

Now for $i=1, \cdots, k$ let $X_{(i)}$ have pdf $f(x, \theta_{[i]})$ defined in (1.1). For arbitrary but fixed ε in $(0, \lambda_i)$ (λ_i was given in (3.1)), define (for $i=1, \cdots, k-1$)

$$(3.9) \qquad \rho_i=\rho_i(\varepsilon, \boldsymbol{\theta})=\inf_{t<0} \{e^{-\varepsilon t}\phi_{\theta_{[k]}}(t)\phi_{\theta_{[i]}}(-t)\},$$

where $\phi_{\theta_{[i]}}(t)$ is the m.g.f. of $T(X_{(i)})$, and

$$(3.10) \qquad M_*=M_*(\varepsilon)=\text{the smallest integer} \geq C(\delta^*, P^*)/\varepsilon .$$

Before stating a main result we first observe a lemma.

LEMMA 3.1. *Assume that $\phi_\theta(t)$, the m.g.f. of $T(X)$ with parameter θ, exists for $t \in (a, b)$ for some $a<0$, $b>0$ (possibly $\pm\infty$) for all $\theta \in \Omega$, and that Condition* A *holds. If $\boldsymbol{\theta}$ satisfies $\theta_{[k]}-\theta_{[k-1]}>0$, then* (a) *the inequality*

$$(3.11) \qquad \mathrm{P}_{(\theta_{[i]}, \theta_{[k]})}\left[Z_{i,n}\leq \frac{C}{n}\right]\leq \rho_i^n$$

holds for all $n>M_*$ for all $i=1, \cdots, k-1$; (b) *the ρ_i's satisfy*

$$(3.12) \qquad 0<\rho_1\leq \cdots \leq \rho_{k-1}<1$$

for every ε in $(0, \lambda_*)$.

PROOF. It is easy to check that Theorem 3.1 applies. The proof follows immediately from that theorem and the fact that $(C(\delta^*, P^*)/n) \leq \varepsilon$ holds for all $n \geq C(\delta^*, P^*)/\varepsilon$, and that ρ_i depends on $\boldsymbol{\theta}$ only through $\theta_{[k]}$ and $\theta_{[i]}$.

For notational convenience ρ_{k-1} will be denoted by ρ_*. We now prove a theorem concerning the behavior of the BKS stopping variable N.

THEOREM 3.2. *Assume that the conditions stated in Lemma 3.1 are met. If $\boldsymbol{\theta}$ satisfies $\theta_{[k]}>\theta_{[k-1]}$, then the following statements are true:* (a) *For every ε in $(0, \lambda_*)$ and for $\rho_i=\rho_i(\varepsilon, \boldsymbol{\theta})$ defined in* (3.9),

$$(3.13) \qquad \mathrm{P}_{\boldsymbol{\theta}}[N>n]\leq 1-\prod_{i=1}^{k-1}(1-\rho_i^n)\leq \sum_{i=1}^{k-1}\rho_i^n$$

holds for every $n>M_*$. In particular,

$$(3.14) \qquad \mathrm{P}_{\boldsymbol{\theta}}[N>n]\leq 1-(1-\rho_*^n)^{k-1}=\sum_{r=1}^{k-1}(-1)^{r-1}\binom{k-1}{r}\rho_*^{rn}\leq (k-1)\rho_*^n$$

holds for all $n>M_*$. (b) *The average sample number is bounded above by*

$$\text{(3.15)} \qquad E_{\boldsymbol{\theta}} N \leq \sum_{i=1}^{k-1} \inf_{0<\varepsilon<\lambda_i} \{1+(C(\delta^*, P^*)/\varepsilon)+1/[1-\rho_i(\varepsilon, \boldsymbol{\theta})]\}$$

and

$$\text{(3.16)} \quad E_{\boldsymbol{\theta}} N \leq \inf_{0<\varepsilon<\lambda_*} \left\{1+(C(\delta^*, P^*)/\varepsilon)+\sum_{r=1}^{k-1}(-1)^{r-1}\binom{k-1}{r}\Big/[1-\rho_*^r(\varepsilon, \boldsymbol{\theta})]\right\}$$
$$\leq \inf_{0<\varepsilon<\lambda_*} \{1+(C(\delta^*, P^*)/\varepsilon)+(k-1)/[1-\rho_*(\varepsilon, \boldsymbol{\theta})]\} .$$

PROOF. (3.13) follows from (2.7) and (3.11), (3.14) follows from (3.13) and (3.12), (3.15) and (3.16) follow from $E_{\boldsymbol{\theta}} N = \sum_{n=1}^{\infty} P_{\boldsymbol{\theta}}[N \geq n]$, (3.13) and (3.14).

Remarks. (a) The inequality given in (3.14) asserts that the probability $P_{\boldsymbol{\theta}}[N>n]$ approaches to zero exponentially in n. Since for $k=2$ the BKS stopping rule N is equivalent to the Sequential Probability Ratio Test, the finding is consistent with existing results in this special case (see, e.g., Wijsman [14] and Lai [7]).

(b) The upper bounds on $E_{\boldsymbol{\theta}} N$ given in (3.15) and (3.16) offer a solution to an open problem in the book of Bechhofer-Kiefer-Sobel ([3], p. 336). Recently Huang [6] made an attempt to solve this problem. Unfortunately the proof of his result contains a key error which cannot be patched up, and a counter-example to his proof was given by Tong [11]. For details, see Tong [11].

Example. Consider k normal distributions with a mean vector $\boldsymbol{\theta}$ satisfying $\theta_{[1]} = \cdots = \theta_{[k-1]}$ and a common known variance σ^2. Let $\lambda_* = \theta_{[k]} - \theta_{[k-1]} > 0$ hold. Then from (3.9) one has, for every ε in $(0, \lambda_*)$,

$$\rho_* = \inf_{t<0} \{\exp[(\lambda_* - \varepsilon)t + \sigma^2 t^2]\} = \exp(-(\lambda_* - \varepsilon)^2/4\sigma^2) .$$

Therefore for every $\boldsymbol{\theta}$ satisfying $\lambda_* > 0$ we have

$$\text{(3.17)} \quad E_{\boldsymbol{\theta}} N \leq \inf_{0<\varepsilon<\lambda_*} \left\{1+(C(\delta^*, P^*)/\varepsilon) + \sum_{r=1}^{k-1}(-1)^{r-1}\binom{k-1}{r}\left[1-\exp\left(-\left(\frac{\lambda_*-\varepsilon}{2\sigma}\right)^2\right]^{-1}\right\} ,$$

or

$$\text{(3.18)} \quad E_{\boldsymbol{\theta}} N \leq \inf_{0<\varepsilon<\lambda_*} \left\{1+(C(\delta^*, P^*)/\varepsilon)+(k-1)\left[1-\exp\left(-\left(\frac{\lambda_*-\varepsilon}{2\sigma}\right)^2\right]^{-1}\right\}$$

if the crude bound is to be used. With $k=4$, $P^*=0.99$, $\delta^*=1/4$, $\lambda_*=1/2$ and $\sigma=1$ an elementary calculation shows that the bounds in (3.17) and (3.18) are 196.7 and 246.0, respectively. Note that when λ_* becomes larger the upper bounds become smaller.

4. The limiting behavior of N and $\mathrm{E}\,N$

We shall now investigate the limiting behavior of the stopping variable N and the average sample number $\mathrm{E}\,N$ as $\delta^* \to 0$ or as $P^* \to 1$. For the sequences of random variables $\{U_{i,m}\}$ $(i=1, \cdots, k-1)$ and $\{\bar{U}_m\}$ defined in Section 2 we first observe an almost sure convergence property.

LEMMA 4.1. *Assume that Condition* A *holds. If* $\boldsymbol{\theta}$ *satisfies* $\theta_{[k]} > \theta_{[k-1]}$, *then*

$$\lim_{m\to\infty} \frac{1}{m} \bar{U}_m = \bar{\lambda} \quad a.s., \tag{4.1}$$

$$\lim_{m\to\infty} \frac{1}{m} U_{k-1,m} = \lambda_* \quad a.s. \tag{4.2}$$

PROOF. Without loss of generality assume that $\theta_i = \theta_{[i]}$ holds for $i=1, \cdots, k$; and let $\varepsilon' > 0$ be arbitrary but fixed. Then, by the Strong Law of Large Numbers, for every ω in the product sample space except possibly in a null set, there exists an $M' = M'(\omega, \varepsilon', \boldsymbol{\theta})$ such that

$$\left| \frac{1}{m} Y_{i,m}(\omega) - \mu(\theta_{[i]}) \right| < \varepsilon'/2$$

holds for $i=1, \cdots, k$ whenever $m > M'$. For $\varepsilon' < \lambda_*$ this implies $Y_{k,m}(\omega) = Y_{[k]m}(\omega)$ and

$$\bar{\lambda} - \varepsilon' \le \frac{1}{m} \bar{U}_m(\omega) = \frac{1}{m} Y_{k,m}(\omega) - \frac{1}{k-1} \sum_{i=1}^{k-1} \frac{1}{m} Y_{i,m}(\omega) \le \bar{\lambda} + \varepsilon'$$

for every $m > M'$. Those in turn imply (4.1). The proof of (4.2) is similar.

We now investigate the limiting behavior of the stopping variables N_1 and N_2 defined in (2.5), (2.6) respectively.

LEMMA 4.2. *Assume that Condition* A *holds. If* $\boldsymbol{\theta}$ *satisfies* $\theta_{[k]} > \theta_{[k-1]}$, *then*

$$\lim_{\delta^*\to 0} N_1 \Big/ \left\{ \frac{1}{\bar{\lambda}} C(\delta^*, P^*) \right\} = \lim_{P^*\to 1} N_1 \Big/ \left\{ \frac{1}{\bar{\lambda}} C(\delta^*, P^*) \right\} = 1 \quad a.s., \tag{4.3}$$

$$\lim_{\delta^*\to 0} N_2 \Big/ \left\{ \frac{1}{\lambda_*} C(\delta^*, P^*) \right\} = \lim_{P^*\to 1} N_2 \Big/ \left\{ \frac{1}{\lambda_*} C(\delta^*, P^*) \right\} = 1 \quad a.s. \tag{4.4}$$

PROOF. Clearly the stopping variable N_1 defined in (2.5) can be rewritten as

$$N_1 = \text{the smallest } n \text{ such that } \bar{\lambda}/(\bar{U}_n/n) \leq n \Big/ \left\{ \frac{1}{\bar{\lambda}} C(\delta^*, P^*) \right\}.$$

Since $C(\delta^*, P^*) \to \infty$ as $\delta^* \to 0$ or as $P^* \to 1$ and since (by Lemma 4.1) the sequence of random variables $\bar{\lambda}/(\bar{U}_n/n)$ converges to 1 a.s., the proof of (4.3) follows from Lemma 1 of [5]. The proof of (4.4) is similar.

THEOREM 4.1. *Assume that Condition* A *holds. If* $\boldsymbol{\theta}$ *satisfies* $\theta_{[k]} > \theta_{[k-1]}$, *then the BKS stopping variable* N *satisfies*

$$\liminf_{\delta^* \to 0} N \Big/ \left\{ \frac{1}{\bar{\lambda}\delta^*} \ln \left[\frac{(k-1)P^*}{1-P^*} \right] \right\} \geq 1 \quad a.s., \tag{4.5}$$

$$\liminf_{P^* \to 1} N \Big/ \left\{ \frac{1}{\bar{\lambda}\delta^*} \ln \left[\frac{(k-1)P^*}{1-P^*} \right] \right\} \geq 1 \quad a.s.\,; \tag{4.6}$$

$$\limsup_{\delta^* \to 0} N \Big/ \left\{ \frac{1}{\lambda_*\delta^*} \ln \left[\frac{(k-1)P^*}{1-P^*} \right] \right\} \leq 1 \quad a.s., \tag{4.7}$$

$$\limsup_{P^* \to 1} N \Big/ \left\{ \frac{1}{\lambda_*\delta^*} \ln \left[\frac{(k-1)P^*}{1-P^*} \right] \right\} \leq 1 \quad a.s. \tag{4.8}$$

Under the slippage configuration when $\boldsymbol{\theta}$ *is of the form* $\theta_{[1]} = \cdots = \theta_{[k-1]} < \theta_{[k]}$ *with* $\lambda_* = \mu(\theta_{[k]}) - \mu(\theta_{[i]})$ $(i=1, \cdots, k-1)$, *we have*

$$\lim_{\delta^* \to 0} N \Big/ \left\{ \frac{1}{\lambda_*\delta^*} \ln \left[\frac{(k-1)P^*}{1-P^*} \right] \right\} = 1 \quad a.s., \tag{4.9}$$

$$\lim_{P^* \to 1} N \Big/ \left\{ \frac{1}{\lambda_*\delta^*} \ln \left[\frac{(k-1)P^*}{1-P^*} \right] \right\} = 1 \quad a.s. \tag{4.10}$$

PROOF. The proof of the theorem follows immediately from Inequality (2.7), Lemmas 4.1 and 4.2.

We are now ready to prove a limiting property of the stopping variable N.

THEOREM 4.2. *Assume that Condition* A *holds. If* $\boldsymbol{\theta}$ *satisfies* $\theta_{[k]} > \theta_{[k-1]}$, *then, for* λ_* *and* $\bar{\lambda}$ *defined in* (3.1) *and* (3.2) *respectively,*

$$\liminf_{\delta^* \to 0} \mathrm{E}_{\boldsymbol{\theta}} N \Big/ \left\{ \frac{1}{\bar{\lambda}\delta^*} \ln \left[\frac{(k-1)P^*}{1-P^*} \right] \right\} \geq 1\,, \tag{4.11}$$

$$\liminf_{P^* \to 1} \mathrm{E}_{\boldsymbol{\theta}} N \Big/ \left\{ \frac{1}{\bar{\lambda}\delta^*} \ln \left[\frac{(k-1)P^*}{1-P^*} \right] \right\} \geq 1\,, \tag{4.12}$$

$$\limsup_{\delta^*\to 0} \mathrm{E}_{\boldsymbol{\theta}} N \Big/ \left\{ \frac{1}{\lambda_* \delta^*} \ln \left[\frac{(k-1)P^*}{1-P^*} \right] \right\} \leq 1 , \tag{4.13}$$

$$\limsup_{P^*\to 1} \mathrm{E}_{\boldsymbol{\theta}} N \Big/ \left\{ \frac{1}{\lambda_* \delta^*} \ln \left[\frac{(k-1)P^*}{1-P^*} \right] \right\} \leq 1 . \tag{4.14}$$

Under the slippage configuration when $\boldsymbol{\theta}$ *is of the form* $\theta_{[1]} = \cdots = \theta_{[k-1]} < \theta_{[k]}$, *we have*

$$\lim_{\delta^*\to 0} \mathrm{E}_{\boldsymbol{\theta}} N \Big/ \left\{ \frac{1}{\lambda_* \delta^*} \ln \left[\frac{(k-1)P^*}{1-P^*} \right] \right\} = 1 , \tag{4.15}$$

$$\lim_{P^*\to 1} \mathrm{E}_{\boldsymbol{\theta}} N \Big/ \left\{ \frac{1}{\lambda_* \delta^*} \ln \left[\frac{(k-1)P^*}{1-P^*} \right] \right\} = 1 . \tag{4.16}$$

PROOF. The proof of (4.11)–(4.14) follows from (4.5)–(4.8) and an application of Fatou's lemma. (4.15) and (4.16) follow from (4.11)–(4.14).

Remark. The result given in Theorem 4.2 also holds when both $\delta^* \to 0$ and $P^* \to 1$ take place at the same time. Note that Theorem 6.4.1 in [3] was obtained for fixed P^*, hence that Theorem does not apply when both δ^* and P^* are subject to vary.

UNIVERSITY OF NEBRASKA

REFERENCES

[1] Bahadur, R. R. (1971). *Some Limit Theorems in Statistics*, NSF Regional Conference Lecture Notes, SIAM, Philadelphia.

[2] Bahadur, R. R. and Rao, R. R. (1960). On deviations of the sample mean, *Ann. Math. Statist.*, **31**, 1015–1027.

[3] Bechhofer, R. E., Kiefer, J. and Sobel, M. (1968). *Sequential Identification and Ranking Procedures*, Institute of Mathematical Statistics Research Monograph Series, Vol. III, Chicago University Press, Chicago.

[4] Chernoff, H. (1952). A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations, *Ann. Math. Statist.*, **23**, 493–507.

[5] Chow, Y. S. and Robbins, H. (1965). On the asymptotic theory of fixed-width sequential confidence intervals for the mean, *Ann. Math. Statist.*, **36**, 457–462.

[6] Huang, D. Y. (1976). A note on an efficiency of the Bechhofer-Kiefer-Sobel sequential selection procedure, *Bull. Inst. Math.*, *Academia Sinica*, **4**, 249–252.

[7] Lai, T. L. (1975). Termination, moments and exponential boundedness of the stopping rule for certain invariant sequential probability ratio tests, *Ann. Statist.*, **3**, 581–598.

[8] Lehmann, E. L. (1959). *Testing Statistical Hypotheses*, John Wiley and Sons, Inc., New York.

[9] Simons, G. (1967). Lower bounds for average sample number of sequential multihypothesis tests, *Ann. Math. Statist.*, **38**, 1343–1364.

[10] Tong, Y. L. (1977). An ordering theorem for conditionally independent and identically distributed random variables, *Ann. Statist.*, **5**, 274–277.

[11] Tong, Y. L. (1979). Review of a paper by D. Y. Huang, *Mathematical Reviews*, **57**, 1423.

[12] Tong, Y. L. (1980). *Probability Inequalities in Multivariate Distributions*, Academic Press, Inc., New York.

[13] Wald, A. (1947). *Sequential Analysis*, John Wiley and Sons, Inc., New York.

[14] Wijsman, R. A. (1968). Bounds on the sample size distribution for a class of invariant sequential probability ratio tests, *Ann. Math. Statist.*, **39**, 1048-1056.