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EXTENSION OF THE INEQUALITY FOR THE VARIANCE OF AN ESTIMATOR BY BAYESIAN PROCESS

ANTONIO DORIVAL CAMPOS

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1. Introduction

Let $X_1, X_2, \cdots$ be a sequence of random variables identically distributed, whose distribution function depends on the parametric vector $\Theta \in \Omega_p \subset R^p$. Let $L_j(x/\Theta)$ and $G(\Theta)$, defined over Ω_p be the conditional likelihood function for

$$x=(x_1, x_2, \cdots, x_j)\ , \qquad j=1, 2, \cdots$$

and the distribution of Θ, respectively. Let us consider the same sequential process described by Wolfowitz [6] and also the existence of a estimating function $g_{N,h}(x, \theta_h)$ for θ_h, with $1\leqq h\leqq p$, defined by Godambe [4].

2. Regularity conditions

We now postulate the following regularity conditions to be satisfied by $L_j(x/\Theta)$ and $g_{N,h}(x, \theta_h)$

i) $\theta_h \in \Omega \subset R$ for $1\leqq h\leqq p$.

ii) There exists a set A_j, $j=1, 2, \cdots$, independent of θ_h, $\Pr(A_j)=0$, such that for all

$$(x_1, x_2, \cdots, x_j) \notin A_j\ , \qquad \text{we have that}$$

$$\frac{\partial^r}{\partial \theta_h^r} L_j(x/\Theta)$$

exists for $h=1, 2, \cdots, p$ and $r=1, 2, \cdots, k$. Now if $L_j(x/\Theta)=0$, we define

$$H_{j,h}^r(x/\Theta)=0\ , \qquad \text{that is,}$$

$H_{j,h}^r(x/\Theta)$ is defined completely for all $\theta_h \in \Omega$ $(1\leqq h\leqq p)$ and for almost all $x \in R_j$, where $H_{j,h}^r(x/\Theta)=(1/L_j(x/\Theta))(\partial^r/\partial\theta_h^r)L_j(x/\Theta)$.

iii) $\mathrm{E}_\Theta \mathrm{E}[g_{N,h}(X, \theta_h)/\Theta]=0$ for $1\leqq h\leqq p$ and all $\Theta \in \Omega_p$.

iv) There exists a set B_j, $j=1, 2, \cdots$, independent of θ_h $(1 \leqq h \leqq p)$, $\Pr(B_j)=0$, such that

$$\frac{\partial^r}{\partial \theta_h^r} g_{j,h}(x, \theta_h)$$

exists for all

$$(x_1, x_2, \cdots, x_j) \notin B_j \quad \text{and} \quad r=1, 2, \cdots, k .$$

v) For $j=1, 2, \cdots, h$ and $r=1, 2, \cdots, k$, let

$$T_j^r(x_1, x_2, \cdots, x_j)$$

be non-negative and L-measurable functions.

a) There exists a set C_j, $j=1, 2, \cdots$ independent of

$$\theta_h \ (1 \leqq h \leqq p) , \qquad \Pr(C_j)=0, \text{ such that,}$$

for all $(x_1, x_2, \cdots, x_j) \notin C_j$, we have

$$\left| \frac{\partial^r}{\partial \theta_h^r} g_{j,h}(x, \theta_h) L_j(x/\Theta) \right| < T_j^r(x_1, x_2, \cdots, x_j) .$$

b) $\int_{R_j} T_j^r(x_1, x_2, \cdots, x_j) \prod_{i=1}^{j} dx_i < \infty$.

vi) Let us define for $1 \leqq h \leqq p$ and $j=1, 2, \cdots$,

$$t_j(\theta_h) = \int_{R_j} g_{j,h}(x, \theta_h) L_j(x/\Theta) \prod_{i=1}^{j} dx_i ,$$

and let us also postulate the uniform convergence of the following series

$$\sum_{j=1}^{\infty} \frac{d^r}{d\theta_h^r} t_j(\theta_h) \qquad \text{for } r=1, 2, \cdots, k .$$

vii) For each $j=1, 2, \cdots$, there exist functions

$$S_j^r(x_1, x_2, \cdots, x_j) \qquad \text{for } r=1, 2, \cdots, k ,$$

such that, when $g_{j,h}(x, \theta_h)$ and

$$T_j^r(x_1, x_2, \cdots, x_j)$$

are replaced by unity and $S_j^r(x_1, x_2, \cdots, x_j)$, respectively, the conditions v) and vi) remain valid.

viii) The covariance matrix $\|\lambda_{rs}\|$ of $H_{N,h}^r(x/\Theta)$, $r=1, 2, \cdots, k$ exists and is non-singular.

3. Theorem

Under the above conditions, we can show that

$$\text{(3.1)} \qquad \mathrm{E}_\Theta \mathrm{E}\,[g^2_{N,h}(X, \theta_h)/\Theta] \geqq \sum_{r,s=1}^{k} \lambda^{rs} \mathrm{E}_\Theta \mathrm{E}\,[W^{r;0}_{N,h}(X, \theta_h)/\Theta] \cdot \mathrm{E}_\Theta \mathrm{E}\,[W^{s;0}_{N,h}(X, \theta_h)/\Theta]\ ,$$

where $W^{k;j}_{N,h}(x, \Theta_h) = H^k_{N,h}(x/\Theta)(\partial^j/\partial\theta^j_h) g_{N,h}(x, \theta_h)$ and λ^{rs} is the element of the rth line and sth column of the inverse matrix of $\|\lambda_{rs}\|$.

PROOF. Let

$$\text{(3.2)} \qquad D = g_{N,h}(x, \theta_h) - \sum_{r=1}^{k} \alpha_r H^r_{N,h}(x/\Theta)$$

where α_r's are independent of X and Θ for $r=1, 2, \cdots, k$. According to the sequential procedure developed by Wolfowitz [6], we can write

$$\text{(3.3)} \qquad \sum_{j=1}^{\infty} \int_{R_j} L_j(x/\Theta) \prod_{i=1}^{j} dx_i = 1\ .$$

Now taking the rth derivative with respect to θ_h $(1 \leqq h \leqq p)$ in both sides of (3.3), we get by virtue of condition vii)

$$\sum_{j=1}^{\infty} \int_{R_j} H^r_{j,h}(x/\Theta) L_j(x/\Theta) \prod_{i=1}^{j} dx_i = 0\ ,$$

that is,

$$\text{(3.4)} \qquad \mathrm{E}\,[H^r_{N,h}(X/\Theta)/\Theta] = 0 \qquad \text{for } r=1, 2, \cdots, k\ .$$

From iii) and (3.4), it follows that

$$\mathrm{E}_\Theta \mathrm{E}\,(D/\Theta) = 0\ .$$

So

$$\text{(3.5)} \quad \mathrm{Var}\,(D) = \mathrm{E}_\Theta \mathrm{E}\,(D^2/\Theta) = \mathrm{E}_\Theta \mathrm{E} \left\{ \left[g_{N,h}(X, \theta_h) - \sum_{r=1}^{k} \alpha_r H^r_{N,h}(X/\Theta) \right]^2 \Big/ \Theta \right\}\ .$$

Equation (3.5) can be written like

$$\text{(3.6)} \qquad \mathrm{Var}\,(D) = \mathrm{E}_\Theta \mathrm{E}\,[g^2_{N,h}(X, \theta_h)/\Theta] - 2 \sum_{r=1}^{k} \alpha_r \Lambda^r + \sum_{r,s=1}^{k} \alpha_r \alpha_s \lambda_{rs}\ ,$$

where

$$\lambda_{rs} = \mathrm{E}_\Theta \mathrm{E}\,[H^r_{N,h}(X/\Theta) \cdot H^s_{N,h}(X/\Theta)/\Theta] = \mathrm{Cov}\,[H^r_{N,h}(X/\Theta), H^s_{N,h}(X/\Theta)]$$

and

$$\Lambda^r = \mathrm{E}_\Theta\, \mathrm{E}\, [W_{N,h}^{r,0}(X, \theta_h)/\Theta] = \mathrm{Cov}\, [g_{N,h}(X, \theta_h), H_{N,h}^r(X/\Theta)] \,.$$

We propose now to determine the values of α_r for $r=1, 2, \cdots, k$ that minimize (3.6). Condition viii) says that $\|\lambda_{rs}\|$ is non-singular; then these values of α_r $(r=1, 2, \cdots, k)$ are given by

$$\alpha_{rs} = \sum_{s=1}^{k} \lambda^{rs} \Lambda^s \,. \tag{3.7}$$

Taking the values of α_r $(r=1, 2, \cdots, k)$ given by (3.7), into (3.6), we get

$$\mathrm{Var}\,(D) = \mathrm{E}_\Theta\, \mathrm{E}\, [g_{N,h}^2(X, \theta_h)/\Theta] - \sum_{r,s=1}^{k} \lambda^{rs} \Lambda^r \Lambda^s \,. \tag{3.8}$$

Since $\mathrm{Var}\,(D)$ is non-negative, from (3.8) it follows that

$$\mathrm{E}_\Theta\, \mathrm{E}\, [g_{N,h}^2(X, \theta_h)/\Theta] \geqq \sum_{r,s=1}^{k} \lambda^{rs} \Lambda^r \Lambda^s \,. \tag{3.9}$$

Obviously, by the definition of Λ^r, equation (3.9) can be written as

$$\begin{aligned} \mathrm{E}_\Theta\, \mathrm{E}\, [g_{N,h}^2(X, \theta_h)/\Theta] \geqq \sum_{r,s=1}^{k} \lambda^{rs}\, \mathrm{E}_\Theta\, \mathrm{E}\, [W_{N,h}^{r,0}(X, \theta_h)/\Theta] \\ \cdot\, \mathrm{E}_\Theta\, \mathrm{E}\, [W_{N,h}^{s,0}(X, \theta_h)/\Theta] \,, \end{aligned}$$

which is the result expressed by (3.1).

4. Application

From inequality (3.1), for $1 \leqq h \leqq s$, let

$$g_{N,h}(X, \theta_h) = \delta_{N,h}(X) - \theta_h - \mathrm{E}\, [\Psi_h(\Theta)] + \mathrm{E}\, (\theta_h)$$

be, such that

$$\mathrm{E}\, [\delta_{N,h}(X)] = \Psi_h(\theta) \,.$$

Thus, we get

$$\begin{aligned} &\mathrm{E}_\Theta\, \mathrm{E}\, \{[\delta_{N,h}(X) - \theta_h]^2/\Theta\} \\ &\geqq \mathrm{E}_\Theta^2\, [\Psi_h(\Theta) - \theta_h] + \sum_{r,s=1}^{k} \lambda^{rs}\, \mathrm{E}_\Theta \left[\frac{\partial^r}{\partial \theta_h^r} \Psi_h(\Theta)\right] \mathrm{E}_\Theta \left[\frac{\partial^s}{\partial \theta_h^s} \Psi_h(\Theta)\right] . \end{aligned} \tag{4.1}$$

When N is fixed and equal to n, (4.1) can be written as

$$\begin{aligned} &\mathrm{E}_\Theta\, \mathrm{E}\, \{[\delta_{n,h}(X) - \theta_h]^2/\Theta\} \\ &\geqq \mathrm{E}_\Theta^2\, [\Psi_h(\Theta) - \theta_h] + \sum_{r,s=1}^{k} \lambda^{rs}\, \mathrm{E}_\Theta \left[\frac{\partial^r}{\partial \theta_h^r} \Psi_h(\Theta)\right] \mathrm{E}_\Theta \left[\frac{\partial^s}{\partial \theta_h^s} \Psi_h(\Theta)\right] \end{aligned} \tag{4.2}$$

which is the inequality obtained by Gart [3]. Now, when Θ is constant and equal to θ for all $1 \leqq h \leqq p$, then (4.1) becomes

$$\text{(4.3)} \qquad \mathrm{E}\{[\delta_N(X)-\theta]^2/\theta\} - \mathrm{E}^2\{\delta_N(X)-\theta]\geqq \sum_{r,s=1}^{k} \lambda^{rs} \frac{d^r}{d\theta^r}\Psi(\theta)\frac{d^r}{d\theta^s}\Psi(\theta) .$$

From (4.3) it follows that

$$\text{(4.4)} \qquad \mathrm{Var}[\delta_N(X)]\geqq \sum_{r,s=1}^{k} \lambda^{rs} \frac{d^r}{d\theta^r}\Psi(\theta)\frac{d^s}{d\theta^s}\Psi(\theta) ,$$

which is the inequality obtained by Seth [5].

Finally, from (4.3) we get inequality proved by Wolfowitz [6] and from (3.1) inequalities obtained by Campos [2], which contains the Bhattacharyya [1] and Godambe [4] inequalities, setting appropriate conditions.

UNIVERSIDADE DE SÃO PAULO

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